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ON SOME INFINITE MATRICES OF BINOMIAL TYPE AND THEIR GENERATING FUNCTIONS

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Abstract. In this study, we aim to detect the generating functions of some special infinite matrices created with the help of special number sequences, then to obtain a formula for the inputs of the infinite matrix. Moreover, we study to determine some known or unknown sequences in each row and column of the infinite matrix. Also, we specify a countable many of sequences with a single formula.

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1. Introduction

The history of generating functions dates back to 1730. In 1730, a French mathematician Abraham De Moivre first used the generating function method to solve the general linear recurrence problem. These functions, which can roughly be considered as the expression of sequences in terms of functions, are widely used tools in studying problems in combinatorics, mathematical analysis, statistics, *etc.* They transform problems related to sequences into problems related to functions, making them relatively easier to solve. So far, there are many great attempts by researchers to develop the studies on generating functions in families of numbers and polynomials some of which can be seen in [1–12].

It has been seen in recent years that new or extended special numerical sequences and generating functions are obtained. Some mathematicians obtained new special sequences *via* adding another variable to known sequences or adding another variable to their generating functions. In this paper, regardless of these trends, we obtain new double sequences with the help of a single recurrence relation, as also in the famous Fibonacci, Lucas, Pell sequences *etc.* Furthermore, we support our obtained results with some concrete examples. Therefore, we think that our study will keep light on forthcoming studies and guide researchers to relevant applications to some areas such as combinatorics, statistics and cryptology.

Fibonacci numbers are represented by recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \in \mathbb{N}$ with initial numbers $F_0 = 0$ and $F_1 = 1$. They are the terms of the sequence $\{0, 1, 1, 2, 3, 5, 8, 13, 21, ...\}$. Also, Fibonacci sequence has the generating function $\dfrac{x}{1-x-x^2}$.

Keywords and phrases: Generating functions, infinite matrices, Fibonacci numbers, Lucas numbers, Pell numbers.

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Lucas numbers are given by recurrence relation $L_{n+2} = L_{n+1} + L_n$ for $n \in \mathbb{N}$ with initial numbers $L_0 = 2$ and $L_1 = 1$. They form the sequence of numbers $\{2, 1, 3, 4, 7, 11, ...\}$ and the sequence has the generating function $\dfrac{2-x}{1-x-x^2}$ [13].

Pell numbers are given by recurrence relation $P_{n+2} = 2P_{n+1} + P_n$ for $n \in \mathbb{N}$ with initial numbers $P_0 = 0$ and $P_1 = 1$. They are the terms of the sequence $\{0, 1, 2, 5, 12, 29, ...\}$ and the sequence has the generating function $\dfrac{x}{1-2x-x^2}$ [14].

Pell-Lucas numbers are defined by the recurrence relation $Q_{n+2} = 2Q_{n+1} + Q_n$ for $n \in \mathbb{N}$ with initial elements $Q_0 = 2$ and $Q_1 = 2$. The function $\dfrac{2-2x}{1-2x-x^2}$ generates Pell-Lucas numbers and this numbers are the terms of the sequence $\{2, 2, 6, 14, 34, 82, ...\}$ [14, 15].

Now, we focus on the key definitions and results used in this work.

In 1962, Gerst [16] mentioned the concept of a bivariate ordinary generating function, which is also associated with double sequences whose definition is as follows:

A double sequence of real numbers is a function $a : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of all natural numbers. We will use the notation $\{a_{n,k}\}_0^\infty$ for a double sequence a throughout this article. The value of double sequence $a = (a_{n,k})$ at a point $(n, k) \in \mathbb{N}^2$ is real number $a_{n,k}$ and it is called (n, k)-th term of a. So, we can think of the elements $a_{n,k}$ of the double sequence $(a_{n,k})$ as a matrix

$$\begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,k} & \cdots \\ a_{1,0} & a_{1,1} & \cdots & a_{1,k} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,k} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}.$$

The bivariate generating function corresponding to the double sequence $f(m, n)$ is given as follows:

Definition 1.1. Given a function $f(m, n)$ of two discrete variable m and n, where $m = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots$, we associate with it the function of two variables defined by the double power series

$$F(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n)\, x^m y^n. \tag{1.1}$$

$F(x, y)$ is said to be the bivariate generating function corresponding to $f(m, n)$, if the series in (1.1) converges absolutely in some neighborhood of the origin, $x = 0$, $y = 0$. We shall also refer to $f(m, n)$ as a double sequence and to $F(x, y)$ as its corresponding transform [16].

