RAIRO-Theor. Inf. Appl. 60 (2026) 11
https://doi.org/10.1051/ita/2026012

RAIRO - Theoretical Informatics and Applications
www.rairo-ita.org

GENERAL POSITION NUMBER OF HIERARCHICAL SCALE-FREE GRAPHS

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Abstract. In this paper, we consider one of graph invariants, called a general position number, of scale-free graphs, where the scale-free property is one of key properties of real-world networks. To understand the tendency of general position numbers of scale-free graphs, we primarily deal with deterministic artificial scale-free graphs, since such graphs are very manageable and artificial models of complex networks mimic many properties of real-world networks. We derive estimates for the general position number of three well-known hierarchical scale-free graphs generated by deterministic models: one model produces fractal graphs, while the other two generate non-fractal graphs.

Mathematics Subject Classification. 05C82.

Received June 5, 2024. Accepted February 22, 2026.

1. Introduction

It is a long-standing challenging problem to analyze the structure of real-world networks. Analytical results for them give us many important applications in the real world. As an example, a complex network has *community structure*, which is a (potentially overlapping) decomposition of the set of nodes such that each component is densely connected internally. Here such a component is called a *community*. If we could find the community structure of a given network, we can efficiently advertise our product in the network by promoting it to a person in each community. Also, graph invariants help us to understand the structure of given networks. A typical one of such invariants is the *betweenness centrality* of nodes. It is a distance-based centrality measure which ranks nodes according to the number of shortest paths passing them. As a community structure, such centrality also has many practical applications. Many research results have been reported so far regarding the structures and invariants introduced above. However, resolving a given network structure remains a challenging problem.

In this paper, we consider the general position number of graphs. A *general position set* is the set S of vertices such that a shortest path between every two vertices in S does not pass another one in S. This is motivated by the century old Dudeney's no-three-in-line problem [1] which asks how many points can be placed in the $n \times n$ grid so that no three points lie on the same line; for example, the set of six points in the 3×3 grid that avoids the three diagonal points is a solution to the problem. The *general position number* of a graph G, $\mathrm{gp}(G)$, is the size of a largest general position set of G. Intuitively, a general position set consists of peripheral nodes of a network, that is, nodes that are far from the hub nodes. In other words, the set does not contain a hub node since many (shortest) paths may pass through a hub node. Thus, the general position number implicitly

Keywords and phrases: General position number, scale-free network, hierarchical network.

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implies how many hub nodes a given network has. This is also one of the significant pieces of information for the structure of a given network.

Our main target in this paper is a *scale-free* graph, which is a graph whose degree distribution follows a power law (at least asymptotically). As a reason for that, we may intuitively expect that the general position number of a scale-free graph will be large since such a graph may have many peripheral nodes; for example, many vertices of degree 1. By the definition, the set of vertices of degree 1 forms a general position set (which might not be largest in general). However, the scale-free property does not always guarantee the existence of vertices of degree 1. In fact, there are many (deterministic) constructions of scale-free graphs without vertices of degree 1 and then we shall estimate the general position number of such graphs generated by deterministic models.

Our main results are summarized below:

- Let F_t be the hierarchical fractal scale-free graph generated by a general model [2] presented in Section 3.1, starting from a generator G, where $t \geq 0$ is the number of generations. It follows that $\mathrm{gp}(F_t) \leq m_F m_G^{t-1} \mathrm{gp}(G)$ (Prop. 3.1), where $m_F = |E(F_0)|$ and $m_G = |E(G)|$, and then we give a sufficient condition for G that the equal sign holds (Thm. 3.3).
- Let H_g be the hierarchical non-fractal scale-free graph with replication factor $m \geq 2$ generated by the hierarchical network model [3] presented in Section 3.2, where $g \geq 1$ is the number of generations. Then $\mathrm{gp}(H_1) = m$ and $\mathrm{gp}(H_g) = m^g - m^{g-1}$ for $g \geq 2$ (Thm. 3.4).
- Let Z_t be the hierarchical non-fractal scale-free graph by another model [4] presented in Section 3.2, where $t \geq 1$ is the number of generations. Then $\mathrm{gp}(Z_1) = 3$ and $\mathrm{gp}(Z_t) = 10 \cdot 4^{t-2}$ for $t \geq 2$ (Thm. 3.5).

