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A NOTE ON WARING–GOLDBACH PROBLEM: ONE SQUARE, FOUR CUBES AND ONE kTH POWER

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Abstract. This paper establishes that for $k \geqslant 4$, all sufficiently large even integers $n \leqslant N$, with at most $O(N^{\frac{1}{2}-\vartheta(k)+\varepsilon})$ exceptions, admit representations as the sum of one square of a prime, four cubes of primes and one kth power of a prime, where the exponent $\vartheta(k)$ relies on k. This result sharpens the bound previously obtained by [J. J. Li, F. Xue and M. Zhang, *Bull. Aust. Math. Soc.* **107** (2023) 416–431].

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1. Introduction

The continuing renaissance in the theory and application of the Hardy–Littlewood method has produced significant advances in Waring's problem, in particular with respect to our understanding of the mixtures of squares, cubes and biquadrates. In 1999, Brüdern and Wooley [1] established that every sufficiently large natural number n possesses $\gg n^{13/12}$ distinct representations as the sum of one square, four cubes and one biquadrate. This lower bound matches precisely the order of magnitude predicted by the formal application of the circle method, suggesting that the corresponding asymptotic formula should hold.

A natural extension of this inquiry concerns the analogous problem wherein all variables are restricted to prime numbers. Specifically, one may ask whether every sufficiently large even integer n can be expressed as

$$n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^4, \tag{1.1}$$

with $p_1, \ldots, p_6$ denoting prime numbers. Throughout this paper, the symbol p, with or without subscripts, always signifies a prime.

Li, Xue and Zhang [2] initiated the study of exceptional sets for equation (1.1). Denoting by $E^*(N)$ the count of positive even integers $n \leqslant N$ not representable in the form (1.1), they proved that $E^*(N) \ll N^{\frac{23}{48}+\varepsilon}$ for any $\varepsilon > 0$.

The present work generalizes and refines this result. We consider the broader family of representations

$$n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^k, \qquad k \geqslant 4, \tag{1.2}$$

and establish the following:

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Theorem 1.1. *For each integer $k \geqslant 4$, let $\mathscr{E}$ denote the set of even positive integers n that cannot be represented in the form (1.2) with prime variables $p_1, \ldots, p_6$. Writing $E(N) = |\mathscr{E} \cap (0, N]|$, we have*

$$E(N) \ll N^{\frac{1}{2} - \vartheta(k) + \varepsilon},$$

where the exponent $\vartheta(k)$ is defined by

$$\vartheta(k) = \begin{cases} \frac{1}{24}, & k = 4; \\ \frac{2}{9m(k)}, & k \geqslant 5, \end{cases} \quad \text{and} \quad m(k) = \begin{cases} 2\left\lceil \left(\frac{2k}{3} + 1 - [\frac{2k}{3}]\right) 2^{[\frac{2k}{3}]-1} \right\rceil, & k = 4, 5; \\ 2\left\lceil \frac{14k}{3} - 20 \right\rceil, & k = 6, 7; \\ 2\left\lceil \left(\frac{2k}{3} - \frac{1}{2}[\frac{2k}{3}]\right)\left([\frac{2k}{3}] + 1\right) \right\rceil, & k \geqslant 8. \end{cases} \tag{1.3}$$

2. Preliminary notation and lemmas

For any subset $\mathcal{A} \subseteq \mathbb{N}$, we designate by $\overline{\mathcal{A}} = \mathbb{N} \setminus \mathcal{A}$ its complementary set within the natural numbers. Given real numbers $a < b$, we write $(\mathcal{A})_a^b = \mathcal{A} \cap (a, b]$ for the truncated set, and $|\mathcal{A}|_a^b$ for its cardinality. For two subsets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$, their sumset is defined in the usual manner:

$$\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A},\ b \in \mathcal{B}\}.$$

Fix a modulus $q \in \mathbb{N}$ and a residue class $\mathfrak{a} \in \{0, 1, \ldots, q-1\}$. We associate to these data the arithmetic progression

$$P_{\mathfrak{a},q} = \{\mathfrak{a} + mq : m \in \mathbb{Z}\}.$$

A set $\mathcal{L} \subseteq \mathbb{N}$ is termed *a union of arithmetic progressions modulo q* if $\mathcal{L} = \bigcup_{\mathfrak{l} \in \mathfrak{L}} P_{\mathfrak{l},q}$ for some subset $\mathfrak{L} \subseteq \{0, 1, \ldots, q-1\}$.