Definition 1.2. The symbol $f \overset{odps}{\leftrightarrow} \{a_{n,k}\}_0^\infty$ means that f is the bivariate (ordinary) generating function in the form of two-variable power series of double sequence $\{a_{n,k}\}_0^\infty$. Namely,

$$f = f(x, y) = \sum_{n,k \geq 0} a_{n,k} x^n y^k = \sum_{n \geq 0} \sum_{k \geq 0} a_{n,k} x^n y^k$$

[16].

Proposition 1.3. *If* $f \overset{odps}{\leftrightarrow} \{a_{n,k}\}_0^\infty$, *then we have*

$$\left(\frac{f-f(0,y)}{x}\right) \overset{odps}{\leftrightarrow} \{a_{n+1,k}\}_0^\infty ,$$

$$\left(\frac{f-f(x,0)}{y}\right) \overset{odps}{\leftrightarrow} \{a_{n,k+1}\}_0^\infty ,$$

$$\left(\frac{f-f(x,0)-f(0,y)+f(0,0)}{xy}\right) \overset{odps}{\leftrightarrow} \{a_{n+1,k+1}\}_0^\infty$$

[16].

Definition 1.4. Let a double sequence $\{a_{n,k}\}_0^\infty$ be given. The double sequence $\left\{a_{n,k}^{(i,0)}\right\}_0^\infty$ is i row down shifted double sequence of $\{a_{n,k}\}_0^\infty$ for integer $i>0$, and it is defined as

$$a_{n,k}^{(i,0)} = \begin{cases} a_{n-i,k}\ , & n \geq i \\ 0, & otherwise. \end{cases}$$

The double sequence $\left\{a_{n,k}^{(0,j)}\right\}_0^\infty$ is j column right shifted double sequence of $\{a_{n,k}\}_0^\infty$ for integer $j>0$, and it is defined as

$$a_{n,k}^{(0,j)} = \begin{cases} a_{n,k-j}\ , & k \geq j \\ 0, & otherwise. \end{cases}$$

The double sequence $\left\{a_{n,k}^{(i,j)}\right\}_0^\infty$ is i row down and j column right shifted double sequence of $\{a_{n,k}\}_0^\infty$ for integers $i,j>0$, and it is defined as

$$a_{n,k}^{(i,j)} = \begin{cases} a_{n-i,k-j}\ , & n \geq i \text{ and } k \geq j \\ 0, & otherwise. \end{cases}$$

Accordingly, $\left\{a_{n,k}^{(0,0)}\right\}_0^\infty = \{a_{n,k}\}_0^\infty$ [17].

For a little illustration of definition above, we might compare double sequences $\{a_{n,k}\}_0^\infty$ and $\left\{a_{n,k}^{(1,2)}\right\}_0^\infty$. If these sequences are written in matrix form, it is seen that $\{a_{n,k}\}_0^\infty$ is in the form of

$$\begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,k} & \cdots \\ a_{1,0} & a_{1,1} & \cdots & a_{1,k} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ a_{n,0} & a_{n,1} & \cdots & a_{n,k} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}$$

and $\left\{a_{n,k}^{(1,2)}\right\}_0^\infty$ is in the form of

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & a_{0,0} & a_{0,1} & \dots & a_{0,k-2} & \dots \\ 0 & 0 & a_{1,0} & a_{1,1} & \dots & a_{1,k-2} & \dots \\ 0 & 0 & a_{2,0} & a_{2,1} & \dots & a_{2,k-2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \dots \\ 0 & 0 & a_{n-1,0} & a_{n-1,1} & \dots & a_{n-1,k-2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \dots \end{bmatrix}.$$

Proposition 1.5. *Let integers $i,j>0$. If $f \overset{odps}{\leftrightarrow} \{a_{n,k}\}_0^\infty$, then $t^i u^j f \overset{odps}{\leftrightarrow} \left\{a_{n,k}^{(i,j)}\right\}_0^\infty$ [17].*

2. Main results

Now, we can move on purpose of this paper, which is getting and determining some double, row and column sequences by the help of a single recurrence relation

$$a_{n+1,k+1} = a_{n+1,k} + a_{n,k+1} \ \ (n,k \geq 0) \tag{2.1}$$

for given first row sequence $\{a_{0,k}\}_0^\infty$ and first column sequence $\{a_{n,0}\}_0^\infty$.

