RAIRO-Theor. Inf. Appl. 60 (2026) 21
https://doi.org/10.1051/ita/2026021

**RAIRO - Theoretical Informatics and Applications**
www.rairo-ita.org

# CATASTROPHIC FAULT PATTERNS AND GLACIAL LANDSCAPES

Tomislav Došlić[1], Luka Podrug[1,*] and Ivana Zubac[2]

**Abstract.** Catastrophic fault patterns are failure patterns resulting in the loss of connectivity of a processor array. We provide bijective correspondences between such patterns in one-dimensional processor arrays and lattice paths satisfying certain inertial criteria. In that way, we obtain new results for the asymptotic behavior of the enumerating sequences of catastrophic fault patterns. We also recover and reinterpret several known enumerative results and open possibilities for obtaining analogous results for more complex, but still essentially one-dimensional arrays.

**Mathematics Subject Classification.** 05A19.

Received March 28, 2026. Accepted June 2, 2026.

## 1. Introduction

An important property of processor interconnecting patterns is their resilience to failures of individual elements. Particularly undesirable are the fault patterns resulting in fragmentation of the processor array, *i.e.*, those resulting in disconnected clusters of processors. For one-dimensional processor arrays, the risk is reduced by introducing additional connections, redundant under normal circumstances, but crucial for maintaining overall connectivity in the event of single or even multiple processor failures. It is, hence, of substantial interest to enumerate the failure patterns leading to the array fragmentation in terms of its defining parameters – the number of processors and the span of its redundant connections. The problem has attracted significant attention, and there are several results enumerating such patterns – catastrophic fault patterns – for one-dimensional arrays [2, 6, 12, 14]. In this paper, we revisit some of those results, interpret them in terms of several classes of restricted lattice paths, emphasize their connections to other combinatorial families, and provide combinatorial bijective proofs.

There are good reasons why the combinatorial, and in particular bijective, proofs are the preferred tool of working discrete mathematicians. Besides establishing the mere fact that two sets are equinumerous, such proofs also provide deeper insights into the nature of the relationship between them. This, in turn, facilitates the transfer of understanding and knowledge from one context to the other. Hence, it is more a rule than an exception that combinatorial proofs are sought after even for the results established by other techniques. There is also another important component – by combining the right amount of parsimony and ingenuity, one can arrive at proofs described as elegant and aesthetically pleasing. Finally, it seems that the nice bijective proofs will, for the time being, remain a human endeavor, as can be concluded from a recent paper reporting that

*Keywords and phrases:* Catastrophic fault patterns, Dyck path, Motznik path.

[1] University of Zagreb Faculty of Civil Engineering, Croatia.
[2] Faculty of Mechanical and Electrical Engineering, University of Mostar, Bosnia and Herzegovina.

[*] Corresponding author: luka.podrug@grad.unizg.hr

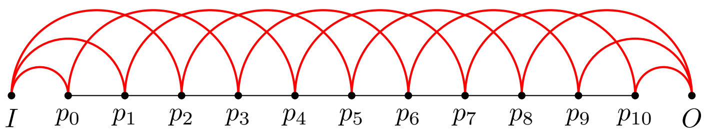


FIGURE 1. A one-dimensional processor array.

bijection discovery is still hard for AI [1]. Hence, we feel justified in revisiting some known results on catastrophic fault patterns and providing bijective proofs relating them to other, better researched, combinatorial classes.

In the next section, we define and introduce the studied objects, along with several classes of lattice paths, and recall the known results on the number of catastrophic fault patterns for one-dimensional arrays with a given length and redundancy. Section 3 presents the bijective proofs, while Section 4 uses the established results to determine the asymptotic behavior of the enumerating sequences. Finally, in the last section, we briefly comment on some possibilities of future research.

## 2. Definitions and preliminary results

### 2.1. Catastrophic fault patterns

We consider a one-dimensional array of $N$ processing elements $p_i$, $i = 0, 1, ..., N-1$. By $p_i p_{i+1}$ we denote the regular links between consecutive processors, and, for a non-negative integer $g \geq 2$, by $p_i p_{i+g}$ we denote the redundant links, also called the bypasses. The bypass links are used for reconfiguration purposes when a fault is detected.

In Figure 1, $I$ denotes the input node, connected to all $p_0, p_1, ...p_{g-1}$, and $O$ denotes the output node, connected to all $p_{N-1}, ...p_{N-g}$. The parameter $g$, the *redundancy*, is equal to 3.

A *catastrophic fault pattern* (a CFP, for short) is a set of integers $F = \{f_0, f_1, ..., f_{m-1}\}$ such that there is no path between $I$ and $O$ when $p_i, i \in F$ are deleted from the array $\{p_0, p_1, ...p_{N-1}\}$. Clearly, a CFP must contain at least $g$ elements. We consider only the smallest cases, $F = \{f_0, ..., f_{g-1}\}$. The *width* of the pattern $F$ is $W_F = f_{g-1} - f_0 + 1$.

