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RESTRICTION OF SUPERCUSPIDAL REPRESENTATIONS

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Abstract. We will show that every countably generated unitarizable representation of a group is decomposable. As an application, Suppose (π, V) is a supercuspidal representation of a reductive linear algebraic group G defined over a p-adic field, H is a closed reductive subgroup of G. Then $(\pi|_H, V)$ can be decomposed into irreducible representations. Moreover, if $Z_H/(H \cap Z_G)$ is compact, then every constituent of $(\pi|_H, V)$ is supercuspidal; if $Z_H/(H \cap Z_G)$ is non-compact, then every constituent is not admissible.

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1. Introduction

By Langlands classification theory, every irreducible admissible $(\mathfrak{g}, K)$-module is equivalent to a quotient of a representation induced parabolically from a tempered representation; every tempered representation is equivalent to a sub-representation of a representation induced parabolically from a discrete series representation; every discrete series representation is equivalent to a sub-representation of a representation induced parabolically from a supercuspidal representation (*cf.* [1]). In this sense, supercuspidal representations can be viewed as fundamental "bricks" in the "building" of admissible representations, its study plays an important role in local harmonic analysis (*cf.* [1–5]).

In the study of intertwining operators attached to representations induced from supercuspidal representations of a Levi subgroup, people need to study the orbital integrals of matrix coefficients of representations on two different types of groups (see for example [5–11]). In the case when one group is a subgroup of another, people may be interested in thinking about the behaviour of these orbital integrals by considering the restriction of the supercuspidal representation on the "larger" group to the "smaller" group, this idea motivates our work in this paper.

The content of the paper is arranged as follows: we firstly prove that a countably generated unitarizable representation of a group is decomposable (Lems. 2.1 and 2.2). If (π, V) is a supercuspidal representation of a reductive algebraic group G, H is a closed subgroup of G, since V carries a G invariant Hermitian form by the supercuspidality of π, V can be decomposed into irreducible representations of H by the admissibility of π. In the case H is also reductive and $Z_H/(H \cap Z_G)$ is compact, where Z is used to denote the center of a group,

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then each constituent of $\pi|_H$ obtained above is supercuspidal. While if $Z_H/(H \cap Z_G)$ is noncompact (H need not be reductive), then each constituent of $\pi|_H$ is not admissible.

2. Decomposition of unitarizable representations

In our settings, representation spaces are always assumed to be defined over the set of complex numbers $\mathbb{C}$, although the results in this section are equally applied to any fields on which the spaces can carry an inner product. Recall a representation space W of a group M is unitarizable if there is a positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on W which is M-invariant, *i.e.*, $\langle \pi(m)u, \pi(m)v \rangle = \langle u, v \rangle, \forall\, m \in M, u, v \in W$. If furthermore, W is a Hilbert space, then W is called unitary (*cf.* p. 140 in [12]).

Lemma 2.1. *Any finitely generated unitarizable representation W of a group M can be decomposed into a finite direct sum of irreducible mutually orthogonal M-subspaces.*

Proof. Suppose $\langle \cdot, \cdot \rangle$ is a positive definite Hermitian form on W which is M-invariant, and W is generated by $\{v_1, v_2, \cdots, v_t\}$ as an M-space. For any $u \in W$, we will let M_u denote the M-subspace generated by u.

(A) We will firstly show that there exists an M-irreducible quotient of W. Suppose on the contrary that there is no such quotient in W. Then W itself if not irreducible, so there is an M-subspace U_1 of W so that $U_1 \neq W$. Since W/U_1 is not irreducible by assumption, there must be an M-subspace U_2 so that $U_1 \subsetneq U_2 \subsetneq W$. Again, since W/U_2 is not irreducible, there is an M-subspace U_3 so that $U_2 \subsetneq U_3 \subsetneq W$. More generally, suppose for a positive integer n, there are M-subspaces U_{n-1} and U_n of W so that $U_{n-1} \subsetneq U_n \subsetneq W$. Since W/U_n is not irreducible, there is an M-subspace U_{n+1} so that $U_n \subsetneq U_{n+1} \subsetneq W$. By induction, we will obtain an infinite increasing sequence of M-subspaces of W :

$$U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_n \subsetneq \cdots .$$

Let

$$V = \bigcup_{n=1}^{\infty} U_n.$$

Then certainly V is also an M-subspace.