Let $\{b_{n,k}\}_0^\infty$ be a double sequence that satisfies the recurrence relation (2.1) with $b_{n,0} = b_{0,k} = 1$ for all $n,k \in \mathbb{N}$. If a few terms of $\{b_{n,k}\}_0^\infty$ are computed and written in rectangular form, then

$n\backslash k$	**0**	**1**	**2**	**3**	**4**
0	1	1	1	1	1
1	1	2	3	4	5
2	1	3	6	10	15
3	1	4	10	20	35
4	1	5	15	35	70

is found. Clearly, it is seen that table above represents Pascal's triangle in rectangular form and $b_{n,k}$ is the well known binomial number $\binom{n+k}{k}$, the generating function of $\{b_{n,k}\}_0^\infty$ is function $\dfrac{1}{1-x-y}$. Furthermore, the row and column sequences of $\{b_{n,k}\}_0^\infty$ are well known.

It should be useful finding an explicit formula for $\{a_{n,k}\}_0^\infty$ in general statement. We will give this as following theorem:

Theorem 2.1. *Let a double sequence $\{a_{n,k}\}_0^\infty$ holds recurrence relation (2.1) for given sequences $\{a_{n,0}\}_0^\infty$ and $\{a_{0,k}\}_0^\infty$, and let $A(x,y) \overset{odps}{\leftrightarrow} \{a_{n,k}\}_0^\infty$. Then, general term of $\{a_{n,k}\}_0^\infty$ and its generating function are as follows, respectively:*

$$a_{n,k} = -a_{0,0}\binom{n+k}{k} + \sum_{j=0}^{k}\binom{n+j}{j}\left(a_{0,k-j} - a_{0,k-j}^{(0,1)}\right) + \sum_{i=0}^{n}\binom{i+k}{k}\left(a_{n-i,0} - a_{n-i,0}^{(1,0)}\right)$$

and

$$A(x,y) = \frac{1}{1-x-y}\left[-A(0,0) + (1-y)A(0,y) + (1-x)A(x,0)\right].$$

Proof. Let the sequences $\{a_{n,0}\}_0^\infty$ and $\{a_{0,k}\}_0^\infty$ be given and let the double sequence $\{a_{n,k}\}_0^\infty$ holds the recurrence relation (2.1). If both sides of (2.1) are multiplied by $x^n y^k$ and then summed over $n,k \geq 0$, we get

$$\sum_{n\geq 0}\sum_{k\geq 0} a_{n+1,k+1}x^n y^k = \sum_{n\geq 0}\sum_{k\geq 0} a_{n+1,k}x^n y^k + \sum_{n\geq 0}\sum_{k\geq 0} a_{n,k+1}x^n y^k.$$

Utilizing Proposition 1.3, we can write

$$\frac{A(x,y) - A(x,0) - A(0,y) + A(0,0)}{xy} = \frac{A(x,y) - A(0,y)}{x} + \frac{A(x,y) - A(x,0)}{y}.$$

This implies that

$$A(x,y) = \frac{1}{1-x-y}\left[-A(0,0) + (1-y)A(0,y) + (1-x)A(x,0)\right]. \tag{2.2}$$

Here, $A(x,0) = \sum_{n\geq 0} a_{n,0}x^n$, $A(0,y) = \sum_{k\geq 0} a_{0,k}y^k$, $A(0,0) = a_{0,0}$ and $\dfrac{1}{1-x-y} = \sum\limits_{n\geq 0}\sum\limits_{k\geq 0}\binom{n+k}{k}x^n y^k$. If we substitute them in equality (2.2), we deduce that

$$\begin{aligned} A(x,y) &= \sum_{n,k\geq 0}\binom{n+k}{k}x^n y^k\left[-a_{0,0} + (1-y)\sum_{k\geq 0}a_{0,k}y^k + (1-x)\sum_{n\geq 0}a_{n,0}x^n\right] \\ &= \sum_{n,k\geq 0} -a_{0,0}\binom{n+k}{k}x^n y^k + \sum_{n,k\geq 0}\left[\begin{array}{c} \sum_{j=0}^{k}\binom{n+j}{j}\left(a_{0,k-j} - a_{0,k-j}^{(0,1)}\right) \\ +\sum_{i=0}^{n}\binom{i+k}{k}\left(a_{n-i,0} - a_{n-i,0}^{(1,0)}\right)\end{array}\right]x^n y^k. \end{aligned}$$

Therefore, the formula for general term of $\{a_{n,k}\}_0^\infty$ is found as

$$a_{n,k} = -a_{0,0}\binom{n+k}{k} + \sum_{j=0}^{k}\binom{n+j}{j}\left(a_{0,k-j} - a_{0,k-j}^{(0,1)}\right) + \sum_{i=0}^{n}\binom{i+k}{k}\left(a_{n-i,0} - a_{n-i,0}^{(1,0)}\right).$$

This finalizes the proof. □

The following result is dedicated to finding the general term and generating function of a double sequence $\{f_{n,k}\}_0^\infty$, which consists of Fibonacci numbers.