Pagli and Pucci considered such patterns in [2] and interpreted the number of minimal CFPs starting with a fixed element $p_i$ for $0 \leq i \leq N - W_F$ in terms of Motzkin numbers $M_{g-1}$ (more on them in the next subsection). In our example, if we take $p_3$ as the starting element in a fault pattern, it should give $M_2 = 2$. Indeed, there are altogether 21 fault patterns of length 3 starting at $p_3$,

$$\underbrace{|\{3,4,*\}|}_{6} + \underbrace{|\{3,5,*\}|}_{5} + ... + \underbrace{|\{3,9,10\}|}_{1} = 21.$$

We claim that only 2 of them are catastrophic. One is, obviously, $\{3, 4, 5\}$, and the second one is $\{3, 5, 7\}$. It's easy to see that the others aren't catastrophic.

### 2.2. Lattice paths, glacial landscapes, and secondary structures

The Motzkin numbers, appearing in the quoted result by Paglia and Pucci, are elements of an extremely versatile nonnegative integer sequence having numerous combinatorial interpretations. Dozens of them are listed in the corresponding entry A001006 in the *The On-Line Encyclopedia of Integer Sequences* [3] and in the references therein. Among the best-known interpretations are those related to some restricted classes of lattice paths. Hence, we start this subsection by introducing two classes of lattice paths, the Dyck paths and the Motzkin paths, which serve as the basis for all our subsequent bijective proofs.

A *Dyck path* of length $2n$ is a lattice path in the coordinate plane $(x, y)$ from the point $(0, 0)$ to $(2n, 0)$ with exactly $n$ *Up-steps* $U = (1, 1)$ and $n$ *Down-steps* $D = (1, -1)$ such that the path never goes below the

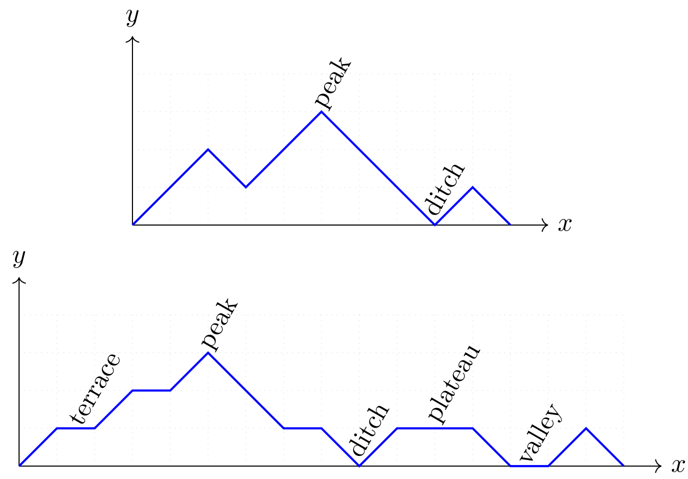


FIGURE 2. A Dyck (above) and a Motzkin (below) path with some landscape-inspired terminology.

$x$-axis. Clearly, the length of a Dyck path is always even. The number of paths of length $2n$ is the $n$−th Catalan number ($C_n$), $C_n = \frac{1}{n+1}\binom{2n}{n}$. The sequence $C_n$, starting with $1, 1, 2, 5, 14, 42, ...$, is one of the best known and most researched integer sequences. It has hundreds of combinatorial interpretations; we refer the reader to entry A000108 in [3] and to the references listed there. The set of all Dyck paths of length $n$ we denote by $\mathcal{D}(n)$. Hence, $|\mathcal{D}(n)| = C_n$. The set of all paths from $\mathcal{D}(n)$ with exactly $k$ Up-steps we denote by $\mathcal{D}_k(n)$.

A *Motzkin path* of length $n$ is a path on the integer lattice $\mathbb{Z} \times \mathbb{Z}$ from the point $(0, 0)$ to $(n, 0)$ with three types of steps: the *Up-step* $U = (1, 1)$, the *Down-step* $D = (1, -1)$ and the *Level step* $L = (1, 0)$ such that the path never goes below the $x$-axis. The set of all Motzkin paths of length $n$ we denote by $\mathcal{M}(n)$. As mentioned above, this set is enumerated by the Motzkin numbers, $|\mathcal{M}(n)| = M_n$. Unlike the Catalan numbers, the Motzkin numbers do not allow for a simple closed formula. There are, however, numerous expressions connecting them with other combinatorial sequences. Probably the best known is the one expressing them in terms of the Catalan numbers and binomial coefficients, $M_n = \sum_{k=0}^{n} \binom{n}{2k} C_k$.

Both Dyck and Motzkin paths resemble schematic drawings or mountain ranges. Two examples, with some orographically inspired terms, are shown in Figure 2. Here we give the formal definitions of those terms. Let $P$ be a Dyck or a Motzkin path. Any non-empty sequence of consecutive Up-steps is called an *ascent*, and any non-empty sequence of consecutive Down-steps is a *descent* in $P$. An Up-step immediately followed by a Down-step is a *peak*; a Down-step immediately followed by an Up-step is a *ditch*. A non-empty sequence of consecutive Level steps preceded by an Up-step and followed by a Down-step is a *plateau*. If a sequence of consecutive Level steps is preceded by a Down-step and followed by an Up-step, it is called a *valley*. Finally, if a non-empty sequence of consecutive Level steps is preceded and followed by the same type of non-Level step, whether Up or Down, it is called a *terrace*.