In the next section, we introduce definitions and foundations for main terminology used in this paper. In Section 3, we prove our main results. In the final section, we conclude this paper and present open problems.

2. Definitions and foundations

In this paper, we deal only with finite simple undirected graphs. In this section, we introduce definitions of notion and terminology, and fundamental facts/results concerning them. For fundamental terminology and notation in graph theory undefined in this paper, we refer the reader to [5].

2.1. General position number

Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and edges. The *distance* of two vertices $u, v \in V(G)$, denoted by $d_G(u,v)$, is the number of edges on a shortest path between u and v in G A shortest (u,v)-path is called a (u,v)-*geodesic* (or simply, *geodesic*) in G. For $S \subseteq V(G)$, a triple set $S_0 \subseteq S$ is *non-geodesic* if no vertex of S_0 lies on any geodesic path between the other two vertices in S_0 in the graph G. Then S is a *general position set* of G if any triple set of S is non-geodesic in G.

Definition 2.1 (General position number)**.** For a graph G, the *general position number* (or *gp-number*) of G, denoted by $\mathrm{gp}(G)$, is defined as

$$\mathrm{gp}(G) = \max\{|S| : S \subseteq V(G) \text{ is a general position set}\}.$$

If $S \subseteq V(G)$ is a general position set of order $\mathrm{gp}(G)$, then S is called a *gp-set*.

A general position set is introduced in [6], but the same concept has already been studied in [7] under the name *geodetic irredundant set*. In subsequent papers, the gp-numbers for several graph classes are determined; for example, subgraphs of particular infinite graphs [8], graphs of small diameter [9], n-dimensional grid graphs [10], *etc.* For some basic graphs, we can easily see their general position number as follows.

Proposition 2.2 ([6])**.** *Let K_n, P_n and C_n be a complete graph, a path and a cycle of order $n \geq 3$, respectively. Then we have the following.*

- $\mathrm{gp}(K_n) = n$.
- $\mathrm{gp}(P_n) = 2$.
- $\mathrm{gp}(C_4) = 2$ *and* $\mathrm{gp}(C_n) = 3$ *for* $n \geq 5$.
- $\mathrm{gp}(T) = \ell$ *for any tree* T *with* ℓ *leaves.*

Finding a gp-set of a given graph G is an NP-hard problem in general [6]. On the other hand, the structure of a general position set is well studied; see [6, 8, 9] and a recent survey [11]. We will introduce one of them, which is a useful lemma for estimating general position numbers.

For a positive integer k, we let $[k] = \{1, 2, \dots, k\}$. For a subset $S \subseteq V(G)$, let $\mathcal{P} = \{S_1, \dots, S_p\}$ be a partition of S. Then $\mathcal{P}$ is *distance-constant* if for any $i, j \in [p]$ with $i \neq j$, the value $d_G(u, v)$ where $u \in S_i$ and $v \in S_j$ is independent of the selection of u and v. If $\mathcal{P}$ is a distance-constant partition, let $d_G(S_i, S_j)$ $(i, j \in [p], i \neq j)$ be the distance between S_i and S_j, that is, the distance between two arbitrary vertices pairwise from them. A distance-constant partition $\mathcal{P}$ is *in-transitive* if $d_G(S_i, S_j) \neq d_G(S_i, S_k) + d_G(S_k, S_j)$ holds for arbitrary pairwise different $i, j, k \in [p]$.

Lemma 2.3 ([9])**.** *Let G be a connected graph. Then $S \subseteq V(G)$ be a general position set of G if and only if all connected components of $G[S]$ are cliques, where $G[S]$ is the subgraph induced by vertices in S, and the vertices of which form an in-transitive distance-constant partition of S.*

2.2. Scale-free property

A graph G is *scale-free* if the degree probability distribution of G follows a power law asymptotically. More precisely, in a scale-free graph, the probability p_k that a vertex has degree k is

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)} \tag{2.1}$$

where γ is the degree exponent and $\zeta(\gamma)$ is the Riemann-zeta function. Note that many real-world scale-free networks have degree exponent between 2 and 3 (*cf.* [12], Sect. 4).