Corollary 2.2. *Let $\{f_{n,0}\}_0^\infty$ and $\{f_{0,k}\}_0^\infty$ be Fibonacci sequences. Assume that $\{f_{n,k}\}_0^\infty$ is a double sequence that satisfies the recurrence relation (2.1). Then, the general term of $\{f_{n,k}\}_0^\infty$ is*

$$f_{n,k} = \sum_{j=0}^{k}\binom{n+j}{j}\left(F_{k-j} - F_{k-j}^{(0,1)}\right) + \sum_{i=0}^{n}\binom{i+k}{k}\left(F_{n-i} - F_{n-i}^{(1,0)}\right)$$

and

$$\left(\frac{1}{1-x-y}\left[\frac{y-y^2}{1-y-y^2} + \frac{x-x^2}{1-x-x^2}\right]\right) \overset{odps}{\leftrightarrow} \{f_{n,k}\}_0^\infty$$

by Theorem 2.1.

If some terms of double sequence $\{f_{n,k}\}_0^\infty$ are computed, then an infinite symmetric matrix is found as follows:

$n \backslash k$	**0**	**1**	**2**	**3**	**4**	**5**	$\cdots$
0	0	1	1	2	3	5	$\cdots$
1	1	2	3	5	8	13	$\cdots$
2	1	3	6	11	19	32	$\cdots$
3	2	5	11	22	41	73	$\cdots$
4	3	8	19	41	82	155	$\cdots$
5	5	13	32	73	155	310	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The triangular form of this matrix is A108617 in [15] and also, it is seen that row sequences are $\{f_{1,k}\}_0^\infty = \{F_{k+2}\}_0^\infty$ and $\{f_{2,k}\}_0^\infty = \{F_{k+4} - 2\}_0^\infty$.

The next corollary gives us an information about the generating function and general term of a new sequence $\{l_{n,k}\}_0^\infty$ derived by Lucas sequences.

Corollary 2.3. *Suppose that $\{l_{n,0}\}_0^\infty$ and $\{l_{0,k}\}_0^\infty$ are Lucas sequences. Let a double sequence $\{l_{n,k}\}_0^\infty$ holds recurrence relation (2.1). Then, the general term of the double sequence is*

$$l_{n,k} = -2\binom{n+k}{k} + \sum_{j=0}^{k} \binom{n+j}{j} \left(L_{k-j} - L_{0,k-j}^{(0,1)}\right) + \sum_{i=0}^{n} \binom{i+k}{k} \left(L_{n-i} - L_{n-i}^{(1,0)}\right)$$

and its generating function is in the form of

$$L(x,y) = \frac{1}{1-x-y}\left[-2 + \frac{2-3y+y^2}{1-y-y^2} + \frac{2-3x+x^2}{1-x-x^2}\right]$$

by Theorem 2.1.

If we calculate the terms of $\{l_{n,k}\}_0^\infty$, then we find the infinite symmetric matrix as follows:

$n \backslash k$	**0**	**1**	**2**	**3**	**4**	**5**	$\cdots$
0	2	1	3	4	7	11	$\cdots$
1	1	2	5	9	16	27	$\cdots$
2	3	5	10	19	35	62	$\cdots$
3	4	9	19	38	73	135	$\cdots$
4	7	16	35	73	146	281	$\cdots$
5	11	27	62	135	281	562	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

If we write this matrix as triangular form, we get A347584 in [15]. Also, row sequence $\{l_{1,k}\}_0^\infty$ is A014739 in [15].

In the following corollary, we obtain double version of generating function of Pell sequence.

Corollary 2.4. *Assume that $\{p_{n,0}\}_0^\infty$ and $\{p_{0,k}\}_0^\infty$ are Pell sequences. Let $\{p_{n,k}\}_0^\infty$ be a double sequence, which satisfies the recurrence relation (2.1). Then, the general term of this double sequence is*

$$p_{n,k} = \sum_{j=0}^{k} \binom{n+j}{j} \left(P_{k-j} - P_{k-j}^{(0,1)}\right) + \sum_{i=0}^{n} \binom{i+k}{k} \left(P_{n-i} - P_{n-i}^{(1,0)}\right),$$

and also

$$\left(\frac{1}{1-x-y}\left[\frac{y-y^2}{1-2y-y^2}+\frac{x-x^2}{1-2x-x^2}\right]\right)\overset{odps}{\leftrightarrow}\{p_{n,k}\}_0^\infty$$

by Theorem 2.1.