Given $\mathcal{C} \subseteq \mathbb{N}$, a union of progressions $\mathcal{L}$ modulo q, and integers a, b, we introduce the quantity

$$\langle \mathcal{C} \wedge \mathcal{L} \rangle_a^b = \min_{\mathfrak{l} \in \mathfrak{L}} |\mathcal{C} \cap P_{\mathfrak{l},q}|_a^b.$$

For $s \in \mathbb{N}$, a subset $\mathcal{Q} \subseteq \{n^s : n \in \mathbb{N}\}$ is called *a high-density subset of sth powers relative to $\mathcal{L}$* provided that, for every $\varepsilon > 0$ and all sufficiently large N, one has $\langle \mathcal{Q} \wedge \mathcal{L} \rangle_0^N \gg_q N^{\frac{1}{s} - \varepsilon}$.

Finally, for $\theta > 0$, a set $\mathcal{R} \subseteq \mathbb{N}$ is said to possess *$\mathcal{L}$-complementary density growth exponent smaller than θ* if there exists $\delta > 0$ such that $|\overline{\mathcal{R}} \cap \mathcal{L}|_0^N < N^{\theta - \delta}$ for all large N.

Lemma 2.1. *Suppose $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are each expressible as unions of arithmetic progressions to some common modulus q, and assume the inclusion $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$ holds. Take $\mathcal{S}$ to be a high-density subset of squares relative to $\mathcal{L}$, and let $\mathcal{A} \subseteq \mathbb{N}$ satisfy the condition that its $\mathcal{M}$-complementary density growth exponent is strictly less than 1. Under these hypotheses, for any $\varepsilon > 0$ and all sufficiently large N depending at most on ε, we have the bound*

$$|\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N} \ll_q N^{\varepsilon - \frac{1}{2}} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N}.$$

Proof. Kawada and Wooley [3], Theorem 2.2 established this estimate. □

Throughout the remainder of this paper, N denotes a sufficiently large integer, and n is confined to the interval $(N/2, N]$. We write $\lceil N \rceil$ for the least integer not smaller than N, and reserve ε for an arbitrarily small positive constant (not necessarily the same at each occurrence) satisfying $\varepsilon < 10^{-10}$. The letter p always signifies a prime number, and $A \sim B$ indicates that $B < A \leqslant 2B$.

We employ the standard exponential notation $e(\alpha) = \mathrm{e}^{2\pi i\alpha}$ and $e_q(\alpha) = e(\alpha/q)$. The Euler totient function and divisor function are denoted by $\varphi(n)$ and $d(n)$ respectively. The symbol χ represents a Dirichlet character modulo q, with χ_0 designating the principal character. Sums of the form $\sum_{r(q)}$ are taken over a complete residue system modulo q. For $k = 2, 3, 4$, put

$$P_k = (N/16)^{1/k}, \quad L = \log N, \quad S_k(\alpha) = \sum_{p\sim P_k} (\log p) e(\alpha p^k),$$
$$G_k(\chi, a) = \sum_{r(q)} \chi(r) e_q(ar^k), \quad S_k^*(q, a) = G_k(\chi_0, a).$$

To establish the Theorem, we analyze the exceptional set associated with the auxiliary equation

$$n = p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^k.$$

Set

$$Q = N^{\frac{1}{2k}}.$$

For $(a, q) = 1$, $1 \leqslant a \leqslant q$, define the major arcs and minor arcs by

$$\mathfrak{M} = \mathfrak{M}(Q) = \bigcup_{1\leqslant q\leqslant Q} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \left(\frac{a}{q} - \frac{Q}{qN}, \frac{a}{q} + \frac{Q}{qN}\right], \quad \mathfrak{m} = [0,1] \setminus \mathfrak{M},$$
$$\mathfrak{N} = \bigcup_{q\leqslant N^{1/8}} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qN^{7/8}}, \frac{a}{q} + \frac{1}{qN^{7/8}}\right], \quad \mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{N}, \quad \mathfrak{m}_2 = \mathfrak{m} \setminus \mathfrak{N}.$$

Then orthogonality yields

$$\begin{aligned} R(n) &= \sum_{\substack{n=p_1^3+p_2^3+p_3^3+p_4^3+p_5^k\\ p_1,p_2,p_3,p_4\sim P_3,\ p_5\sim P_k}} (\log p_1)(\log p_2)(\log p_3)(\log p_4)(\log p_5) \\ &= \int_0^1 S_3^4(\alpha) S_k(\alpha) e(-\alpha n) \mathrm{d}\alpha = \left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right) S_3^4(\alpha) S_k(\alpha) e(-\alpha n) d\alpha. \end{aligned}$$

The remaining lemmas required are stated below.