A scale-free property is one of three cardinal properties of real-world networks, namely a scale-free property, a small-world property and a high clustering coefficient. Analyzing real-world networks is very challenging, and hence it is reasonable to consider a graph with only one of the above three properties. To do this, many researchers have given deterministic/random constructions of such graphs. The most famous one of those constructions is the Barabási-Albert model, which generates a random scale-free graph [13]. So far, many such constructions are developed and the structure of graphs generated by them are well studied in mathematical aspects (see a survey [14]).

In graph theory, it is an interesting problem to estimate graph invariants of scale-free graphs. There are several known results for some such invariants; chromatic number [15, 16], domination [17, 18], *etc.* Such researches may help us to recognize real-world scale-free networks since they are expected to have similar behavior for the same graph invariants. Thus, we expect the evaluation of general position number of scale-free graphs will give us a new helpful feature of such graphs.

3. Main results

In this section, we introduce three constructions of hierarchical deterministic scale-free graphs and give bounds of their general position numbers.

3.1. Hierarchical fractal scale-free graph

Firstly we focus on a general model generating hierarchical fractal scale-free graphs introduced by Yakubo and Fujiki [2].

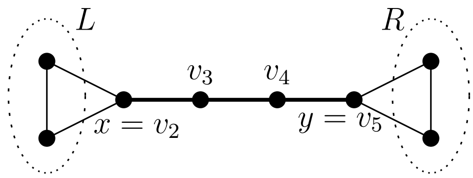

FIGURE 1. A middle path $v_2v_3v_4v_5$ of P_ℓ^* (bold line) with $\ell = 5$ and $L = R = K_2$.

Let G be a connected graph of small order, called a *generator*, in which two particular vertices of G are specified as *root vertices*. We denote by F_t ($t \geq 0$) the graph in the t-th generation. For $t \geq 1$, the graph F_t is constructed by replacing every edge e in F_{t-1} with the generator G so that two end vertices of e coincide with the root vertices of G. Note that when $G = F_0 = K_3$, F_t is a famous pseudo-fractal scale-free graph [19]. In order to obtain a fractal scale-free network, the generator G must satisfy the following three conditions [2]:

(i) The degree of the root vertices is at least 2.
(ii) The shortest-path distance between the two root vertices is at least 2.
(iii) The two root vertices are symmetric to each other in G.

By the definition of F_t, F_t contains exactly $|E(F_0)| \cdot |E(G)|^{t-1}$ disjoint copies of G when $t \geq 1$. Hence we can immediately have the following result.

Proposition 3.1. *Let F_t ($t \geq 1$) be the graph constructed as above, and let $m_F = |E(F_0)|$ and $m_G = |E(G)|$, where G is the generator. Then $\mathrm{gp}(F_t) \leq m_F m_G^{t-1} \mathrm{gp}(G)$.*

Proof. Suppose to the contrary that F_t has a general position set S with $|S| \geq m_F m_G^{t-1}\mathrm{gp}(G) + 1$. Since there are $m_F m_G^{t-1}$ copies of G in F_t, there is a copy of G which has a general position set with at least $\mathrm{gp}(G) + 1$ vertices, a contradiction. □

Therefore, we are interested in the following problem.

Problem 3.2. For which generators G does the equality $\mathrm{gp}(F_t) = |E(F_0)| \cdot |E(G)|^{t-1}\mathrm{gp}(G)$ hold?

The next theorem is an answer for this problem. Let P_ℓ^* be a connected graph obtained from a path $P_\ell = v_1v_2 \dots v_{\ell+1}$ with $\ell \geq 5$ by replacing two leaves v_1 and $v_{\ell+1}$ with two isomorphic graphs L and R of order at least one, respectively, in which v_2 and v_ℓ are adjacent to all vertices of L and R, respectively. Here v_2 and v_ℓ are called *support vertices*. Note that if $G \cong P_\ell^*$ and root vertices are support vertices, then G satisfies the above three conditions (i)–(iii).