Also, the infinite symmetric matrix

$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**5**	$\cdots$
0	0	1	2	5	12	29	$\cdots$
1	1	2	4	9	21	50	$\cdots$
2	2	4	8	17	38	88	$\cdots$
3	5	9	17	34	72	160	$\cdots$
4	12	21	38	72	144	304	$\cdots$
5	29	50	88	160	304	608	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

consists of the terms of $\{p_{n,k}\}_0^\infty$. From this matrix, we see that row sequence $\{p_{1,k}\}_0^\infty$ is A024837 and row sequence $\{p_{2,k}\}_0^\infty$ is A100131 in [15]. Other column and row sequences are not included in [15].

In the next result, we find some terms of the double sequence formed by expanding the Pell-Lucas sequence.

Corollary 2.5. *Let $\{q_{n,0}\}_0^\infty$ and $\{q_{0,k}\}_0^\infty$ be Pell-Lucas sequences. Assume that a double sequence $\{q_{n,k}\}_0^\infty$ holds the recurrence relation (2.1). Then, the general term of $\{q_{n,k}\}_0^\infty$ is*

$$q_{n,k}=-2\binom{n+k}{k}+\sum_{j=0}^{k}\binom{n+j}{j}\left(Q_{k-j}-Q_{k-j}^{(0,1)}\right)+\sum_{i=0}^{n}\binom{i+k}{k}\left(Q_{n-i}-Q_{n-i}^{(1,0)}\right)$$

and its generating function is

$$P\left(x,y\right)=\frac{1}{1-x-y}\left[-2+\frac{2-4y+2y^2}{1-2y-y^2}+\frac{2-4x+2x^2}{1-2x-x^2}\right]$$

by Theorem 2.1. Besides, some terms of double sequence $\{q_{n,k}\}_0^\infty$ form an infinite symmetric matrix as follows:

$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**5**	$\cdots$
0	2	2	6	14	34	82	$\cdots$
1	2	4	10	24	58	140	$\cdots$
2	6	10	20	44	102	242	$\cdots$
3	14	24	44	88	190	432	$\cdots$
4	34	58	102	190	380	812	$\cdots$
5	82	140	242	432	812	1624	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

From this matrix, one can see that row sequence $\{q_{1,k}\}_0^\infty$ is left shifted sequence of A052542 and A163271 in [15]. Other column and row sequences are not found in [15].

In Corollary 5, we present the generating function of $\{c_{n,k}\}_0^\infty$, which is the combination of Fibonacci and Pell sequences.

Corollary 2.6. *Let $\{c_{n,0}\}_0^\infty$ be Fibonacci sequence and let $\{c_{0,k}\}_0^\infty$ be Pell sequence. Suppose that a double sequence $\{c_{n,k}\}_0^\infty$ holds the recurrence relation (2.1). Then, its general term is*

$$c_{n,k} = \sum_{j=0}^{k} \binom{n+j}{j} \left(P_{k-j} - P_{k-j}^{(0,1)}\right) + \sum_{i=0}^{n} \binom{i+k}{k} \left(F_{n-i} - F_{n-i}^{(1,0)}\right),$$

and also

$$\left(\frac{1}{1-x-y}\left[\frac{y-y^2}{1-2y-y^2} + \frac{x-x^2}{1-x-x^2}\right]\right) \overset{odps}{\leftrightarrow} \{c_{n,k}\}_0^\infty$$

by Theorem 2.1.

Also, we find the corresponding infinite matrix as

$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**5**	$\cdots$
0	0	1	2	5	12	29	$\cdots$
1	1	2	4	9	21	50	$\cdots$
2	1	3	7	16	37	87	$\cdots$
3	2	5	12	28	65	152	$\cdots$
4	3	8	20	48	113	265	$\cdots$
5	5	13	33	81	194	459	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

by computing some terms of double sequence $\{c_{n,k}\}_0^\infty$. From this matrix, it is seen that row sequence $\{c_{1,k}\}_0^\infty$ is A024537 in [15] and column sequences are $\{c_{n,1}\}_0^\infty = \{F_{n+2}\}_0^\infty$ and $\{c_{n,2}\}_0^\infty = \{F_{n+4} - 1\}_0^\infty$. Other column and row sequences are not found in [15].

The subsequent result introduces the generating function and general term of a new double sequence obtained by Lucas and Pell-Lucas numbers.