A **glacial landscape** of length $n$ is a Motzkin path of length $n$ without peaks and ditches. The name is motivated by the resemblance of such paths to the landscapes eroded by the action of glaciers; the peaks are flattened, and the ditches are widened into valleys. The set of all glacial landscapes of length $n$ we denote by $\mathcal{G}(n)$. Clearly, $\mathcal{G}(n) \subseteq \mathcal{M}_n^{(1)}$. An example is shown in Figure 3. The set of all glacial landscapes of length $n$ in which all plateaus and all valleys have width at least $l$ is denoted by $\mathcal{G}^{(l)}(n)$, and its cardinality is denoted by $G^{(l)}(n) = \left|\mathcal{G}^{(l)}(n)\right|$. The numbers $G^{(l)}(n)$ can be interpreted as the number of all walks of length $n$, consisting of steps $1, 0, -1$ on the non-negative integers, which start and end at 0, and have *inertia* $l$, *i.e.*, they cannot easily change direction.

The mountain-related terminology will be handy for explaining connections between the Dyck and Motzkin paths and some other classes of combinatorial object such as, *e.g.*, the secondary structures of single-stranded

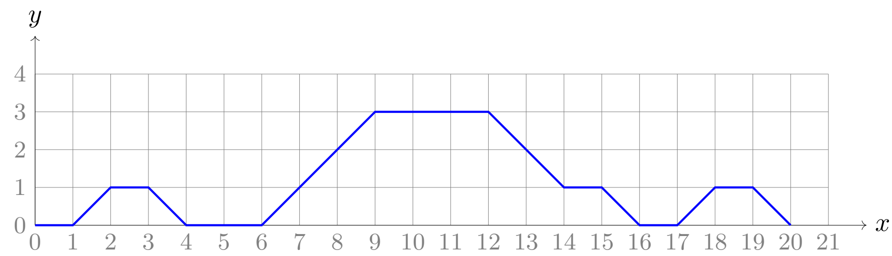


FIGURE 3. A glacial landscape of length 20.

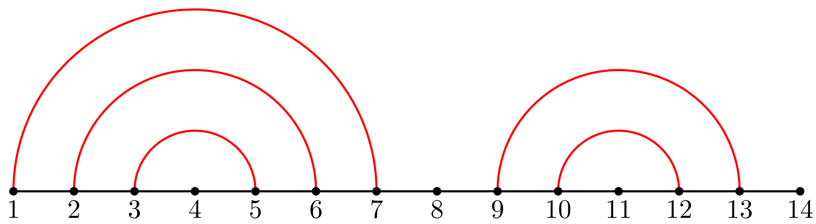


FIGURE 4. A secondary structure of length 14, rank 1 and order 5.

RNA molecules. RNA molecules are single-stranded chains of nucleotides composed of four types of bases: A, C, G, and U. Through complementary base-pairing, these sequences fold into energetically favorable configurations known as *secondary structures*. These are often constrained by stereochemical rules, such as the non-crossing of bonds and minimum loop lengths.

From a mathematical perspective, RNA secondary structures can be represented as planar graphs with two specific types of edges: the backbone bonds, also called $p$-bonds, connecting adjacent nucleotides, and hydrogen bonds (or $h$-bonds), forming arcs over the linear backbone. The stiffness of the backbone is characterized by a positive integer parameter $l$.

A *secondary structure* $S$ of length $n$ and rank $\ell$ is a labeled, undirected graph $G = (V, E)$, with a set of vertices $V = \{1, 2, ..., n\}$ and set of edges $E = P \cup H$ which satisfies:

(a) $\{i, i+1\} \in P, \ \forall\, 1 \leq i < n-1$;
(b) $\{i, j\} \in H \ \&\ \{i, k\} \in H \Longrightarrow j = k$;
(c) $\{i, j\} \in H \Longrightarrow |i - j| > l$;
(d) $\{i, j\} \in H,\ \{p, q\} \in H \ \&\ i \leq p < j \Longrightarrow i \leq q < j$.

The number of nucleotide bases $n$ is the *length* and the number of $h$-bonds, $|H|$, is called the *order* of the secondary structure. The stiffness parameter $l$ is called the *rank*. The secondary structure shown in Figure 4 has the length 14, rank 1, and order 5. Notice that the rank of a secondary structure plays a role similar to the inertia parameter in the walks corresponding to glacial landscapes, with the important difference that here inertia is experienced in one direction only: a walk going upwards cannot change direction before spending at least $l$ steps at the same level, while a walk going downwards can reverse its direction already in the next step.

The set of all secondary structures of length $n$, order $k$, and rank $l$ is denoted by $\mathcal{S}^{(l)}(n)$, and the set of all such structures with exactly $k$ $h$-bonds is denoted by $\mathcal{S}_k^{(l)}(n)$.

There is a simple bijection between $\mathcal{S}^{(l)}(n)$ and the set of all peakless Motzkin paths whose all plateaus are at least $l$ steps wide, denoted by $\mathcal{M}^{(l)}(n)$. The result belongs to mathematical folklore and has been independently rediscovered by several authors.