(i) If $V = W$. Then for any $v_i, 1 \leq i \leq t$, there is n_i so that $v_i \in U_{n_i}$. In particular, there is N so that $v_i \in U_N, \forall\, 1 \leq i \leq t$. Hence, $U_N = W$. But that conflicts the infiniteness of the sequence $\{U_n\}$.

(ii) If $V \subsetneq W$. Let $V^\perp$ be the orthogonal complement of V in W. For any positive integer j, let $U'_j = U_j + V^\perp$. Then we also have an infinite increasing sequence of M-subspaces:

$$U'_1 \subsetneq U'_2 \subsetneq \cdots \subsetneq U'_n \subsetneq \cdots ,$$

and

$$\bigcup_{n=1}^{\infty} U'_n = V + V^\perp = W.$$

By (i), this is impossible.

(B) Now we have an M-irreducible quotient of W, say $U_0 = W/W'$. Since W is a Hermitian space, U_0 can be realized as the orthogonal complement of W' in W, then U_0 is an M-subspace of W. Let W_1 be a maximal M-subspace of W that can be written as a direct sum of mutually orthogonal irreducible M-subspaces, and let $W_1^\perp$ be the orthogonal complement of W_1 in W. Then $W_1^\perp$ is also an M-subspace. Choose a $0 \neq v \in W_1^\perp$, then the above statement shows that M_v has an irreducible M-subspace U'. Then $U' + W_1$ can be written as a direct sum of mutually orthogonal irreducible M-subspaces. That conflicts the maximality of W_1. Hence, W can be written as a direct sum of mutually orthogonal irreducible M-subspaces.

(C) Suppose

$$W = \bigoplus_{j \in \Lambda} V_j, \tag{2.1}$$

is a sum of W in mutually orthogonal irreducible M-subspaces, where Λ is a set of indices. Then for any $v_i, 1 \leq i \leq t$, there are finite number of subspaces $V_{i_1}, V_{i_2}, \cdots, V_{i_s}$ so that $M_{v_i} = V_{i_1} + V_{i_2} + \cdots + V_{i_s}$. Therefore, the set of generators $\{v_1, v_2, \cdots, v_t\}$ must belong to a finite sum of subspaces on the right hand side of (2.1). Since W is generated by $\{v_1, v_2, \cdots, v_t\}$, the sum in (2.1) is finite.

(D) According to (C), there is an irreducible direct sum of W as:

$$W = \bigoplus_{j=1}^{m} V_j^{a_j}, \tag{2.2}$$

where each V_j is an isomorphism class of irreducible representation in W, a_j is the multiplicity of V_j in W, m is the number of isomorphism classes in W. Suppose there is another irreducible direct sum (not necessary mutually orthogonal) of W as:

$$W = \bigoplus_{i=1}^{k} W_i^{b_i}, \tag{2.3}$$

where W_i, b_i and k have the same meanings as those of V_j, a_j and m as above, respectively.

For each V_j above, let P_j be the projection map from W onto $V_j^{a_j}$ according to (2.2). Then

$$\sum_{i=1}^{m} P_j = \mathrm{Id}_W,$$

where Id_W is the identity map on W. For each W_i,

$$W_i = \mathrm{Id}_W(W_i) = \sum_{i=1}^{m} P_j(W_i).$$

Then there is at least one j such that $P_j(W_i) \neq 0$. On the other hand, if $P_j(W_i) \neq 0$, then by Schur's Lemma, $W_i \cong P_j(W_i) \subset V_j^{a_j}$. Thus, $W_i \cong V_j$ since every irreducible sub-representation $(P_j(W_i))$ in $V_j^{a_j}$ is isomorphic to V_j. As the V_j's on the right hand side of (2.2) are distinct isomorphism classes, we can see that there is only one j such that $P_j(W_i) \neq 0$. Hence, $W_i \subset V_j^{a_j}$, and consequently, $W_i^{b_i} \subset V_j^{a_j}$.

