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k-FREE NUMBERS IN A GENERALIZED PIATETSKI-SHAPIRO SEQUENCE

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Abstract. A generalized Piatetski-Shapiro sequence is of the form

$$\{\lfloor \alpha n^c + \beta n + \theta \rfloor\}_{n=1}^{\infty} \quad (c > 1, c \notin \mathbb{N})$$

where α, β, θ are real numbers and $\alpha > 0$. The case when $\alpha = 1, \beta = \theta = 0$ is the normal Piatetski-Shapiro sequence. In this article, we estimate the k-free numbers in the generalized Piatetski-Shapiro sequence, namely

$$A(x) = \sum_{\substack{n \leqslant x \\ \lfloor \alpha n^c + \beta n + \theta \rfloor \text{ is } k\text{-free}}} 1$$

We obtain the main term and establish bounds for the error term depending on k and c.

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1. Introduction

The Piatetski-Shapiro sequences are defined by

$$\mathcal{N}^{(c)} = (\lfloor n^c \rfloor)_{n=1}^{\infty} \qquad (c > 1,\ c \notin \mathbb{N}).$$

Here we denote by $\lfloor t \rfloor$ the integral part of t. We also denote by $\{t\}$ the fractional part of t. These sequences have attracted considerable attention since Piatetski-Shapiro [1] proved that the sequence $\mathcal{N}^{(c)}$ contains infinitely many primes for $c \in (1, 12/11)$. More precisely, the counting function

$$\pi^{(c)}(x) = |\{\text{prime } p \leqslant x : p \in \mathcal{N}^{(c)}\}|$$

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satisfies $\pi^{(c)}(x) \sim x^{1/c}/\log x$ as $x \to \infty$. Many other properties of these sequences have been extensively studied, including smooth numbers [2], Waring problems [3], character sums [4], and small gaps [5] and so on. In this paper we focus on k-free numbers in a generalized Piatetski-Shapiro sequence.

Recall that for an integer $k \geqslant 2$, a positive integer n is called k-free if it has no divisor $d > 1$ that is a perfect k-th power. The counting function

$$Q_k(x) = |\{n \leqslant x : n \text{ is } k\text{-free}\}|$$

satisfies

$$\lim_{x\to\infty} \frac{Q_k(x)}{x} = \frac{1}{\zeta(k)},$$

and the error term $R_k(x) = Q_k(x) - x/\zeta(k)$ is known to satisfy $R_k(x) = O(x^{1/k})$ and $R_k(x) = \Omega(x^{1/(2k)})$, where the symbol "$\Omega$" means

$$\limsup_{x\to+\infty} \frac{|R_k(x)|}{x^{1/2k}} > 0.$$

We refer readers to [6, 7] for further background.

Turning to k-free numbers in Piatetski-Shapiro sequences, Stux [8] studied squarefree numbers and showed that for $1 < c < 4/3$,

$$\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor \text{ is squarefree}}} 1 = \frac{x}{\zeta(2)} + o(1). \tag{1.1}$$

Rieger [9] extended the range to $1 < c < 3/2$, obtained a definite estimate of the error with a polynomial saving, and remarked that the method can be adapted to cube-free numbers. Cao and Zhai [10] further improved the admissible range for (1.1) to $1 < c < 61/36$, and later [11] proved that for $1 < c < 149/87$ the Piatetski-Shapiro sequences contain infinitely many squarefree numbers.

For cube-free numbers, Zhang and Li [12] (following the method of Cao and Zhai [10]) proved that for any $\delta < 10^{-10}$ and $1 < c < 11/6$,

$$\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor \text{ is cube-free}}} 1 = \frac{x}{\zeta(3)} + o(x^{-\delta}).$$

Using an approach similar to Rieger's, for $1 < c < 2$, Deshouillers [13] obtained

$$\sum_{\substack{n\leqslant x\\ \lfloor n^c\rfloor \text{ is cube-free}}} 1 = \frac{x}{\zeta(3)} + O\big(x^{(c+1)/3}\log x\big).$$

In the present work we consider a generalized Piatetski-Shapiro sequence of the form

$$\{\lfloor \alpha n^c + \beta n + \theta\rfloor\}_{n=1}^{\infty},$$

where α, β, θ are real numbers and $\alpha > 0$. The case when $\alpha = 1, \beta = \theta = 0$ is the normal Piatetski-Shapiro sequence. We are interested in the following counting function

$$A(x) = \sum_{\substack{n \leqslant x \\ \lfloor \alpha n^c + \beta n + \theta \rfloor \text{ is } k\text{-free}}} 1$$

We study k-free numbers appearing in such sequences and prove the following theorem.