**Lemma 2.1.** *There is a bijection between* $\mathcal{S}^{(l)}(n)$ *and* $\mathcal{M}^{(l)}(n)$.

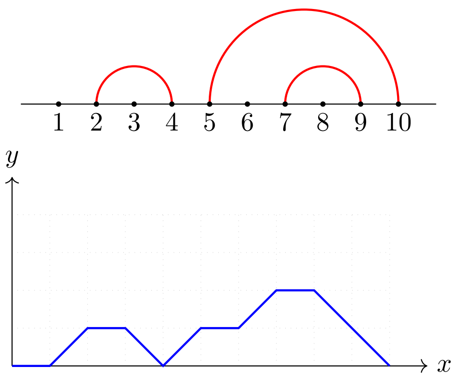


Figure 5. A secondary structure and the corresponding Motzkin path.

*Proof.* For a given secondary structure from $\mathcal{S}^{(l)}(n)$, one can construct the corresponding path by scanning the structure from left to right, assigning the Level step to each base not incident with an $h$-bond, the Up step to a base in which an $h$-bond starts, and the Down step to each base in which an already encountered $h$-bond terminates. Clearly, the constructed path will contain only the three types of steps allowed for Motzkin paths. It will remain above the $x$-axis, as no $h$-bond finishes before it starts, it will terminate at the $x$-axis, since the Up and Down steps come in pairs, and all its plateaus will be made of at least $l$ consecutive Level steps. Hence, it will belong to $\mathcal{M}^{(l)}(n)$. As the construction can be easily inverted, the claim follows. □

An example of the correspondence is shown in Figure 5.

Secondary structures are much better researched than the catastrophic fault patterns. In particular, they were enumerated with respect to their defining parameters $n$, $l$, and $k$ in [4]. As in the case of the Motzkin numbers, no closed-form expressions exist for $|\mathcal{S}_k^{(l)}(n)|$. However, their enumerating sequences are completely described by the generating functions given in [4]. Those generating functions provide, among other things, precise asymptotics for $|\mathcal{S}_k^{(l)}(n)|$. Also, there are formulas expressing those numbers in terms of sums involving binomial coefficients and an important sequence of numbers known as the Narayana numbers.

The *Narayana numbers* $N(n,k)$ are defined for positive integers $n$ and $k$ as

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}.$$

By definition, $N(0,0) = 1, N(n,0) = 0$, and $N(n,1) = 1$ for $n \geq 1$.

Among several other combinatorial structures, they count Dyck paths of length $2n$ with exactly $k$ peaks for $n, k \geq 1$. Hence,

$$\sum_k N(n,k) = C_n$$

As mentioned above, they also appear in the enumeration of secondary structures.

**Theorem 2.2.** *[4]*

$$S_k^{(l)}(n) = \sum_{p=1}^{k} N(p,k)\binom{n-lp}{2k}$$

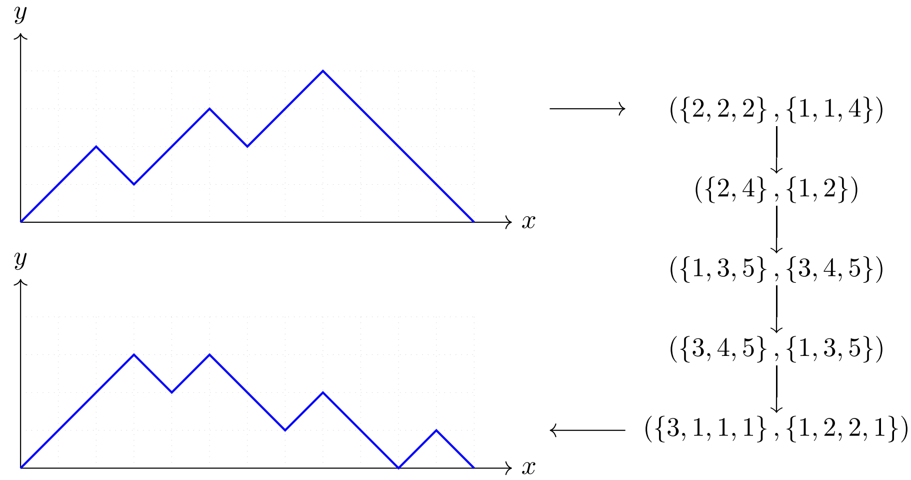


Figure 6. Symmetry of Narayana numbers – example 1.

**Corollary 2.3.** *[4]*

$$S_k^{(1)}(n) = N(n-k, k+1).$$

We will see shortly that the Narayana numbers also appear in enumerating formulas for glacial landscapes and CFPs. The results of the next section will elucidate the nature of the connection and enable us to extend the known results on the asymptotics of the secondary structure numbers also to the catastrophic fault patterns.

## 3. Bijective proofs

We start this section by providing a bijective proof of the symmetry of Narayana numbers. The result itself can be easily obtained by direct computation, but the bijection of the next Lemma will also be useful later in our other proofs.

**Lemma 3.1.** *[5]*

$$N(n, k) = N(n, n+1-k)$$

*Proof.* We start with a Dyck path of length $2n$ with $k$ peaks. To this path we assign an ordered pair of compositions of $n$ in $k$ parts: The first one is made by listing the ascents in order of their appearance, the second one by listing the descents. Then we form an ordered pair of sets of their partial sums, excluding the last element. The next step is to take the ordered pair of their complements within the set $[n] = \{1, \dots, n-1\}$. We switch the complements as components of the ordered pair. We retain the first element of each component, then fill in the differences between consecutive elements to obtain compositions of $n$. Finally, we construct a Dyck path by reading off its ascents and descents from the ordered pair of compositions of $n$. As all steps are invertible, we have a bijection.