Theorem 3.3. *Let F_0 be a connected graph and let F_t ($t \geq 1$) be the graph constructed as above. Suppose that the generator G is isomorphic to P_ℓ^* which is obtained from P_ℓ, L, R, and both root vertices are support vertices. Let $m_F = |E(F_0)|$ and $m_G = |E(G)|$. Then,*

$$\mathrm{gp}(F_t) = m_F m_G^{t-1}\mathrm{gp}(G).$$

Proof. By Proposition 3.1, it suffices to prove $\mathrm{gp}(F_t) \geq m_F m_G^{t-1}\mathrm{gp}(G)$ with $t \geq 1$. Let $\{x, y\}$ be the set of root vertices of the generator $G \cong P_\ell^*$, that is, $v_2 = x$ and $v_\ell = y$. The remaining path $v_2v_3 \dots v_\ell$ obtained from P_ℓ^* by removing all vertices in $V(L) \cup V(R) \cup \{x, y\}$ is called the *middle path* (see Fig. 1).

We claim that P_ℓ^* has a gp-set which contains neither any root vertex nor any vertex of the middle path. By symmetry, suppose a gp-set T contains a vertex u which is in the middle path or $\{x, y\}$. First suppose $u \notin \{x, y\}$. By the definition of a gp-set, $\mathrm{gp}(P_\ell^*) = 2$ in this case. Thus, another set T' consisting of a vertex in L and one in R is a desired gp-set.

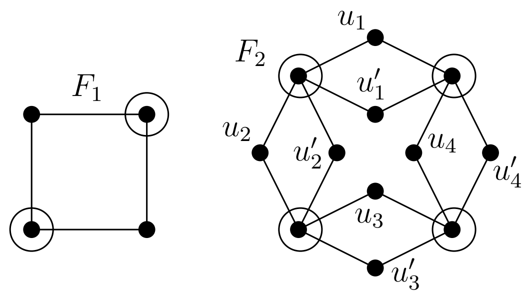

FIGURE 2. F_1 and F_2 when $F_0 \cong K_2$ and $G \cong C_4$. Circled vertices denote root vertices. Note that F_1 is obtained from $K_2 = uv$ by removing an edge uv and identifying root vertices of C_4 with u, v.

Therefore, we may assume $u = x$ by symmetry. By the definition of a gp-set, T satisfies

1. T contains only x and vertices in $V(L)$.
2. T contains only x and vertices in $V(R)$.
3. T contains only x and y.

In the first case (resp. second case), $T' = (T \cup \{z\}) \setminus \{x\}$ is also a desired gp-set of P_ℓ^* where $z \in V(R)$ (resp. $z \in V(L)$). In the case 3, $T' = (T \cup \{z, z'\}) \setminus \{x, y\}$ is also a desired gp-set of P_ℓ^* where $z \in V(L)$ and $z' \in V(R)$. Therefore, the claim holds.

Let S_G be a gp-set of $G \cong P_\ell^*$ which contains neither any root vertex nor any vertex of the middle path. Note that there are $m_F m_G^{t-1}$ copies of G in F_t. Then the set S of $m_F m_G^{t-1}$ copies of the gp-set S_G in each copy of G forms a general position set of F_t, since any shortest path between two vertices in S passes only root vertices and middle paths. Therefore, we have $\mathrm{gp}(F_t) \geq m_F m_G^{t-1}\mathrm{gp}(G)$, and hence the theorem holds. □

If we drop our condition, then we can find a pair (F_0, G) satisfying $\mathrm{gp}(F_t) < |E(F_0)| \cdot |E(G)|^{t-1}\mathrm{gp}(G)$, even if G satisfies three conditions (i)–(iii). Let us consider F_2 such that $F_0 \cong K_2$, $G \cong C_4$ and two root vertices of G are not adjacent (see Fig. 2). By Proposition 2.2, $\mathrm{gp}(C_4) = 2$ and hence $|E(F_0)| \cdot |E(G)| \cdot \mathrm{gp}(G) = 8$.

Suppose to the contrary that the F_2 has a general position set S with $|S| \geq 8$. If u_i (or u_i') and u_{i+2} (or u_{i+2}') are in S, then a shortest path between the two vertices must pass a vertex in S, a contradiction. Thus, since the total number of root vertices in F_2 is four, without loss of generality, we have $\{u_1, u_1', u_2, u_2'\} \subseteq S$ and all root vertices in S. However, this is also a contradiction since a (u_1, u_2)-geodesic must pass a root vertex. Therefore, we have $\mathrm{gp}(F_2) < 8 = |E(F_0)| \cdot |E(G)| \cdot \mathrm{gp}(G)$ in this case.