Theorem 1.1. *Let α, β, θ be real numbers and $\alpha > 0$. It gives that*

$$A(x) = \frac{x}{\zeta(k)} + O\big(\Phi(x)\big),$$

where $\Phi(x)$ is a function depending on k and c as follows.

- $k = 2$*: We have that*

$$\Phi(x) = x^{(2c+1)/4}, \quad \text{if } 1 < c < \tfrac{3}{2}.$$

- $k = 3$*: We have that*

$$\Phi(x) = x^{(c+1)/3} \log x, \quad \text{if } 1 < c < 2.$$

- $3 < k < 7$*: We have that*

$$\Phi(x) = \begin{cases} x^{(c+1)/3}, & \text{if } 1 < c < \frac{5}{4}; \\ x^{3/4 + (4c-5)/4k}, & \text{if } \frac{5}{4} \leqslant c < 2. \end{cases}$$

- $k \geqslant 7$*: We have that*

$$\Phi(x) = \begin{cases} x^{(c+1)/3}, & \text{if } 1 < c < \frac{5}{4}; \\ x^{(c+4)/7} \log x, & \text{if } \frac{5}{4} \leqslant c < 2. \end{cases}$$

The following proposition is a key part to prove Theorem 1.1.

Proposition 1.2. *For $1 < c < 2$, suppose a and q are two integers such that $0 \leqslant a < q \leqslant x^c$. Then*

$$\left| \mathcal{N}_c(x; q, a) - \frac{x}{q} \right| \ll \min\left(\frac{x^c}{q}, \frac{x^{(c+1)/3}}{q^{1/3}}, \frac{x^{(4+c)/7}}{q^{1/7}} \right),$$

where

$$\mathcal{N}_c(x; q, a) = \sum_{\substack{n \leqslant x \\ \lfloor \alpha n^c + \beta n + \theta \rfloor \equiv a \ (\mathrm{mod}\ q)}} 1.$$

The proof of Proposition 1.2 follows the ideas in the proof of Theorem 2 by Deshouillers [13] but yields a much better estimation. This is also a generalization of Claim 1 in [13]. The following Proposition plays an important role in the work of Proposition 1.2.

Proposition 1.3. *Let x be a real number large enough. Given a_1, a_2, a_3, a_4 are four fixed positive real numbers with $a_2 > 1$, $0 < a_3 < 1$, $0 < a_4 < 1$ and*

$$a_1 > \frac{a_2 - a_3}{1 - a_4}.$$

Let q, a be two integers such that $0 \leqslant a < q \leqslant x^{a_1}$. Let $f(x)$ be a real function defined on $\mathbb{R}$ satisfying that $f(x) \asymp x^{a_2}$, which means there exists a positive constant C, such that $|f(x)| = Cx^{a_2}$. Define

$$\mathcal{A}_f(x; q, a) = |\{n \leqslant x : \lfloor f(n) \rfloor \equiv a \pmod{q}\}|.$$

If we have that

$$\Big| \sum_{\substack{N < n \leqslant 2N \\ \lfloor f(n) \rfloor \equiv a \bmod q}} 1 - \frac{N}{q} \Big| \ll \frac{N^{a_3}}{q^{a_4}},$$

it follows that

$$\Big| \mathcal{A}_f(x; q, a) - \frac{x}{q} \Big| \ll \min\Big(\frac{x^{a_2}}{q}, \frac{x^{a_3}}{q^{a_4}} \Big).$$

We mention that this proposition also follows an idea by Deshouillers [13] but provides a more general result.

2. Technical lemmas

We write $e(t) = \mathrm{e}^{2\pi i t}$. We need the following well-known Erdős–Turán inequality which is Theorem of [14].