Examples of the described bijective correspondence are shown in Figures 6 and 7.

□

Narayana numbers have many combinatorial interpretations; see entry A001263 in [3]. However, the following one is not listed there.

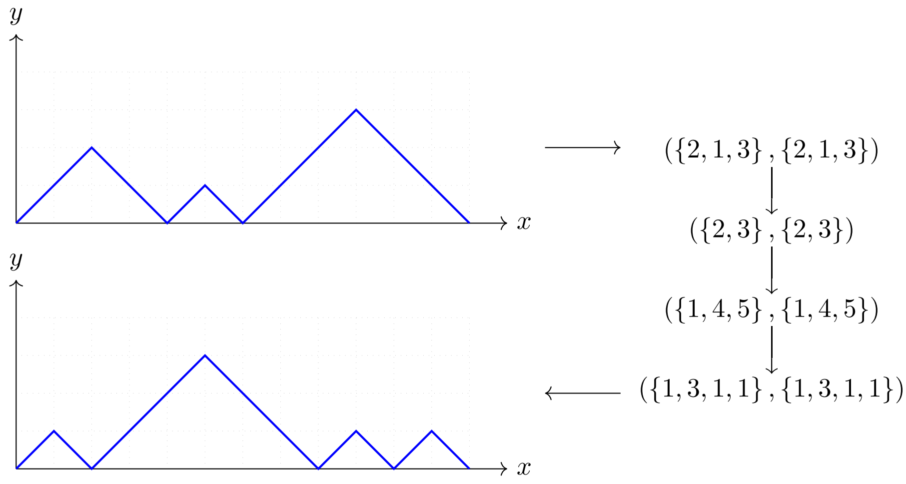


FIGURE 7. Symmetry of Narayana numbers – example 2.

**Lemma 3.2.** *Narayana numbers $N(n,k)$ count the number of Dyck paths of length $2n$ with exactly $2(k-1)$ internal positions that are neither peaks nor ditches.*

*Proof.* There are exactly $N(n,k)$ Dyck paths of length $2n$ with exactly $k$ peaks. By symmetry, this is also the number of paths from $\mathcal{D}(n)$ with exactly $n+1-k$ peaks. Since between any two consecutive peaks must be exactly one ditch, the total number of peaks and ditches is equal to $2n-2k+1$. Hence, out of the total $2n-1$ internal positions, there are exactly $2n-1-(2n-2k+1)=2(k-1)$ positions which are neither peaks nor ditches. □

Next, we show that the Narayana numbers are instrumental in enumerating glacial landscapes. Let $\mathcal{G}_k(n)$ denote the set of all paths from $\mathcal{G}(n)$ with exactly $k$ steps of type $U$.

**Proposition 3.3.**

$$|\mathcal{G}_k(n)| = \sum_{j=1}^{k} N(k,j)\binom{n-2j+1}{2k}$$

*Proof.* We consider $N(k,j)$ Dyck paths of length $2k$ with exactly $j$ peaks (and therefore exactly $j-1$ ditches). By inserting a single Level horizontal step $L$ at each peak and at each ditch, we obtain $N(k,j)$ glacial landscapes of length $2k+2j-1$, in which all plateaus and all valleys have width exactly 1.

Now, let us take additional $m \geq 0$ Level steps and distribute them arbitrarily throughout the landscape. This can be done in $\binom{m+2k}{2k}$ ways. The claim now follows by denoting the total number of steps by $n$ and summing over all $j$. □

The formula for $|\mathcal{G}_k(n)|$ has the same structure as the formula for $S_k^{(2)}(n+1)$. By using a similar argument as in Proposition 3.3, we can prove a more general claim for the number $G^{(l)}(n) = \left|\mathcal{G}^{(l)}(n)\right|$, *i.e.*, the number of glacial landscapes in which all plateaus and all valleys have width at least $l$.

**Theorem 3.4.**

$$G^{(l)}(n) = |\mathcal{G}_k(n)| = S^{(2l)}(n+l)$$

As mentioned above, the number $G^{(l)}(n)$ can be interpreted as the number of all walks of length $n$, consisting of steps $1, 0, -1$ on the non-negative integers, which start and end at 0, and have *inertia* $l$. This brings us, finally, back to the CFPs. Namely, such walks were used by Maity, Roy and Nayak in [6] where they started from an array with redundancy $G = \{1, g\}$ and generalized the results to the case $\{1, 2, ...k, g\}$ for some $1 < k < g$. According to their notation from the above work,

$$\gamma(1,2,g) = 1 + \sum_{k=1}^{\lfloor \frac{g-2}{2} \rfloor} \sum_{r=1}^{n} \underbrace{\left[ \binom{n-1}{r-1}^2 - \binom{n-1}{r-2}\binom{n-1}{r} \right]}_{N(n,r)} \binom{g-2-2(n-r)}{2n}.$$

The expression in the square brackets is equal to $N(n, r)$, and this observation provides another connection to the secondary structures and related combinatorial objects. Namely, the above expression has the same structure as the expression for $S^{(l)}(n)$ for some suitably defined parameters $l$ and $n$. That motivates the following conjecture.