Corollary 2.7. *Assume that $\{d_{n,0}\}_0^\infty$ is Lucas sequence and $\{d_{0,k}\}_0^\infty$ is Pell-Lucas sequence. Let a double sequence $\{d_{n,k}\}_0^\infty$ holds the recurrence relation (2.1). Then, its general term*

$$d_{n,k} = -2\binom{n+k}{k} + \sum_{j=0}^{k} \binom{n+j}{j} \left(Q_{k-j} - Q_{k-j}^{(0,1)}\right) + \sum_{i=0}^{n} \binom{i+k}{k} \left(L_{n-i} - L_{n-i}^{(1,0)}\right),$$

and also

$$\left(\frac{1}{1-x-y}\left[-2 + \frac{2-4y+2y^2}{1-2y-y^2} + \frac{2-3x+x^2}{1-x-x^2}\right]\right) \overset{odps}{\leftrightarrow} \{d_{n,k}\}_0^\infty$$

by Theorem 2.1.

If we write some terms of double sequence $\{d_{n,k}\}_0^\infty$, then we obtain the corresponding infinite matrix as follows:

$n\backslash k$	***0***	***1***	***2***	***3***	***4***	***5***	$\cdots$
0	*2*	*2*	*6*	*14*	*34*	*82*	$\cdots$
1	*1*	*3*	*9*	*23*	*57*	*129*	$\cdots$
2	*3*	*6*	*15*	*38*	*95*	*224*	$\cdots$
3	*4*	*10*	*25*	*63*	*158*	*382*	$\cdots$
4	*7*	*17*	*52*	*115*	*273*	*655*	$\cdots$
5	*11*	*28*	*80*	*195*	*468*	*1123*	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

In addition, we see that column sequence $\{d_{n,1}\}_0^\infty$ is left shifted sequence of A001610 and it is also two times left shifted sequence of A324015 in [15]. Other column and row sequences are not found in [15].

We continue the article with some special results obtained by taking $\{a_{0,k}\}_0^\infty = \{1\}_{k=0}^\infty$.

Theorem 2.8. *Let a double sequence $\{a_{n,k}\}_{n,k=0}^\infty$ holds the recurrence relation (2.1) for given sequences $\{a_{n,0}\}_0^\infty$ and $\{a_{0,k}\}_0^\infty = \{1\}_{k=0}^\infty$, and let $A(x,y) \overset{odps}{\leftrightarrow} \{a_{n,k}\}_0^\infty$. Then, we have*

$$A(x,y) = \frac{(1-x)A(x,0)}{1-x-y}$$

and the general term of $\{a_{n,k}\}_{n,k=0}^\infty$ is in the form

$$a_{n,k} = \sum_{i=0}^{n} \binom{i+k-1}{i} a_{n-i,0}.$$

Here k-th column sequence $\{a_{n,k}\}_{n=0}^\infty$ is k times partial summed sequence of $\{a_{n,0}\}_0^\infty$. Additionally, k-th column generating function is

$$A_k(x) = \sum_{n\geq 0} a_{n,k}x^n = \frac{A(x,0)}{(1-x)^k}.$$

Proof. Since $A(0,y) = \sum_{k\geq 0} a_{0,k}y^k = \sum_{k\geq 0} y^k = \frac{1}{1-y}$ and $A(0,0) = 1$, by Theorem 2.1, it is easily seen that

$$A(x,y) = \frac{1}{1-x-y}\left[-1 + (1-y)\frac{1}{1-y} + (1-x)A(x,0)\right] = \frac{(1-x)A(x,0)}{1-x-y}.$$

If this function is expanded in series in powers of y, then

$$\begin{aligned} A(x,y) &= A(x,0)\frac{1}{\frac{1-x-y}{1-x}} \\ &= A(x,0)\frac{1}{1-\frac{y}{1-x}} \end{aligned}$$

$$= A(x,0)\sum_{k\geq 0}\left(\frac{y}{1-x}\right)^k$$
$$= \frac{A(x,0)}{(1-x)^k}\sum_{k\geq 0} y^k$$

is obtained. This shows that $A_k(x) = \dfrac{A(x,0)}{(1-x)^k}$. Furthermore, by virtue of rule 5 in [18], k-th column sequence $\{a_{n,k}\}_{n=0}^{\infty}$ is k times partial summed sequence of $\{a_{n,0}\}_0^{\infty}$, since

$$A_k(x) = \sum_{n\geq 0} a_{n,k}x^n = \frac{1}{(1-x)^k}\sum_{n\geq 0} a_{n,0}x^n.$$