3.2. Hierarchical non-fractal scale-free graphs

In the previous subsection, we considered fractal scale-free graphs, but almost all real-world networks are non-fractal (*cf.* [20]). Hence, in this subsection, we estimate the general position number of a hierarchical non-fractal scale-free graph.

The first deterministic model is originally proposed by Ravasz and Barabási [3], and more formally defined by Noh [21] as follows:

The hierarchical graph is characterized by the number of generations $t \geq 1$ and a replication factor $m \geq 2$ recursively. We denote by H_t the graph in the t-th generation having m^t vertices which will be labeled by the coordinate t tuples of integers $[\mathbf{x}_t] = [x_t x_{t-1} \ldots x_1]$ with $0 \leq x_i < m$. The first generation consists of one central vertex $[0]$ referred as a *hub*, and $(m-1)$ peripheral vertices $[y]$ with $1 \leq y < m$ which are adjacent to the central vertex $[0]$. The set of all m vertices induces a clique. Suppose we have H_{t-1} for $t > 1$ and each vertex of the H_{t-1} is assigned to a coordinate $[\mathbf{x}_{t-1}]$. In the next generation, $(m-1)$ copies of H_{t-1} are added to the graph

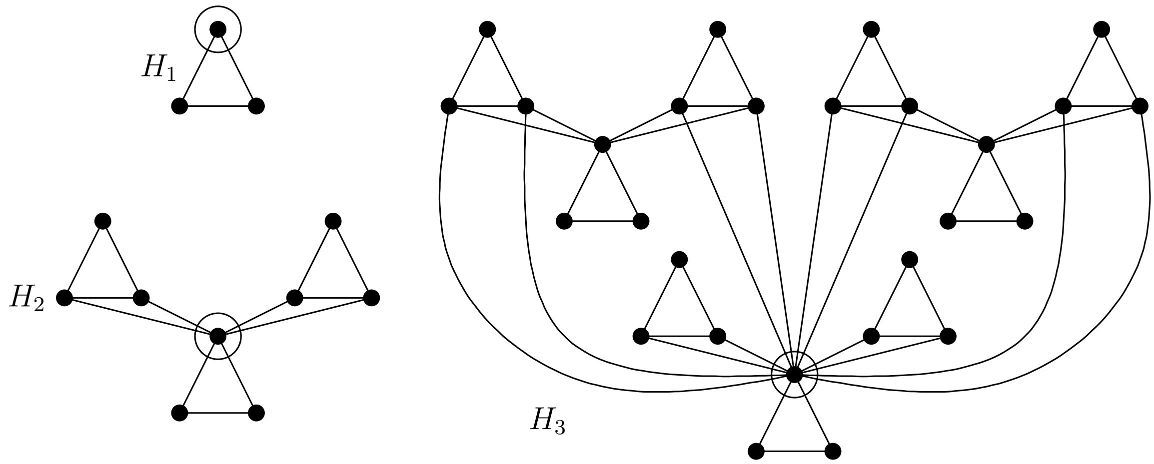

FIGURE 3. The graphs H_t generated by the hierarchical network model ($m = 3$) where the circled vertex is the hub of H_t.

and all their peripheral vertices are connected to the hub of the original unit. After that, vertices in the original unit are assigned to $[0\mathbf{x}_{t-1}]$ and the copies to $[y\mathbf{x}_{t-1}]$ with $y = 1, \ldots, m-1$, respectively. The resulting graph is H_t, where vertices $[y_t y_{t-1} \ldots y_1]$ with $y_i \neq 0$ for all i are peripheral and $[\mathbf{0}_t]$ is the hub of H_t (see Fig. 3).

It is known in [21] that H_t is scale-free since the degree distribution follows a power law with degree exponent $\gamma = 1 + \mathrm{In}\, m/\mathrm{In}(m-1) > 2$, where $\mathrm{In} x = \log_e x$. The following is our first main result.