**Conjecture 3.5.**

$$\gamma(1,2,...,k,g) = S^{(2k-2)}(g + k - 2).$$

Maity, Roy, and Nayak proved their result in [6] by constructing a bijection between CFPs and walks in the one-dimensional integer lattice with steps 1, 0, and -1. Proposition 3 from the mentioned paper gives the conditions for the sequence $\{m_1, ..., m_{g-1}\}$ to be a catastrophic sequence of minimal CFP. In the following table, we translate their conditions into the language of lattice paths with only Up, Down, and Level steps.

| # | Integer walk condition | Lattice path interpretation |
|---|---|---|
| (1) | $m_j \in \{1, 0, -1\}$ for $j = 1, 2, \dots, g-k$ | steps $\diagup, -, \diagdown$ |
| (2) | $m_{n-1} = m_{n-2} = \dots = m_{n-k+1} = 0$ | tail of $k-1$ horizontal steps |
| (3) | $\sum_{j=1}^{k} m_j \geq 0,\ \forall k = 1, 2, \dots, g-k-1$ | path remains above the $x$-axis |
| (4) | $\sum_{j=1}^{g-k} m_j = 0$ | path ends on the $x$-axis |
| (5) | $m_i + m_{i+1} + \dots + m_{i+s} \in \{-1, 0, 1\}$, $\forall i = 1, 2, \dots, g-k-s,\ \forall s = 1, 2, \dots, k-1$ | there are at least $k-1$ horizontal steps between any two $\diagup$ or $\diagdown$ steps |

Let us denote by $\mathcal{T}_{g-1}^k$ the set of all paths that satisfy the conditions listed in the right-hand column of the above table. Conjecture 3.5 will follow if we establish a bijection between $\mathcal{T}_{g-1}^k$ and $\mathcal{S}^{(2k-2)}(g + k - 2)$. We do this in the following theorem.

**Theorem 3.6.**

$$\gamma(1,2,...,k,g) = \left|\mathcal{T}_{g-1}^k\right| = S^{(2k-2)}(g + k - 2). \tag{3.1}$$

*Proof.* The left equality was established by Maity, Roy, and Nayak in [6]. It remains to prove the right equality. To this end, we construct a bijection between $\mathcal{T}_{g-1}^k$ and $\mathcal{S}^{(2k-2)}(g + k - 2)$.

Let us consider a secondary structure of rank $2k - 2$ on $g + k - 2$ bases. This is a lattice path with $g + k - 2$ steps, where each plateau is at least $2k - 2$ steps long (Figure 8). From the first plateau of this structure, we take $2k - 2$ horizontal steps: we set aside $k - 1$ of them and discard the remaining $k - 1$. This secondary structure originated from a Dyck path, which we will call the *master path* $D(S)$.

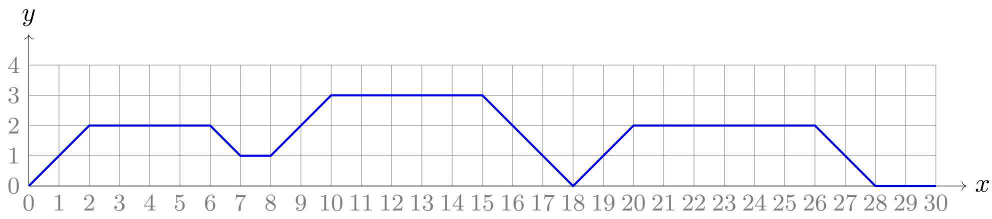


FIGURE 8. A secondary structure $S$ of rank 4 on $g + k - 2 = 30$ bases with 3 plateaus.

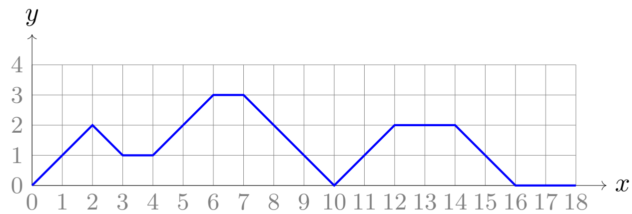


FIGURE 9. Removing blocks of 4 horizontal steps from each plateau.

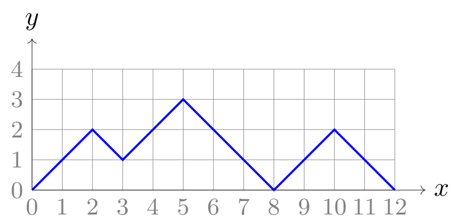


FIGURE 10. Removing remaining horizontal steps.

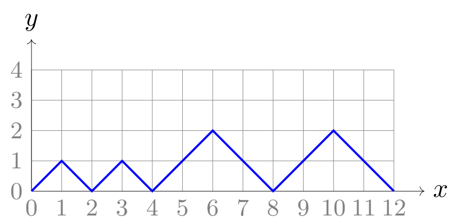


FIGURE 11. Applying the symmetry.