Lemma 2.1. *Let $\mathbb{T}$ denote $\mathbb{R}/\mathbb{Z}$. Let $(u_n)_{n=1}^N$ be a sequence of points of $\mathbb{T}$ and $J = [\alpha, \beta]$ be an arc of $\mathbb{T}$ with $\alpha \leqslant \beta \leqslant \alpha + 1$. Then for any positive integers N and K,*

$$|D(N; \alpha, \beta)| \leqslant \frac{N}{K+1} + 2\sum_{k=1}^{K} \left(\frac{1}{K+1} + \min\Big(\beta - \alpha, \frac{1}{\pi k}\Big) \right) \left| \sum_{n=1}^{N} e(k u_n) \right|,$$

where

$$D(N; \alpha, \beta) = Z(N; \alpha, \beta) - (\beta - \alpha) N$$

and

$$Z(N; \alpha, \beta) = |\{n \in \mathbb{Z} : 1 \leqslant n \leqslant N,\ u_n \in [\alpha, \beta] \bmod 1\}|.$$

Lemma 2.2. *Suppose that f is a real valued function with two continuous derivatives on an interval I. Suppose also that there is some $\lambda > 0$ and for some $\alpha \geqslant 1$ such that*

$$\lambda \leqslant |f''(x)| \leqslant \alpha\lambda$$

on I. Then

$$\sum_{n \in I} e(f(n)) \ll \alpha |I| \lambda^{1/2} + \lambda^{-1/2}.$$

Proof. See Theorem 2.2 of [15]. □

Lemma 2.3. *Suppose that f is a real valued function with three continuous derivatives on I. Suppose also that for some $0<\lambda<1$ and for some $\alpha \geqslant 1$,*

$$\lambda \leqslant |f^{(3)}(x)| \leqslant \alpha\lambda$$

on I. Then

$$\sum_{n\in I} e(f(n)) \ll \alpha|I|\lambda^{1/6} + \lambda^{-1/3}.$$

Proof. See Corollary 4.2 of [16]. □

3. Proof of Proposition 1.3

Since $f(x) \asymp x^{a_2}$ for x large enough and $a_2>1$, the cardinality of $\mathcal{A}_f(x;q,a)$ is at most the number of natural numbers not exceeding $K_f x^{a_2}$ that are congruent to a modulo q where K_f is a constant depending only on $f(x)$. It follows that

$$\Big|\mathcal{A}_f(x;q,a) - \frac{x}{q}\Big| \leqslant \mathcal{A}_f(x;q,a) + \frac{x}{q} \ll \frac{x^{a_2}}{q}. \tag{3.1}$$

Since we have

$$\frac{x^{a_3}}{q^{a_4}} \leqslant \frac{x^{a_2}}{q} \quad \text{when} \quad q \leqslant x^{(a_2-a_3)/(1-a_4)},$$

it is sufficient to show that

$$\Big|\mathcal{A}_f(x;q,a) - \frac{x}{q}\Big| \ll \frac{x^{a_3}}{q^{a_4}} \quad \text{for} \quad q \leqslant x^{(a_2-a_3)/(1-a_4)}.$$

We divide the interval $[1,x]$ by the intervals $(2^{-l}x, 2^{-l+1}x]$, for l from 1 to $\lfloor \log x/\log 2\rfloor + 1$. We define s_1 and s_2 with $x2^{-s_1} = q^{s_2}$. We first assume that

$$q < (x2^{-s_1})^{a_1} \tag{3.2}$$

to make sure that Proposition 1.2 can be applied on the interval $(x2^{-s_1}, x]$. It follows that

$$\Big|\sum_{\substack{x2^{-\lfloor s_1\rfloor}<n\leqslant x\\ \lfloor f(n)\rfloor \equiv a \bmod q}} 1 - \frac{x-x2^{-\lfloor s_1\rfloor}}{q}\Big| \leqslant \sum_{l=1}^{\lfloor s_1\rfloor}\Big|\sum_{\substack{x2^{-l}<n\leqslant x2^{-l+1}\\ \lfloor f(n)\rfloor \equiv a \bmod q}} 1 - \frac{x2^{-l}}{q}\Big| \leqslant \sum_{l=1}^{\lfloor s_1\rfloor} K_{a_3,a_4}\frac{x^{a_3}}{q^{a_4}}2^{-a_3 l}. \tag{3.3}$$

Since the series $\sum 2^{-a_3 l}$ is convergent, there exist a constant K_{a_1,a_3,a_4} depending only on a_1,a_3,a_4 such that

$$\Big|\sum_{\substack{x2^{-\lfloor s_1\rfloor}<n\leqslant x\\ \lfloor f(n)\rfloor \equiv a \bmod q}} 1 - \frac{x-x2^{-\lfloor s_1\rfloor}}{q}\Big| \leqslant K_{a_1,a_3,a_4}\frac{x^{a_3}}{q^{a_4}}.$$