Conversely, we can show that $V_j^{a_j} \subset W_i^{b_i}$. Therefore, $W_i^{b_i} = V_j^{a_j}$. Thus, $W_i \cong V_j$ and $a_j = b_i$. Hence, $m = k$, and the decomposition in (2.3) is the same as that in (2.2) by rearranging the order of subspaces $W_i^{b_i}$, *i.e.*, the decomposition of W in (2.1) is unique up to isomorphism. □

Lemma 2.2. *Suppose W is a unitarizable representation of a group M which is generated by a countable set of elements. Then W can be decomposed into countable irreducible sub-representations of M.*

Proof. (A) Let W_0 be a maximal subspace (under inclusion) of W that can be written as a direct sum of irreducible subspaces. Since by Lemma 2.1, any finitely generated subspace of W can be so decomposed, we know that $W_0 \neq 0$.

If $W_0 \neq W$, let $W_0^{\perp}$ be the orthogonal complement of W_0 in W. Suppose $x \in W_0^{\perp}$, then by Lemma 2.1, M_x can be decomposed into a direct sum of irreducible subspaces, so is $W_0 + M_x$. As $W_0 \subsetneq W_0 + M_x$, that is a contradiction to the maximality of W_0. Hence, $W = W_0$.

(B) By Lemma 2.1, every M_x can be decomposed into finitely many irreducible subspaces, as x runs through a set of countable generators of W, the number of such irreducible subspaces are certainly countable.

(C) Suppose W can be written as:

$$W = \bigoplus_{j \in \Lambda} \mathcal{V}_j, \tag{2.4}$$

where each $\mathcal{V}_j$ is a class of isomorphic irreducible representations in W, and Λ is the set of indices of $\mathcal{V}_j$. Suppose there is another direct sum of W as:

$$W = \bigoplus_{i \in \Xi}^{k} \mathcal{W}_i, \tag{2.5}$$

where $\mathcal{W}_i$ and Ξ have the same meanings of $\mathcal{V}_j$ and Λ, respectively. Then a similar proof as part (D) in the proof of Lemma 2.1 shows that for any $\mathcal{V}_j$, there is one and only one (irreducible representation in different $\mathcal{V}_j$'s is non-isomorphic to each other) $\mathcal{W}_i$ so that $\mathcal{V}_j = \mathcal{W}_i$, and vice versa. So there is a one to one correspondence between $\mathcal{V}_j$'s in (2.4) and $\mathcal{W}_i$'s in (2.5).

Meanwhile, if the multiplicity of irreducible representations in $\mathcal{V}_j$ is finite, then part (D) in the proof of Lemma 2.1 shows that it equals to the multiplicity of those in $\mathcal{W}_i$, and there is an M-isomorphism on $\mathcal{V}_j = \mathcal{W}_i$ that establishes a one to one map between irreducible copies in $\mathcal{V}_j$ to those in $\mathcal{W}_i$. If the multiplicity of irreducible representations in $\mathcal{V}_j$ is infinite, then it must be countable by part (B), so is the multiplicity of irreducible representations in $\mathcal{W}_i$ by part (D) in the proof of Lemma 2.1. Again there is an M-isomorphism on $\mathcal{V}_j$ $(= \mathcal{W}_i)$ that makes a one to one correspondence between irreducible copies in $\mathcal{V}_j$ to those in $\mathcal{W}_i$.

Therefore, there is an M-automorphism on W that maps irreducible representations in (2.4) one to one to those in (2.5), *i.e.*, the decomposition of W into irreducible representations is unique up to isomorphism. □

3. Restriction of supercuspidal representations

Suppose F is a non-Archimedean field of characteristic zero, and G is a reductive algebraic group defined over F. An admissible representation (π, V) (V is a complex space in the context of this paper) of G is called supercuspidal if any matrix coefficient of V has compact support on G/Z_G, where Z_G is the center of G. If (π, V) is supercuspidal, then by Proposition 2.5.4 in [2], there is a positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on V so that V is unitarizable.