The peaks of this master path correspond to the plateaus in $S$. From each of the plateaus in $S$ except for the first one, we remove $2k - 2$ horizontal steps and set them aside (Figure 9). We are left with the master path along with some remaining horizontal steps (Figure 10). We record the positions and the lengths of these horizontal step blocks.

We then map the master path $D(S)$ to its image *via* the bijection $\mathcal{D}_k(n) \leftrightarrow \mathcal{D}_{n+1-k}(n)$, which is based on the symmetry of the Narayana numbers (Figure 11). Next, we restore the removed horizontal step blocks exactly in the positions from which they were taken (Figure 12). After this, we begin inserting the blocks of $2k - 2$ steps removed from the plateaus: $k - 1$ are placed between two consecutive $U$ (up) steps, and $k - 1$ are placed between two consecutive $D$ (down) steps. According to Lemma 3.2, there are exactly enough positions for this insertion.

Finally, we take the $k - 1$ horizontal steps that we set aside from the first plateau and attach them at the end of the path (the tail) (Figure 13). All steps in this construction are reversible, which ensures that the mapping is indeed a bijection. $\square$

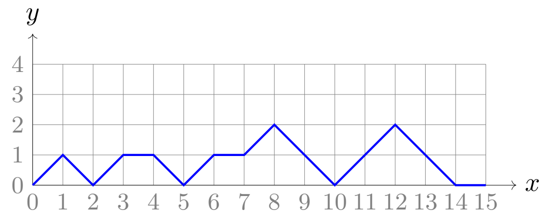


FIGURE 12. Restoring the removed blocks.

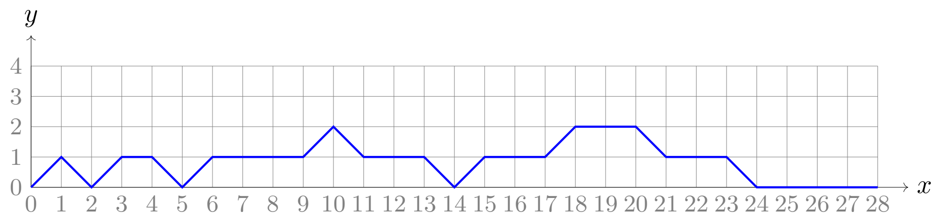


FIGURE 13. Inserting the horizontal steps taken from the plateaus.

**Example 3.7.** Let us illustrate the steps of the preceding theorem on an example: let $g = 29$, $k = 3$, so $g + k - 2 = 30$ and $g - 1 = 28$.

We remove a block of 4 horizontal steps from each plateau.

We record the positions and lengths of the remaining horizontal steps and remove them. The recorded positions and lengths are: position 3 – length 1, position 6 – length 1, position 12 – length 2, and position 16 – length 2. For the master path, we have $D(S) \in D_3(6)$.

We apply the symmetry, and then restore the removed blocks and insert the horizontal steps that were taken from the plateaus.

A final clarification: why do we have exactly the number of removed steps needed? If in a path of length $2n + 2$ there are $2k$ positions which are neither peaks nor ditches, then there are $2n + 1 - 2k$ positions which are either peaks or ditches. Out of these, there are exactly $n + 1 - k$ peaks and $n - k$ ditches.

## 4. CONSEQUENCES

In this section, we list some consequences of the established connections between CFPs and secondary structures. We start by quoting some results on the generating functions of the sequences $S^{(l)}(n)$ and on their asymptotics.

**Theorem 4.1.** *[4] The generating function $S_l(x)$ of the sequence $S^{(l)}(n)$ is given by*

$$S_l(x) = \frac{-\omega_l(x) - \sqrt{\omega_l^2(x) - 4x^2}}{2x^2},$$

*where*

$$\omega_l(x) = x - 1 - x^2 - x^3 - \ldots - x^{l+1} = x - 1 - x^2 \frac{1 - x^l}{1 - x}.$$

From the above explicit form of the generating function $S_l(x)$, one can readily get the asymptotics for the numbers $S^{(l)}(n)$ by applying a version of Darboux's theorem [7].

**Proposition 4.2.** *[4]*

$$S^{(l)}(n) \sim \frac{h_l(r_l)n^{-3/2}}{\Gamma(-1/2)r_l^n},$$

*where*

$$h_l(x) = -\frac{\sqrt{-\omega_l(x)+2x}\sqrt{w_l(x)}}{2x^2},$$

$$w_l(x) = \frac{-\omega_l(x)-2x}{1-\frac{x}{r_l}},$$

*and $r_l$ is the smallest positive root of the discriminant $\Delta_l(x) = \omega_l^2(x) - 4x^2 = 0$.*

Hence, $S^{(l)}(n) \sim \frac{h_l(r_l)}{\Gamma(-1/2)} n^{-3/2}\alpha_l^n$, where $\alpha_l = 1/r_l$.

By translating the above results into the context of CFPs, we obtain the asymptotic behavior of their enumerating sequences.