Lemma 2.2. *For every $k \geqslant 4$, one has the mean value estimate*

$$\int_0^1 |S_k(\alpha)|^{m(k)} \mathrm{d}\alpha \ll N^{\frac{m(k)}{k} - \frac{2}{3} + \varepsilon},$$

where $m(k)$ is as given by (1.3).

Proof. For $1 \leqslant i \leqslant 3$, $a_i \leqslant m(k) \leqslant b_i$,

$$\begin{aligned} \int_0^1 |S_k(\alpha)|^{m(k)} \mathrm{d}\alpha &= \left(\int_0^1 |S_k(\alpha)|^{a_i} \mathrm{d}\alpha\right)^{\frac{b_i - m(k)}{b_i - a_i}} \left(\int_0^1 |S_k(\alpha)|^{b_i} \mathrm{d}\alpha\right)^{\frac{m(k)-a_i}{b_i-a_i}} \\ &=: A_i. \end{aligned} \tag{2.1}$$

□

Case 1 ($k=4,5$). Take $a_1=2^{[\frac{2k}{3}]}$ and $b_1=2^{[\frac{2k}{3}]+1}$. The definition of $m(k)$ gives

$$m(k)\geqslant\left(\frac{2k}{3}+1+\left[\frac{2k}{3}\right]\right)2^{[\frac{2k}{3}]}. \tag{2.2}$$

Applying Hölder's inequality together with Hua's Lemma, and appealing to (2.1)–(2.2), we obtain

$$A_1\ll N^{\frac{m(k)}{k}-\frac{1}{k}\left([\frac{2k}{3}]+2^{-[\frac{2k}{3}]}m(k)-1\right)+\varepsilon}\ll N^{\frac{m(k)}{k}-\frac{2}{3}+\varepsilon}. \tag{2.3}$$

Case 2 ($k=6,7$). Put $a_2=16$ and $b_2=30$. From the definition of $m(k)$ one has

$$m(k)\geqslant\frac{28k}{3}-40. \tag{2.4}$$

Combining Hölder's inequality, Hua's Lemma and (2.1), (2.4) yields

$$A_2\ll N^{\frac{m(k)}{k}-\frac{m(k)+40}{14k}+\varepsilon}\ll N^{\frac{m(k)}{k}-\frac{2}{3}+\varepsilon}. \tag{2.5}$$

Case 3 ($k\geqslant 8$). Set $a_3=[\frac{2k}{3}]([\frac{2k}{3}]+1)$ and $b_3=([\frac{2k}{3}]+1)([\frac{2k}{3}]+2)$. One readily verifies that

$$m(k)\geqslant 2\left(\frac{2k}{3}-\frac{1}{2}\left[\frac{2k}{3}\right]\right)\left(\left[\frac{2k}{3}\right]+1\right). \tag{2.6}$$

Invoking Hölder's inequality together with Hua's Lemma, and making use of (2.1) and (2.6), we arrive at

$$A_3\ll N^{\frac{m(k)}{k}-\frac{1}{2k}\left(\frac{m(k)}{[\frac{2k}{3}]+1}+[\frac{2k}{3}]\right)+\varepsilon}\ll N^{\frac{m(k)}{k}-\frac{2}{3}+\varepsilon}. \tag{2.7}$$

Collecting (2.3), (2.5) and (2.7) completes the proof of Lemma 2.2.

Lemma 2.3. *For $k\geqslant 4$, we have*

$$\int_0^1\left|S_3^4(\alpha)S_k^{m(k)}(\alpha)\right|\mathrm{d}\alpha\ll N^{\frac{m(k)}{k}+\frac{1}{3}+\varepsilon},$$

with $m(k)$ given by (1.3).

Proof. Counting solutions of the underlying Diophantine inequality gives

$$\int_0^1\left|S_3^4(\alpha)S_k^{m(k)}(\alpha)\right|\mathrm{d}\alpha\ll N^{\varepsilon}\int_0^1\left|f_3^4(\alpha)S_k^{m(k)}(\alpha)\right|\mathrm{d}\alpha,$$

where $f_3(\alpha)=\sum_{t\sim P_3}e(\alpha t^3)$. By Lemma 2.3 in Vaughan [4],

$$\int_0^1\left|S_3^4(\alpha)S_k^{m(k)}(\alpha)\right|\mathrm{d}\alpha\ll N^{\frac{1}{3}}\mathfrak{N}(p),$$

where $\mathfrak{N}(p)$ counts solutions to

$$\Delta=3h_1h_2(2t+h_1+h_2)=p_1^k+p_2^k+\cdots+p_{m(k)/2}^k-p_{m(k)/2+1}^k-\cdots-p_{m(k)}^k \tag{2.8}$$