Let $C_c^{\infty}(G)$ be the space of locally constant functions with compact support on G modulo Z_G. Let $L^2(G)$ be the completion of $C_c^{\infty}(G)$ with respect to the norm, denoted by $\| \ \|$, and induced from an inner product defined by:

$$\langle f(x), g(x) \rangle = \int_{G/Z_G} \overline{f(x)} g(x) \mathrm{d}x, \quad \forall\ f(x), g(x) \in C_c^{\infty}(G).$$

Let $C_c^{\infty}(G, \chi)$ and $L^2(G, \chi)$ be the subspace of $C_c^{\infty}(G)$ and $L^2(G)$, respectively, where Z_G acts as a unitary character χ.

We will also define the following (also see chapter 3 in part I of [3]):

Definition 3.1. Let f be a locally constant function on G. f is said to be a supercusp form if
(i) supp(f) is compact modulo Z_G,
(ii) for all parabolic subgroups $P \neq G$,

$$\int_N f(xn)\mathrm{d}n = 0, \quad \forall\, x \in G,$$

where N is the unipotent radical of P, dn is a Haar measure on N.

We will use $^\circ C_c^\infty(G)$ to denote the space of supercusp forms of G, and use $^\circ C_c^\infty(G,\chi)$ to denote the subspace of $^\circ C_c^\infty(G)$ on which Z_G acting as a unitary character χ.

Theorem 3.2. *Suppose H is a reductive algebraic group defined over F, σ is an irreducible unitarizable representation of H on a Hermitian space V (with inner product $\langle\cdot,\cdot\rangle$), such that $\sigma(Z_H)$ acts on V as a unitary character χ, and the matrix coefficient of σ defined by any pair $(u,v) \in V \times V$ is compact modulo Z_H. Then σ is supercuspidal.*

Proof. Let $P = MN$ be a parabolic subgroup of H, where M and N are the Levi subgroup and unipotent radical of P, respectively. Let $V(N)$ be the subspace of V spanned by the elements $\{\sigma(n)v - v |\ n \in N, v \in V\}$, and let $J_N = V/V(N)$ be the corresponding Jacquet module. Similarly, if N_1 is a compact subgroup of N, we will let $V(N_1)$ be the subspace of V spanned by elements $\{\sigma(n)v - v |\ n \in N_1, v \in V\}$. Then

$$V(N) = \bigcup V(N_0),$$

where the union is over all open compact subgroups of N. Since V is Hermitian, we may take J_N to be the orthogonal complement of $V(N)$ in V. For any $w \in J_N$ and $n \in N$, we may write

$$\sigma(n)w = (\sigma(n)w - w) + w \in V(N) \oplus J_N = V.$$

Then

$$\begin{aligned}\parallel w \parallel^2 &= \langle w, w\rangle = \langle \sigma(n)w, \sigma(n)w\rangle \\ &= \langle (\sigma(n)w - w) + w, (\sigma(n)w - w) + w\rangle \\ &= \parallel \sigma(n)w - w \parallel^2 + \parallel w \parallel^2 .\end{aligned}$$

Therefore, $\parallel \sigma(n)w - w \parallel = 0$, which implies $\sigma(n)w - w = 0$, *i.e.*, $\sigma(n)w = w$. So if $J_N \neq 0$, we may pick a nonzero $w \in J_N$. However, the matrix coefficient

$$m(x) = \langle w, \sigma(x)w\rangle, \quad \forall\, x \in H,$$

will be a nonzero constant $\parallel w \parallel^2$ on N, which conflicts the compactness of the support of $m(x)$ modulo Z_H. Thus, $J_N = 0$ and $V(N) = V$.

Fix any two vectors $u, v \in V$, let

$$f(x) = \langle u, \sigma(x)v\rangle, \quad \forall\, x \in H.$$

Let $\mathcal{M}_\sigma$ be the subspace of $C_c^\infty(H)$ generated by $f(x)$ under the left regular representation $\mathfrak{l}$ of H, then $\mathcal{M}_\sigma \cong V$.