Finally if we expand $A_k(x)$ in series in powers of x, then

$$\begin{aligned}
A(x,y) &= \sum_{k\geq 0}\frac{A(x,0)}{(1-x)^k}y^k \\
&= \sum_{k\geq 0}\frac{1}{(1-x)^k}\sum_{n\geq 0} a_{n,0}x^n y^k \\
&= \sum_{k\geq 0}\left(\left(\sum_{n\geq 0}\binom{n+k-1}{n}x^n\right)\left(\sum_{n\geq 0} a_{n,0}x^n\right)\right)y^k \\
&= \sum_{k\geq 0}\sum_{n\geq 0}\left(\sum_{i=0}^{n}\binom{i+k-1}{i}a_{n-i,0}\right)x^n y^k.
\end{aligned}$$

This shows that $a_{n,k} = \sum\limits_{i=0}^{n}\binom{i+k-1}{i}a_{n-i,0}$. So, the proof is completed. □

Corollary 2.9. *Suppose that a double sequence $\{r_{n,k}\}_{n,k=0}^{\infty}$ holds the recurrence relation (2.1), where $\{r_{n,0}\}_0^{\infty} = \{F_{n+1}\}_0^{\infty}$ and $\{r_{0,k}\}_0^{\infty} = \{1\}_{k=0}^{\infty}$. Let $R(x,y) \overset{odps}{\leftrightarrow} \{r_{n,k}\}_0^{\infty}$. Then, $R(x,y)$ is in the form*

$$R(x,y) = \frac{(1-x)}{(1-x-y)(1-x-x^2)}$$

and the general term of $\{r_{n,k}\}_{n,k=0}^{\infty}$ is

$$r_{n,k} = \sum_{i=0}^{n}\binom{i+k-1}{i}F_{n+1-i}.$$

In addition, we get the corresponding infinite matrix

$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	$\cdots$
0	1	1	1	1	1	1	1	$\cdots$
1	1	2	3	4	5	6	7	$\cdots$
2	2	4	7	11	16	22	29	$\cdots$
3	3	7	14	25	41	63	92	$\cdots$
4	5	12	26	51	92	155	247	$\cdots$
5	8	20	46	97	189	344	591	$\cdots$
6	13	33	79	176	365	709	1300	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

with some terms of $\{r_{n,k}\}_0^\infty$. If this matrix is written as triangular form, one can see that the triangular form is A105809 in [15]. Also, the sequence $\{r_{n,1}\}_0^\infty$ is partial sums sequence of Fibonacci sequence $\{F_{n+3}\}_0^\infty$, which is the sequence $\{F_{n+2}-1\}_0^\infty$. Further, if we look at the matrix, we see row and column sequences, which correspond to some sequences in [15]. For example; $\{r_{n,2}\}_0^\infty$ is A001924, $\{r_{n,3}\}_0^\infty$ is A014162, $\{r_{n,4}\}_0^\infty$ is A014166, $\{r_{n,5}\}_0^\infty$ is A053739, $\{r_{n,6}\}_0^\infty$ is A053295, $\{r_{2,k}\}_0^\infty$ is A000124, $\{r_{3,k}\}_0^\infty$ is A004006, $\{r_{4,k}\}_0^\infty$ is A027927, $\{r_{5,k}\}_0^\infty$ is A027528, $\{r_{6,k}\}_0^\infty$ is A027929 in [15]. On the other hand, in this matrix, each column sequence is partial sums sequence of its left column sequence.

Corollary 2.10. *Let a double sequence $\{s_{n,k}\}_{n,k=0}^\infty$ holds recurrence relation (2.1), where $\{s_{n,0}\}_0^\infty=\{L_{n+1}\}_0^\infty$ and $\{s_{0,k}\}_0^\infty=\{1\}_{k=0}^\infty$, and let $S(x,y)\overset{odps}{\leftrightarrow}\{s_{n,k}\}_0^\infty$. Then, we have*

$$S(x,y)=\frac{(1-x)(1+2x)}{(1-x-y)(1-x-x^2)}$$

and general term of $\{s_{n,k}\}_{n,k=0}^\infty$ is in the form of

$$s_{n,k}=\sum_{i=0}^{n}\binom{i+k-1}{i}L_{n+1-i}.$$

In addition, if the terms of double sequence $\{s_{n,k}\}_0^\infty$ are calculated, then the corresponding infinite matrix is as follows:

$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	$\cdots$
0	1	1	1	1	1	1	1	$\cdots$
1	3	4	5	6	7	8	9	$\cdots$
2	4	8	13	19	26	34	43	$\cdots$
3	7	15	28	47	73	107	150	$\cdots$
4	11	26	54	101	174	281	431	$\cdots$
5	18	44	98	199	373	654	1085	$\cdots$
6	29	73	171	370	743	1397	2482	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