with $t \sim P_3$, $|h_j| < P_3$ $(j = 1, 2)$, $p_l \sim P_k$ $(l = 1, \cdots, m(k))$. When the right side of (2.8) vanishes, $\mathfrak{N}(p) \ll N^{\frac{2}{3}+\varepsilon} \int_0^1 |S_k(\alpha)|^{m(k)} \mathrm{d}\alpha$; otherwise $\mathfrak{N}(p) \ll N^{\frac{m(k)}{k}+\varepsilon}$. Using Lemma 2.2,

$$\int_0^1 \left|S_3^4(\alpha) S_k^{m(k)}(\alpha)\right| \mathrm{d}\alpha \ll N^{\frac{1}{3}+\varepsilon} N^{\frac{m(k)}{k}} + N^{1+\varepsilon} \int_0^1 |S_k(\alpha)|^{m(k)} \mathrm{d}\alpha \ll N^{\frac{m(k)}{k}+\frac{1}{3}+\varepsilon}.$$

□

Lemma 2.4. *The estimate $S_3(\alpha) \ll N^{\frac{11}{36}+\varepsilon}$ holds uniformly for $\alpha \in \mathfrak{m}_2$.*

Proof. This bound appears as Lemma 2.5 in Liu [5]. □

Lemma 2.5. *For parameters satisfying $N^{\frac{1}{2k}} \ll K \ll N^{\frac{1}{8}}$ and frequencies $\alpha \in \mathfrak{M}(2K) \setminus \mathfrak{M}(K)$, we have*

$$S_3(\alpha) \ll N^{\frac{1}{3}+\varepsilon} K^{-\frac{1}{2}}.$$

Proof. This follows from Kumchev [6], Theorem 2. □

Lemma 2.6. *Define $\omega_3(q) = 3q^{-\frac{1}{2}}$. Then the following hold:*

(i) For $\gamma \in \mathbb{R}$, set

$$\mathcal{L}(\gamma) = \sum_{q \leq P_3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{|\alpha - \frac{a}{q}| \leq P_3} \frac{w_3^2(q) d^c(q) \left| \sum_{p \sim P_3} e(p^3(\alpha+\gamma)) \right|^2}{1 + P_3^3 |\alpha - \frac{a}{q}|} \mathrm{d}\alpha.$$

Then

$$\mathcal{L}(\gamma) \ll N^{-\frac{1}{3}+\varepsilon}.$$

(ii) Let $\mathcal{M}$ denote the union of intervals $\mathcal{M}(q,a)$ for $1 \leqslant a \leqslant q \leqslant P_3^{3/4}$ with $(a,q) = 1$, where $\mathcal{M}(q,a) = \{\alpha : |q\alpha - a| \leqslant P_3^{-9/4}\}$. Suppose $G(\alpha)$ and $h(\alpha)$ are integrable 1-periodic functions and $\mathfrak{m} \subseteq [0,1)$ is measurable. Then

$$\int_{\mathfrak{m}} S_3(\alpha) G(\alpha) h(\alpha) \mathrm{d}\alpha \ll P_3 J^{\frac{1}{4}} \left(\int_{\mathfrak{m}} |G(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{4}} \mathcal{J}^{\frac{1}{2}} + P_3^{\frac{7}{8}+\varepsilon} \mathcal{J},$$

where

$$J = \sup_{\beta \in [0,1]} \int_{\mathcal{M}} \frac{\omega_3^2(q) |h(\alpha+\beta)|^2}{(1 + P_3^3 |\alpha - \frac{a}{q}|)^2} \mathrm{d}\alpha, \qquad \mathcal{J} = \int_{\mathfrak{m}} |G(\alpha) h(\alpha)| \mathrm{d}\alpha.$$

Proof. (i) can be obtained by taking $k = 3$ and $P = Q = P_3$ in Zhao [7], Lemma 2.2. (ii) is Zhao [7] Lemma 3.1. □

3. Auxiliary estimates

This section develops three propositions needed for the proof of the Theorem.