Fix any $x \in H$, and consider the function $f(xn)$ in terms of n. Then this function has a compact support $N_1 \subset N$. Let $\bar{N} = xNx^{-1}$, $\bar{N}_1 = xN_1x^{-1}$, and $\bar{n} = xnx^{-1}$. Then $f(xn) = f(\bar{n}x)$, and the support of $f(\bar{n}x)$ is contained in $\bar{N}_1$. Since $V = V(\bar{N})$ also holds, there is a subgroup $\bar{N}_0$ of $\bar{N}$ such that $u \in V(\bar{N}_0)$. Let $\bar{N}_2$ be an open compact subgroup of $\bar{N}$ containing both $\bar{N}_0$ and $\bar{N}_1$. Then $V(\bar{N}_0) \subset V(\bar{N}_2)$, and

$$\int_N f(xn)\mathrm{d}n = \int_{N_1} f(xn)\mathrm{d}n = \int_{\bar{N}_1} f(\bar{n}x)\mathrm{d}\bar{n}$$

$$= \int_{\bar{N}_2} f(\bar{n}x)\mathrm{d}\bar{n} = \int_{\bar{N}_2} \langle \sigma(\bar{n}^{-1})u, \sigma(x)v\rangle \mathrm{d}\bar{n}$$

$$= \langle \int_{\bar{N}_2} \sigma(\bar{n}^{-1})u\mathrm{d}\bar{n}, \sigma(n)v\rangle = 0.$$

As P and x are arbitrarily chosen, one sees that $f(x) \in {}^{\circ}C_c^{\infty}(H,\chi)$.

Let λ (on Hilbert space, to distinguish from $\mathfrak{l}$ above) denote the left regular representation of H on $L^2(H,\chi)$. Let $\mathcal{H}_f$ be the smallest closed subspace of $L^2(H,\chi)$ which is stable under λ and contains f. Then by Theorem 6 in [3], there is a mutually orthogonal, λ-stable irreducible decomposition:

$$(\lambda, \mathcal{H}_f) = \bigoplus_{i=1}^{r} (\sigma_i, V_i),$$

where each $\sigma_i, 1 \leq i \leq r$, is supercuspidal and is actually the restriction of λ on V_i. In particular, each V_i is admissible under $\lambda(H)$ (for example, applying Thm. 4 in [3] by taking the trivial representation for any open compact subgroup K of H), so is $\mathcal{H}_f$. As $f \in \mathcal{H}_f$, it is obvious that $(\mathfrak{l}, \mathcal{M}_\sigma) \subset (\lambda, \mathcal{H}_f)$. Therefore, $\mathcal{M}_\sigma \cong V$ is also admissible. As we have shown above that $J_N = 0$ for any parabolic subgroup $P = MN$ of H, V is supercuspidal by an equivalent definition of supercuspidal representations in Section 5.1 of [2]. □

Theorem 3.3. *Suppose (π, V) is an irreducible supercuspidal representation of a reductive algebraic group G defined over F, H is a closed reductive subgroup of G such that $Z_H/(Z_H \cap Z_G)$ is compact. Then V can be decomposed into a countable direct sum of mutually orthogonal supercuspidal representations of H.*

Proof. Let $\langle\cdot,\cdot\rangle$ be the G-invariant Hermitian inner product of V determined by Proposition 2.5.4 in [2]. We will use Z_H^* to denote the set of characters of H (the set of multiplicative functions from H to $\mathbf{C}^\times$).

We observe that as $Z_G \subset Z_H$, then $H \cap Z_G \subset Z_G \subset Z_H$. Therefore,

$$Z_H \cap Z_G \subset H \cap Z_G = (H \cap Z_G) \cap Z_G \subset Z_H \cap Z_G.$$

Hence, the second inclusion above must be an identity, *i.e.*,

$$Z_H \cap Z_G = H \cap Z_G. \tag{3.1}$$

(A) Since V is an irreducible admissible p-adic represnetation, it has countable dimension. So as an H-space, it is certainly countably generated.

(B) Then by Lemma 2.2, V can be decomposed into a countable direct sum of mutually orthogonal irreducible representations of H :

$$V = \bigoplus_{i\in I} W_i, \tag{3.2}$$

where each W_i has an inner product inherited from that on V. Moreover, for any $v \in W_i$, v is fixed by an open subgroup K_v of G since V is smooth. Then v is fixed by $H \cap K_v$, which is open in H. Hence, W_i is smooth under $\pi(H)$.