If we indicate this matrix as triangular form, then we obtain that it is A125608 in [15]. Besides, the sequence $\{s_{n,1}\}_0^\infty$ is partial sums sequence of Lucas sequence $\{L_{n+1}\}_0^\infty$, which is the sequence $\{L_{n+2}-3\}_0^\infty$. Additionally, if we examine row and column sequences in the matrix, we see that they correspond to some sequences in [15]. Namely; $\{s_{n,2}\}_0^\infty$ is A023537, $\{s_{n,3}\}_0^\infty$ is A027963, $\{s_{n,4}\}_0^\infty$ is A02767, $\{s_{n,5}\}_0^\infty$ is A053298, $\{s_{2,k}\}_0^\infty$ is A034856, $\{s_{3,k}\}_0^\infty$ is A027965, $\{s_{4,k}\}_0^\infty$ is A027966, $\{s_{5,k}\}_0^\infty$ is A027967, $\{s_{6,k}\}_0^\infty$ is A027968 in [15]. Notice that each column sequence is partial sums sequence of its left column sequence in this matrix.

The following proposition reveals the relationship between double sequences of bivariate polynomials in [17] and their classical versions in the present study.

Proposition 2.11. *Let* $\{f_{n,k}\}_0^\infty$, $\{l_{n,k}\}_0^\infty$, $\{p_{n,k}\}_0^\infty$, $\{q_{n,k}\}_0^\infty$, $\{c_{n,k}\}_0^\infty$, $\{d_{n,k}\}_0^\infty$ *and* $\{r_{n,k}\}_0^\infty$ *be double sequences, which are given in corollaries of this paper. Let double sequences of bivariate polynomials* $\{f_{n,k}(x,y)\}_0^\infty$, $\{J_{n,k}(x,y)\}_0^\infty$, $\{l_{n,k}(x,y)\}_0^\infty$, $\{p_{n,k}(x,y)\}_0^\infty$, $\{q_{n,k}(x,y)\}_0^\infty$, $\{c_{n,k}(x,y)\}_0^\infty$, $\{d_{n,k}(x,y)\}_0^\infty$ *and* $\{r_{n,k}(x,y)\}_0^\infty$ *be double sequences of bivariate polynomials, which are given in corollaries of [17]. Then, for all* $n,k \geq 0$, *the following equalities are satisfied:*

$$\begin{aligned} f_{n,k}(1,1) &= J_{n,k}(1,1) = f_{n,k}, \\ l_{n,k}(1,1) &= l_{n,k}, \\ p_{n,k}(1,1) &= p_{n,k}, \\ q_{n,k}(1,1) &= q_{n,k}, \\ c_{n,k}(1,1) &= c_{n,k}, \\ d_{n,k}(1,1) &= d_{n,k}, \\ r_{n,k}(1,1) &= r_{n,k}. \end{aligned}$$

Proof. If we substitute $x = y = 1$ in bivariate generating functions $F(x,y;t,u)$, $J(x,y;t,u)$, $L(x,y;t,u)$, $P(x,y;t,u)$, $Q(x,y;t,u)$, $C(x,y;t,u)$, $D(x,y;t,u)$ and $R(x,y;t,u)$ obtained in [17], then one can easily see that

$$\begin{aligned} F(1,1;t,u) &= J(1,1;t,u) = F(t,u), \\ L(1,1;t,u) &= L(t,u), \\ P(1,1;t,u) &= P(t,u), \\ Q(1,1;t,u) &= Q(t,u), \\ C(1,1;t,u) &= C(t,u), \\ D(1,1;t,u) &= D(t,u), \\ R(1,1;t,u) &= R(t,u). \end{aligned}$$

Since $F(t,u) \overset{odps}{\leftrightarrow} \{f_{n,k}\}_0^\infty$, $L(t,u) \overset{odps}{\leftrightarrow} \{l_{n,k}\}_0^\infty$, $P(t,u) \overset{odps}{\leftrightarrow} \{p_{n,k}\}_0^\infty$, $Q(t,u) \overset{odps}{\leftrightarrow} \{q_{n,k}\}_0^\infty$, $C(t,u) \overset{odps}{\leftrightarrow} \{c_{n,k}\}_0^\infty$, $D(t,u) \overset{odps}{\leftrightarrow} \{d_{n,k}\}_0^\infty$ and $R(t,u) \overset{odps}{\leftrightarrow} \{r_{n,k}\}_0^\infty$, it follows that the desired equalities are satisfied. This completes the proof. □

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