Theorem 3.4. *Let H_t be the graph with replication factor $m \geq 2$ generated as above. Then,*

$$\mathrm{gp}(H_t) = \begin{cases} m & (t = 1) \\ m^t - m^{t-1} & (t > 1) \end{cases}$$

Proof. By Proposition 2.2, we immediately have $\mathrm{gp}(H_1) = m$. Thus, we assume $t \geq 2$. Moreover, by the construction of H_t, the set of all peripheral vertices of all units forms a general position set since for any two peripheral vertices u and v, every shortest path between them does not pass any other peripheral vertex. Therefore, we have $\mathrm{gp}(H_t) \geq m^t - m^{t-1}$ (note that $|V(H_t)| = m^t$), and hence, it suffices to prove that $\mathrm{gp}(H_t) \leq m^t - m^{t-1}$ for $t \geq 2$.

We prove the equality holds by induction on t. If H_2 has a general position set S with size at least $m^2 - m + 1$, since $m \geq 2$, we have

$$\frac{m^2 - m + 1}{m} = m - 1 + \frac{1}{m}$$

which means that all vertices of some clique K of size m are in S and at least one vertex z of other clique is also in S. Therefore, a shortest path between the central vertex c of K and z has to pass a peripheral vertex of K that is in S, a contradiction. Thus, $\mathrm{gp}(H_2) \leq m^2 - m$.

Now we suppose $t \geq 3$ and for smaller t the inequality holds, *i.e.*, for every integer $2 \leq s \leq t-1$, $\mathrm{gp}(H_s) \leq m^s - m^{s-1}$. Suppose to the contrary that H_t has a general position set S consisting of at least $m^t - m^{t-1} + 1$ vertices. Note that H_t contains (vertex disjoint) m units each of which is isomorphic to H_{t-1}. Then, we have

$$\frac{m^t - m^{t-1} + 1}{m} = m^{t-1} - m^{t-2} + \frac{1}{m}$$

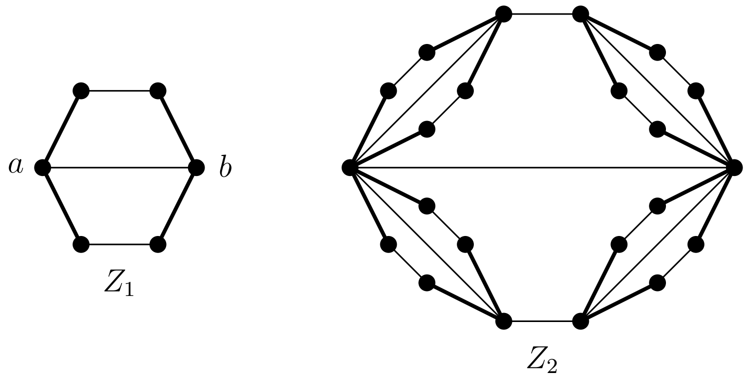

FIGURE 4. The graphs Z_t; bold edges are iterative edges.

which implies that some unit $K \cong H_{t-1}$ has at least $m^{t-1} - m^{t-2} + 1$ vertices in S. However, this contradicts the hypothesis, and hence the theorem holds. □

Next we introduce another construction of hierarchical non-fractal scale-free graphs defined by Miralles *et al.* [4]. We denote by Z_t ($t \geq 1$) the graph in the t-th generation. (Originally, the generation starts from $t = 0$ but we deal with only $t \geq 1$ since $Z_0 \cong K_2$.) For $t = 1$, Z_1 is isomorphic to the graph shown in the left of Figure 4. In the figure, four bold edges are *iterative edges*. For $t > 0$, Z_t is obtained from Z_{t-1} by replacing each iterative edge uv of Z_{t-1} with Z_1 so that $u = a$ and $v = b$ where a, b are two vertices of degree 3 in Z_1; see Figure 4 again.

Note that Z_t has 4^t iterative edges, and the number is the same as the number of disjoint Z_1's in Z_{t+1}. Using this, we can estimate the general position number of Z_t as follows.