Now we will fix an arbitrary W_k and show that $\pi(Z_H)$ acts on it as a character. First of all, since (π, V) is irreducible and admissible, $\pi(Z_G)$ acts on V as a character, denoted as χ_G. We will fix representatives of $Q := Z_H/(Z_H \cap Z_G)$ in Z_H, and write $Z_H = Q(Z_H \cap Z_G)$. We will also take Q in the quotient topology, *i.e.*, a subset X of Q is defined to be open if and only if its preimage under the projection map

$$P : \quad Z_H \to Q$$

is open in Z_H. Then by the assumption of the theorem, Q is compact abelian (as Z_H is abelian).

Pick an arbitrary nonzero element $v \in W_k$, since (π, V) is admissible, there is an open subgroup K_v in G that fixes v. Let $Z_H^v = K_v \cap Z_H$, then Z_H^v is an open subgroup of Z_H under the relative topology of Z_H. Thus, $Q_v = P(Z_H^v)$ is an open subgroup of Q. Hence, Q/Q_v is finite abelian (since as pointed above, Q is compact). Moreover, since $\pi(Z_G)$ acts as a character on v, we know that the one dimensional linear space $L(v) := \{cv|\ c \in \mathbb{C}\}$ is invariant under $\pi(Q_v Z_G)$. Since

$$Z_H/(Q_v Z_G) = (QZ_G)/(Q_v Z_G) \cong Q/Q_v,$$

the Z_H-module

$$N_v := \{\pi(z)u|\ z \in Z_H, u \in L(v)\}$$

generated by v is finite dimensional. Hence, there are $\psi_i \in Z_H^*$ so that

$$N_v = \bigoplus_{i=1}^{s} N_{\psi_i},$$

where $N_{\psi_i} \subset N_v$ is defined by:

$$N_{\psi_i} = \{u \in N_v|\ \pi(z)u = \psi_i(z)u,\ \forall\ z \in Z_H\}.$$

For each i with $1 \le i \le s$, let W_{ψ_i} be the H-module generated by N_{ψ_i}. Then

$$\sum_{i=1}^{s} W_{\psi_i} \subset W_k. \tag{3.3}$$

Notice since Z_H is the center of H, for any $z \in Z_H$ and $w \in W_{\psi_i}$, $\pi(z)w = \psi_i(z)w$. While if $\psi_i \ne \psi_j$, then there is $z' \in Z_H$ so that $\psi_i(z') \ne \psi_j(z')$. Therefore, $W_{\psi_i} \cap W_{\psi_j} = \{0\}$. Thus, the sum in (3.3) is a direct sum. Since W_k itself is H-irreducible, there is only one W_{ψ_i} appearing in (3.3) and $W_k = W_{\psi_i}$. Hence, $\pi(Z_H)$ acts on W_k as a character ψ_i.

The matrix coefficient defined by any pair $(v', v) \in W_k \times W_k$ has compact support on $H/(H \cap Z_G)$ because π is suppercuspidal. Since

$$\begin{aligned} H/(H \cap Z_G) &\cong [H/Z_H] \times [Z_H/(H \cap Z_G)] \\ &\overset{\text{by}(3.1)}{=} [H/Z_H] \times [Z_H/(Z_H \cap Z_G)], \end{aligned}$$

and since $Z_H/(Z_H \cap Z_G)$ is compact, this matrix coefficient also has compact support on H/Z_H. Then by Theorem 3.2, W_k is supercuspidal. Thus, each irreducible subspace W_i appearing in (3.2) is supercuspidal since W_k above is arbitrarily chosen. □

Proposition 3.4. *Let G, H and (π, V) be the same as in Theorem 3.3. Suppose W is an H-stable subspace of V, then W can be decomposed into a countable direct sum of mutually orthogonal supercuspidal representations of H.*

Proof. Replacing V by W in the proof of Theorem 3.3 will complete the proof. □

Theorem 3.5. *Suppose (π, V) is a supercuspidal representation of a reductive algebraic group G defined over F, H is a closed reductive subgroup of G such that $Z_H/(Z_H \cap Z_G)$ is noncompact. Then V can be decomposed into a countable direct sum of irreducible representations of H, and none of these irreducible constituents is admissible.*