For the interval $[1, x2^{-s_1}]$, we further assume that

$$s_2 < \frac{1-a_4}{a_2-a_3}. \tag{3.4}$$

This is equivalent to

$$\frac{(2^{-s_1}x)^{a_2}}{q} < \frac{x^{a_3}}{q^{a_4}}$$

by using the definition of s_1. This means that

$$\Big|\sum_{\substack{1\leqslant n\leqslant 2^{-\lfloor s_1\rfloor}x\\ \lfloor f(n)\rfloor\equiv a \bmod q}} 1 - \frac{2^{-\lfloor s_1\rfloor}x}{q}\Big| \leqslant \frac{2(2^{-s_1})^{a_2}x}{q} \ll \frac{x^{a_3}}{q^{a_4}}. \tag{3.5}$$

Combine (3.3) and (3.5) to obtain the stated result. The condition

$$a_1 > \frac{a_2-a_3}{1-a_4}.$$

guarantees the existence of s_1 and s_2 under the conditions (3.2) and (3.4).

4. Proof of Proposition 1.2

We prove the following lemma firstly.

Lemma 4.1. *Let c be in $(1,2)$. For any real number M and integers a and q satisfying*

$$0 \leqslant a < q < M^c,$$

one has

$$\Big|\sum_{\substack{M<m\leqslant 2M\\ \lfloor \alpha m^c+\beta m+\theta\rfloor\equiv a \bmod q}} 1 - \frac{M}{q}\Big| \ll \min\Big(\frac{M^c}{q}, \frac{M^{(c+1)/3}}{q^{1/3}}, \frac{M^{(c+4)/7}}{q^{1/7}}\Big). \tag{4.1}$$

Proof. We first note that

$$\lfloor \alpha m^c+\beta m+\theta\rfloor \equiv a \pmod q \iff \Big\{\frac{\alpha m^c+\beta m+\theta}{q}\Big\} \in \Big[\frac{a}{q}, \frac{a+1}{q}\Big),$$

By Lemma 2.1, for any positive integer H we obtain

$$\begin{aligned}\Big|\sum_{\substack{M<m\leqslant 2M\\ \lfloor \alpha m^c+\beta m+\theta\rfloor\equiv a \bmod q}} 1 - \frac{M}{q}\Big| \\ \leqslant \frac{M}{H+1} + 2\sum_{h=1}^{H}\Big(\frac{1}{H+1} + \min\Big(\frac{1}{q}, \frac{1}{\pi h}\Big)\Big)\Big|\sum_{M<m\leqslant 2M} e\Big(\frac{\alpha h m^c+\beta h m+h\theta}{q}\Big)\Big|. \end{aligned} \tag{4.2}$$

Now we deal with the estimation of the trigonometric sum occurring in (4.2). Letting

$$f(t) = (h/q)(\alpha t^c + \beta t + \theta),$$

and

$$f''(t) \asymp (h/q)t^{c-2}.$$

By Lemma 2.2 we have

$$\sum_{M<m\leqslant 2M} e\Big(\frac{\alpha hm^c + \beta hm + h\theta}{q}\Big) \ll_c (h/q)^{1/2}M^{c/2} + (q/h)^{1/2}M^{1-c/2}. \tag{4.3}$$

For the second part of the right-hand side of (4.2), we consider two cases according to q and choose

$$H = \lfloor q^{1/3}M^{(2-c)/3} \rfloor.$$

Case I: $q < M^{1-c/2}$. It is equivalent to $q < H$. Inserting (4.3) into (4.2), one can obtain that (4.2) is

$$\begin{aligned}
&\ll \frac{M}{H+1} + \sum_{h=1}^{q} \frac{1}{q}\Big| \sum_{M<m\leqslant 2M} e\Big(\frac{\alpha hm^c + \beta hm + h\theta}{q}\Big)\Big| \\
&\qquad + \sum_{h=q+1}^{H} \frac{1}{h}\Big| \sum_{M<m\leqslant 2M} e\Big(\frac{\alpha hm^c + \beta hm + h\theta}{q}\Big)\Big| \\
&\ll q^{-1/3}M^{(c+1)/3} + \sum_{h=1}^{q}(h^{1/2}q^{-3/2}M^{c/2} + q^{-1/2}h^{-1/2}M^{1-c/2}) \\
&\qquad + \sum_{h=q+1}^{H} (q^{-1/2}h^{-1/2}M^{c/2} + q^{1/2}h^{-3/2}M^{1-c/2}) \\
&\ll q^{-1/3}M^{(c+1)/3} + M^{c/2} + M^{1-c/2} + q^{1/3}M^{(2-c)/3} \ll q^{-1/3}M^{(c+1)/3}.
\end{aligned} \tag{4.4}$$