Theorem 3.5. *Let Z_t ($t \geq 1$) be the graph constructed as above. Then*

$$\mathrm{gp}(Z_t) = \begin{cases} 3 & (t = 1) \\ 10 \cdot 4^{t-2} & (t \geq 2) \end{cases}$$

Proof. We take three vertices from the 6-cycle of Z_1 alternately. The set of three vertices forms a general position set. On the other hand, any set of four vertices does not clearly form a general position set. Thus, we have $\mathrm{gp}(Z_1) = 3$. Then we first prove that $\mathrm{gp}(Z_2) = 10 \cdot 4^0 = 10$. We label vertices of Z_2 as shown in Figure 5 It is easy to check that the set $\{v_1, v_3, c, u_1, u_3, d, r_1, r_3, s_1, s_3\}$ is a general position set. Hence we have $\mathrm{gp}(Z_2) \geq 10$.

Therefore we suffice to prove $\mathrm{gp}(Z_2) \leq 10$. Suppose to the contrary that Z_2 has a general position set S with $|S| \geq 11$.

Claim 3.6. *Each component in $G[S]$ is either K_1 or K_2.*

Proof. This claim immediately holds by Lemma 2.3 since Z_2 has no triangle. □

Let $W_a = \{a, v_i, u_i, c, d : 1 \leq i \leq 4\}$ and $W_b = \{b, r_i, s_i, c', d' : 1 \leq i \leq 4\}$. Let $A_1 = \{a, v_i, c : 1 \leq i \leq 4\}$ and $A_2 = \{a, u_i, d : 1 \leq i \leq 4\}$.

Claim 3.7. $|S \cap W_a| \leq 6$ *and* $|S \cap W_b| \leq 6$.

Proof. Suppose by symmetry that $|S \cap W_a| \geq 7$. For at least one of subsets A_1 and A_2, say A_1, $|S \cap A_1| \geq 4$. By Claim 3.6, we have $S \cap A_1 = \{v_i : 1 \leq i \leq 4\}$. However, a shortest path between u_1 and u_4 must pass u_2 or u_3, a contradiction. □

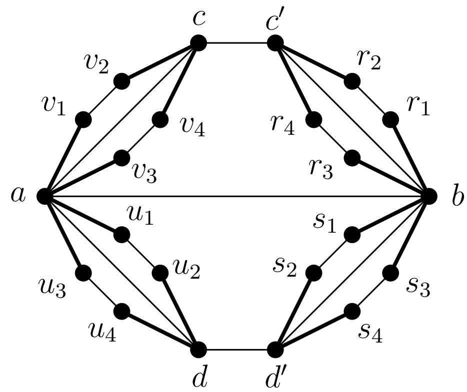

FIGURE 5. The graph Z_2 with vertex labels.

Claim 3.8. *Suppose* $|S \cap W_a| = 6$*. Then* $S \cap W_a = \{v_1, v_3, c, u_1, u_3, d\}$*. Similarly, if* $|S \cap W_b| = 6$*, then* $S \cap W_b = \{r_1, r_3, c', s_1, s_3, d'\}$*.*

Proof. By symmetry, we suffice to prove for $S \cap W_a$. It is easy to see that $a \notin S \cap W_a$; otherwise, $|(S \cap A_1) \setminus \{a\}| \leq 2$ and $|S \cap A_2 \setminus \{a\}| \leq 2$ by Claim 3.6, contrary to the assumption of the claim. If $|S \cap A_1| \geq 4$, we have $S \cap A_1 = \{v_i : 1 \leq i \leq 4\}$ similarly to the proof of Claim 3.7, a contradiction. Thus, $|S \cap A_1| = |S \cap A_2| = 3$.

If $c \notin S \cap A_1$, then we have a contradiction as above. Thus, by Claim 3.6 and symmetry, either $S \cap A_1 = \{v_1, c, v_4\}$ or $S \cap A_1 = \{v_1, v_3, c\}$ holds. However, in the former case, a shortest path between v_1 and v_4 passes c, a contradiction. Therefore, we have $S \cap A_1 = \{v_1, v_3, c\}$ and also $S \cap A_2 = \{u_1, u_3, d\}$ by symmetry. □

Since now $|S| \geq 11$, we may assume by symmetry that $|S \cap W_a| = 6$. By Claim 3.8, $S \cap W_a = \{v_1, v_3, c, u_1, u_3, d\}$. If one of r_2, r_4, s_2, s_4 is in S, say r_2, then a shortest path between v_1 and r_2 passes c since $d_{Z_2}(v_1, r_2) = 4$, a contradiction. By a similar argument, $c', d' \notin S$. Therefore, $S \cap W_b = \{r_1, r_3, b, s_1, s_3\}$ by $|S| \geq 11$. However, this contradicts to Claim 3.6, which implies that $\mathrm{gp}(Z_2) \leq 10$.