**Corollary 4.3.**

$$\gamma\,(1,2,...,k,g) \sim C(g,k)\alpha_{2k-2}^{g+k-2},$$

*where $\alpha_{2k-2} = 1/r_{2k-2}$, $r_{2k-2}$ is the smallest positive root of $\omega_{2k-2}^2(x) - 4x^2 = 0$, and $C(g,k)$ is a subexponential term, which can be explicitly computed from Proposition 4.2.*

The above result is of significant practical value, since it provides a quick insight into the behavior of $\gamma\,(1,2,...,k,g)$ for large values of $g$, which would be otherwise difficult to assess from the explicit expressions. For small values of $k$, $k \geq 4$, the corresponding values of $r_{2k-2}$ can be explicitly computed [4]. For larger values of $k$, $r_{2k-2}$ can be given in terms of generalized binomial series. The generalized binomial series $\mathcal{B}_t(z)$ is defined for arbitrary complex values of $t$ and $z$ by

$$\mathcal{B}_t(z) = \sum_{k\geq 0} (tk)^{\underline{k-1}} \frac{z^k}{k!},$$

where $x^{\underline{m}}$ denotes the falling factorial, $x^{\underline{m}} = x(x-1)\dots(x-m+1)$ ([8], p. 47).

**Corollary 4.4.**

$$r_{2k-2} = \frac{1}{2}\mathcal{B}_k\left(-\frac{1}{2^k}\right).$$

The claim follows by taking $l = 2k-2$ in Proposition 2.3 of reference [4].

## 5. Concluding remarks

In this note, we have revisited some known results on the number of catastrophic fault patterns in a one-dimensional array of processors with a given length and redundancy. We have constructed bijections between such arrays and suitably defined classes of lattice paths, providing thus combinatorial proofs of some known results, and interpreting them in a different context. As a consequence, we derived explicit asymptotics for the enumerating sequences of CFPs with given parameters.

We feel that it would not be too difficult to extend our results to the 1-dimensional structures obtained as Cartesian products of some small graphs ($P_m$, $C_m$, $K_m$, *etc.*) with paths $P_n$ [13]. Another interesting possibility would be to explore whether our results facilitate studying the link failures, a problem much less researched than the node failures.

Our results can also be exploited in other directions. For example, one could look at the series of papers by Deutsch and Elizalde [9, 10] concerned with bargraphs and their connections with Dyck and Motzkin paths. The statistics on the bargraphs studied in [9] could be reinterpreted in the CFP context and provide relevant information. Another possibly profitable direction could be considering other types of peakless and ditchless lattice paths, such as, *e.g.*, the Delannoy paths considered in [11].

## Acknowledgments

We are thankful to Marie-Sophie Litz for contributing constructive suggestions. T.D. gratefully acknowledges the hospitality and support of Professor Yang Zhao at the Nanyang Technological University Singapore.

## Funding

This work was partially supported by the Croatian Science Foundation under the project number HRZZ-IP-2024-05-2130 and in part by the Federal Ministry of Education and Science of Bosnia and Herzegovina (under 260 the Project, Contract No. 05-35-3504-1/25).

## Conflicts of interest

The authors have nothing to disclose.

## Data availability statement

This article has no associated data generated or analyzed.

## Author contribution statement

All authors have contributed to this paper equally.

## References

[1] D. Brown, J. He, H. Jenne, H. Kvinge and M. Vargas, Even with AI, bijection discovery is still hard: the opportunities and challenges of OpenEvolve for novel bijection construction. arXiv:2511.20987v1, preprint.

[2] L. Pagli and G. Pucci, Counting the number of fault patterns in redundant VLSI arrays. *Inf. Proc. Lett.* **50** (1994) 337–342.

[3] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org (2023).

[4] T. Došlić, D. Svrtan and D. Veljan, Enumerative aspects of secondary structures. *Discrete Math.* **285** (2004) 67–82.

[5] D. Veljan, On some primary and secondary structures in combinatorics. *Math. Commun.* **6** (2001) 217–232.

[6] S. Maity, B.K. Roy and A. Nayak, On enumeration of catastrophic fault patterns. *Inf. Proc. Lett.* **81** (2002) 209–212.

[7] A.M. Odlyzko, Asymptotic enumeration methods, in *Handbook of Combinatorics*, vol. 2, edited by R. Graham, M. Grötschel and L. Lovász. Elsevier, Amsterdam (1995).

[8] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*. Addison-Wesley, Reading (1988).

[9] E. Deutsch and S. Elizalde, Statistics on bargraphs viewed as cornerless Motzkin paths. *Discrete Appl. Math.* **221** (2017) 54–66.

[10] E. Deutsch and S. Elizalde, A bijection between bargraphs and Dyck paths. *Discrete Appl. Math.* **251** (2018) 340–344.

[11] S. Seo and H. Shin, On Delannoy paths without peaks and valleys. *Discrete Math.* **346** (2023) 113399.

[12] V. Acciaro and A. Nayak, Characterization of catastrophic faults in reconfigurable systolic arrays. *VLSI Des.* **7** (1998) 143–150.
[13] S. Maity, A. Nayak and B.K. Roy, Characterization of catastrophic faults in two-dimensional reconfigurable systolic arrays with unidirectional links. *Inf. Proc. Lett.* **92** (2004) 189–197.
[14] S. Maity, B.K. Roy and A. Nayak, Enumerating catastrophic fault patterns in VLSI arrays with both uni- and bidirectional links. *Integration* **30** (2001) 157–168.