Case II: $q \geqslant M^{1-c/2}$. This is the same to $q \geqslant H$. Similarly, we can get that (4.2) is

$$\begin{aligned}
&\ll \frac{M}{H+1} + \sum_{h=1}^{H} H^{-1}((h/q)^{1/2}M^{c/2} + (q/h)^{1/2}M^{1-c/2}) \\
&\ll q^{-1/3}M^{(c+1)/3} + H^{-1/2}q^{1/2}M^{1-c/2} \ll q^{-1/3}M^{(c+1)/3}
\end{aligned} \tag{4.5}$$

provided that $q < M^{c-1/2}$.

Meanwhile, for the estimation of the trigonometric sum in (4.2) we consider another way to handle it. It can be known that the third derivative of the function $f(t)$ is

$$f'''(t) \asymp (h/q)t^{c-3},$$

where $F \asymp G$ means $F \ll G \ll F$. By Lemma 2.3 we have

$$\sum_{M<m\leqslant 2M} e\Big(\frac{\alpha hm^c+\beta hm+h\theta}{q}\Big) \ll_c (h/q)^{1/6}M^{(3+c)/6}+(q/h)^{1/3}M^{(3-c)/3}. \tag{4.6}$$

Choosing

$$H=\lfloor M^{(3-c)/7}q^{1/7}\rfloor,$$

we can get the following.

Case I: $q<M^{(3-c)/6}$. It is equivalent to $q<H$. Inserting (4.6) into (4.2), one can obtain that (4.2) is

$$\begin{aligned}
&\ll \frac{M}{H+1}+\sum_{h=1}^{q}\frac{1}{q}\Big|\sum_{M<m\leqslant 2M} e\Big(\frac{\alpha hm^c+\beta hm+h\theta}{q}\Big)\Big| \\
&\qquad +\sum_{h=q+1}^{H}\frac{1}{h}\Big|\sum_{M<m\leqslant 2M} e\Big(\frac{\alpha hm^c+\beta hm+h\theta}{q}\Big)\Big| \\
&\ll q^{-1/7}M^{(4+c)/7}+\sum_{h=1}^{q}(h^{1/6}q^{-7/6}M^{(3+c)/6}+q^{-2/3}h^{-1/3}M^{(3-c)/3}) \\
&\qquad +\sum_{h=q+1}^{H}(h^{-5/6}q^{-1/6}M^{(3+c)/6}+q^{1/3}h^{-4/3}M^{(3-c)/3}) \\
&\ll q^{-1/7}M^{(4+c)/7}+M^{(3+c)/6}+M^{(3-c)/3}+q^{1/7}M^{(6-c)/7} \ll q^{-1/7}M^{(4+c)/7}.
\end{aligned} \tag{4.7}$$

Case II: $q\geqslant M^{(3-c)/6}$. We have that $q\geqslant H$. Similarly, we can get that (4.2) is

$$\begin{aligned}
&\ll \frac{M}{H+1}+\sum_{h=1}^{H}H^{-1}((h/q)^{1/6}M^{(3+c)/6}+(q/h)^{1/3}M^{(3-c)/3}) \\
&\ll q^{-1/7}M^{(4+c)/7}+q^{2/7}M^{(6-2c)/7} \ll q^{-1/7}M^{(4+c)/7}
\end{aligned} \tag{4.8}$$

provided that $q<M^{c-2/3}$. Note that

$$\frac{M^c}{q}\leqslant \frac{M^{(c+1)/3}}{q^{1/3}} \quad \text{for} \quad M^{c-1/2}\leqslant q<M^c$$

and

$$\frac{M^c}{q}\leqslant \frac{M^{(c+4)/7}}{q^{1/7}} \quad \text{for} \quad M^{c-2/3}\leqslant q<M^c.$$

Hence we obtain the stated result. □

Using Lemma 4.1, it is sufficient to show that (4.1) holds for any $m\leqslant x$ instead of $M<m\leqslant 2M$. Next we deduce Proposition 1.2 from Proposition 1.3. Applying Proposition 1.3 with

$$(a_1,a_2,a_3,a_4)=(c,c,\frac{c+1}{3},\frac{1}{3})$$

and

$$(a_1, a_2, a_3, a_4) = (c, c, \frac{c+4}{7}, \frac{1}{7})$$

respectively, we finish the proof of Proposition 1.2.