Now we have prepared to prove the theorem for $t \geq 3$. Observe that Z_t has 4^{t-2} disjoint copies of Z_2, denoted by $Z_2^1, Z_2^2, \ldots, Z_2^{4^{t-1}}$, Assume that every copy Z_2^i is labeled as shown in Figure 5 with a superscript i; for example, a^i, v_1^i.

If Z_t has a general position set S with $|S| > 10 \cdot 4^{t-2}$, then some copy Z_2^i contains at least 11 vertices in S, a contradiction. Therefore, $\mathrm{gp}(Z_t) \leq 10 \cdot 4^{t-2}$, and hence we suffice to prove that Z_t has a general position set S with $|S| = 10 \cdot 4^{t-2}$.

For Z_2^i, the set $\{v_1^i, v_3^i, c^i, u_1^i, u_3^i, d^i, r_1^i, r_3^i, s_1^i, s_3^i\}$ is called the *standard gp-set* of Z_2^i. Here $v_1^i, v_3^i, u_1^i, u_3^i, r_1^i, r_3^i, s_1^i, s_3^i$ are called *light vertices* and c^i, d^i are called *middle vertices*. We show that the union of standard gp-sets for all copies Z_2^i's, denoted by S, forms a general position set of Z_t.

Let $x, y \in S$ be two vertices belonging to distinct copies Z_2^i and Z_2^j, respectively. Observe that any shortest path P between x and y does not pass light vertices; otherwise, for example, if P passes v_1^h for some $h \in \{1, 2, \ldots, 4^{t-2}\}$, then P must passes a path $a^h v_1^h v_2^h c^h$, and hence, we can find a shorter path passing the edge $a^h c^h$ instead of $a^h v_1^h v_2^h c^h$, contrary to that P is shortest. If a shortest path P between x and y passes c^h by symmetry, then without loss of generality, $P = \ldots a^h c^h (c')^h r_2^h r_1^h$ or $P = r_1^h r_2^h (c')^h c^h a^h \ldots$ (where $h \in \{i, j\}$ in this case). However, in either case, we can find a shorter path $P' = \ldots a^h b^h r_1^h$ or $P' = r_1^h b^h a^h \ldots$, contrary to that P is shortest. Therefore, S is a general position set of Z_t, which completes the proof since $|S| = 10 \cdot 4^{t-2}$. □

4. CONCLUSION

In this paper, we estimate the general position number of hierarchical scale-free graphs generated by three deterministic models. The overall trend is that their general position numbers are asymptotically the same as the number of "peripheral" vertices of those graphs. As described in introduction, the scale-free property intuitively guarantees that there are many vertices of low degree, which implies its gp-number to be large. However, those "peripheral" vertices are not necessarily of low degree (as in the construction in Sect. 3.2). This implies that the set of vertices of low degree is not necessarily a general position set itself.

More specifically, we can describe this phenomenon using the *betweenness centrality* of vertices in graphs, which is one of the useful centrality of vertices. Intuitively, the betweenness centrality of a vertex v is the number of shortest paths between any two vertices passing v. We can regard a "peripheral" vertex as a vertex with low betweenness centrality. Indeed, we may guess that a set of vertices with low betweenness centrality forms a general position set. (The relation between betweenness centrality and other centrality notions has been studied; for example, see [22].) Thus we may consider the following problem.

Problem 4.1. For a scale-free graph G, find a real number $r(G) > 0$ such that $\mathrm{gp}(G) = |S|$ where $S \subseteq V(G)$ is the set of vertices with betweenness centrality at most $r(G)$.

Funding

The author is supported by JSPS Grant-in-Aid for Scientifific Research (C) 22K11911.

Conflicts of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Data availability statement

No data were generated or analyzed in this study.

Author contribution statement

The author solely conceived the study, conducted the analysis, and wrote the manuscript.

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