Proof. The first assertion of the theorem follows from Lemma 2.2. Now suppose $(\pi|_H, V)$ has a decomposition as (3.2). Pick up an arbitrary $W = W_i$ in (3.2), fix nonzero elements $u, v \in W \subset V$, and let

$$f(x) = \langle u, \pi(x)v\rangle, \quad \forall\, x \in H.$$

Then by the supercuspidality of π, $f(x)$ is a locally constant function with compact support on H modulo $H \cap Z_G$. Let $\mathcal{M}_W$ be the space generated by $f(x)$ under the right regular representation ρ of H, then $(\rho, \mathcal{M}_W) \cong (\pi|_H, W)$.

As shown in part (B) of the proof of Theorem 3.3, W is H-smooth. Choose an open compact subgroup O_v of H that fixes v. Let

$$W^{O_v} = \{w \in W |\ \pi(x)w = w,\ \forall\, x \in O_v\}.$$

Then $v \in W^{O_v}$ by definition, and $\forall\, z \in Z_H$, it is directly seen that $\pi(z)v \in W^{O_v}$. Thus, we will define:

$$N_v = \{\pi(z)v |\ z \in Z_H\},$$

i.e., N_v is the Z_H-module generated by v. We will show that W is not H-admissible by showing that N_v is not finite dimensional.

Since the support of $f(x)$ is compact on $H/(H \cap Z_G)$, we may choose a compact set $C \subset H$ so that $\mathrm{supp}\{f(x)\} \subset C(H \cap Z_G)$. As Z_H is closed in H, $Z_H/(H \cap Z_G)$ is closed in $H/(H \cap Z_G)$ (quotient topology). By the assumption of this theorem, $Z_H/(H \cap Z_G)$ is noncompact, it is therefore unbounded. Thus, there is $z_1 \in Z_H$ such that

$$C(H \cap Z_G) \cap z_1 C(H \cap Z_G) = \emptyset.$$

We will inductively choose a sequence $\{z_n |\ z_n \in Z_H\}$ as follows:

(i) Let $z_0 = 1$ (the identity of H), and choose z_1 as above.

(ii) Suppose for any positive integer n, a set of elements $\{z_k |\ 0 \le k \le n\}$ in Z_H has been chosen. As $Z_H/(H \cap Z_G)$ is unbounded, we may choose $z_{n+1} \in Z_H$ so that

$$z_{n+1} C(H \cap Z_G) \bigcap \left(\bigcup_{k=0}^{n} z_k C(H \cap Z_G) \right) = \emptyset. \tag{3.4}$$

For any nonnegative integer i, let

$$f_i(x) = f(z_i^{-1}x) = \langle u, \pi(z_i^{-1}x)v\rangle$$
$$= \langle u, \pi(x)[\pi(z_i^{-1})v]\rangle.$$

Then one checks immediately that

$$\mathrm{supp}\{f_i(x)\} = z_i \mathrm{supp}\{f(x)\}.$$

For any positive integer m, suppose $a_0, a_1, \cdots, a_m \in \mathbb{C}$ so that

$$F(x) := \sum_{i=0}^{m} a_i f_i(x) = 0.$$

By considering the value of $F(x)$ on $z_m C(H \cap Z_G), z_{m-1} C(H \cap Z_G), \cdots, z_0 C(H \cap Z_G)$ in turn, we get $a_m = 0, a_{m-1} = 0, \cdots, a_0 = 0$ iteratively by (3.4). Hence, $f_0(x), f_1(x), \cdots, f_m(x)$ are linearly independent. Consequently, $\pi(z_0^{-1})v, \pi(z_1^{-1})v, \cdots \pi(z_m^{-1})v$ are linearly independent. Since m could be any positive integer, one sees that N_v is infinitely dimensional. Therefore, W^{O_v} is also infinitely dimensional. Thus, W is not admissible, *i.e.*, any constituent appearing in (3.2) is not admissible. □

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Conflicts of interest

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