5. Proof of Theorem 1.1

It is known that

$$\sum_{d^k|n} \mu(d) = \begin{cases} 1, & \text{if } n \text{ is } k\text{-free;} \\ 0, & \text{otherwise .} \end{cases}$$

It follows that

$$A(x) = \sum_{n \leqslant x} \sum_{d^k | \lfloor \alpha n^c + \beta n + \theta \rfloor} \mu(d) = \sum_{d \leqslant (\alpha x^c + \beta x + \theta)^{1/k}} \mu(d) \mathcal{N}_c(x; d^k, 0).$$

Using Proposition 1.2, we get

$$\begin{aligned} A(x) = x & \sum_{d \leqslant (\alpha x^c + \beta x + \theta)^{1/k}} \frac{\mu(d)}{d^k} \\ & + O\Big(\sum_{d \leqslant (\alpha x^c + \beta x + \theta)^{1/k}} \min\Big(\frac{x^c}{d^k}, \frac{x^{(c+1)/3}}{d^{k/3}}, \frac{x^{(c+4)/7}}{d^{k/7}} \Big) \Big). \end{aligned} \tag{5.1}$$

For the main term, we have that

$$x \sum_{d \leqslant (\alpha x^c + \beta x + \theta)^{1/k}} \frac{\mu(d)}{d^k} = \frac{x}{\zeta(k)} - x \sum_{d > (\alpha x^c + \beta x + \theta)^{1/k}} \frac{\mu(d)}{d^k} = \frac{x}{\zeta(k)} + O\big(x^{1-2c/3}\big).$$

The sum in the error term of (5.1) can be discussed in several cases. We first have that

$$\begin{aligned} & \sum_{d \leqslant (\alpha x^c + \beta x + \theta)^{1/k}} \min\Big(\frac{x^c}{d^k}, \frac{x^{(c+1)/3}}{d^{k/3}}, \frac{x^{(c+4)/7}}{d^{k/7}} \Big) \\ & \ll \sum_{d \leqslant x^{(4c-5)/4k}} \frac{x^{(c+4)/7}}{d^{k/7}} + \sum_{x^{(4c-5)/4k} < d \leqslant x^{(2c-1)/2k}} \frac{x^{(c+1)/3}}{d^{k/3}} \\ & \quad + \sum_{x^{(2c-1)/2k} < d \leqslant (\alpha x^c + \beta x + \theta)^{1/k}} \frac{x^c}{d^k}. \end{aligned} \tag{5.2}$$

When $k = 2$, we obtain that the right-hand side of (5.2)

$$\ll x^{(4c+1)/8} + x^{(2c+1)/4} + x^{1/2} \ll x^{(2c+1)/4}.$$

When $k = 3$, we prove that the right-hand side of (5.2)

$$\ll x^{(c+1)/3} + x^{(c+1)/3} \log x + x^{1/2} \ll x^{(c+1)/3} \log x.$$

Note that the first sum in (5.2) is empty for $1 < c < 5/4$. Thus when $k > 3$ and $1 < c < 5/4$, we arrive that the right-hand side of (5.2) is

$$\ll x^{(c+1)/3} + x^{1/2} \ll x^{(c+1)/3}.$$

If $5/4 \leqslant c < 2$, when $3 < k < 7$, we prove that the right-hand side of (5.2) is

$$\ll x^{3/4+(4c-5)/4k} + x^{3/4} + x^{1/2} \ll x^{3/4+(4c-5)/4k}$$

If $5/4 \leqslant c < 2$ and $k \geqslant 7$, we have that the right-hand side of (5.2) is

$$\ll x^{(c+4)/7} \log x + x^{3/4} + x^{1/2} \ll x^{(c+4)/7} \log x.$$

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Conflicts of interest

Authors state no conflict of interest.

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Author contribution statement

All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results and approved the final version of the manuscript. All authors wrote the manuscript and are considered to have equal contributions.

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