Proposition 3.1. *For $n \in (N/2, N]$, we have*

$$\int_{\mathfrak{M}} S_3^4(\alpha) S_k(\alpha) e(-n\alpha) \mathrm{d}\alpha = \frac{1}{81k} \mathfrak{S}(n) \mathfrak{J}(n) + O(N^{\frac{1}{k}+\frac{1}{3}} L^{-A}),$$

where

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{a(q)^*} \frac{S_3^{*4}(q,a) S_k^*(q,a) e_q(-an)}{\varphi^5(q)}$$

converges absolutely and satisfies

$$(\log\log n)^{-c^*} \ll \mathfrak{S}(n) \ll d(n)$$

for any odd integer n, with $c^* > 0$ a fixed constant; while $\mathfrak{J}(n)$ is defined by

$$\sum_{\substack{n_1 = m_1 + m_2 + m_3 + m_4 + m_5 \\ P_3^2 < m_1, m_2, m_3, m_4 \leqslant (2P_3)^2, P_k^2 < m_5 \leqslant (2P_k)^2}} m_1^{-2/3} m_2^{-2/3} m_3^{-2/3} m_4^{-2/3} m_k^{1/k-1},$$

and satisfies

$$\mathfrak{J}(n) \asymp N^{\frac{1}{k}+\frac{1}{3}}.$$

Proof. This follows from the standard endgame analysis in the Hardy–Littlewood method; see Hua [8], Chapter 7, Sections 7–9 for a comprehensive exposition. □

Proposition 3.2. *For $k \geqslant 4$, we have*

$$\int_{\mathfrak{m}} |S_3^8(\alpha) S_k^2(\alpha)| \mathrm{d}\alpha \ll N^{\frac{5}{3}+\frac{2}{k}-\vartheta(k)+\varepsilon},$$

where ϑ_k is given by (1.3).

Proof. Handling $\mathfrak{m}_1$ requires the bound

$$\int_{\mathfrak{M}(2K)\setminus\mathfrak{M}(K)} |S_3^8(\alpha) S_k^2(\alpha)| \mathrm{d}\alpha \ll N^{\frac{5}{3}+\frac{2}{k}-\vartheta(k)+\varepsilon},$$

where $N^{\frac{1}{2k}} \ll K \ll N^{\frac{1}{8}}$. By Lemma 2.5 and Hoffman and Yu [9] Lemma 5.2,

$$\begin{aligned}\int_{\mathfrak{M}(2K)\setminus\mathfrak{M}(K)} |S_3^8(\alpha) S_k^2(\alpha)| \mathrm{d}\alpha &\ll N^{\frac{8}{3}+\varepsilon} K^{-4} \left(N^{-1} K (N^{\frac{1}{k}} K + N^{\frac{2}{k}})\right) \\ &\ll N^{\frac{5}{3}+\frac{2}{k}-\frac{3}{2k}+\varepsilon} \ll N^{\frac{5}{3}+\frac{2}{k}-\vartheta(k)+\varepsilon}. \qquad (3.1)\end{aligned}$$

□

For $\alpha \in \mathfrak{m}_2$, we treat two ranges of k separately.

Case 1 ($k=4$). Taking $h(\alpha)=S_3(\alpha)$ and $G(\alpha)=|S_3^6(\alpha)S_4^2(\alpha)|$ in Lemma 2.6 gives

$$\begin{aligned}\int_{\mathfrak{m}_2}|S_3^8(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha&=\int_{\mathfrak{m}_2}|S_3(\alpha)G(\alpha)h(\alpha)|\mathrm{d}\alpha\\&\ll N^{\frac13}J^{\frac14}\left(\int_{\mathfrak{m}_2}|S_3^{12}(\alpha)S_4^4(\alpha)|\mathrm{d}\alpha\right)^{\frac14}\mathcal{J}^{\frac12}+N^{\frac{7}{24}+\varepsilon}\mathcal{J},\end{aligned}\tag{3.2}$$

with

$$J=\sup_{\beta\in[0,1]}\int_{\mathcal{M}}\frac{\omega_3^2(q)|S_3(\alpha+\beta)|^2}{(1+P_3^3|\alpha-\frac aq|)^2}\mathrm{d}\alpha,\qquad \mathcal{J}=\int_{\mathfrak{m}_2}|S_3^7(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha.$$

From Lemma 2.6 (i),

$$J\ll\sup_{\beta\in[0,1)}\sum_{q\le P_3^{\frac34}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{|a-\frac aq|\le q^{-1}P_3^{-\frac94}}\frac{w_3^2(q)\left|\sum_{p\sim P_3}e(p^3(\alpha+\beta))\right|^2}{(1+P_3^3|\alpha-\frac aq|)^2}\mathrm{d}\alpha\ll\sup_{\beta\in[0,1)}\mathcal{L}(\beta)\ll N^{-\frac13+\varepsilon}.\tag{3.3}$$

By Hölder's inequality, Hua's Lemma and Lemma 2.3,

$$\mathcal{J}\leqslant\left(\int_0^1|S_3^4(\alpha)S_4^8(\alpha)|\mathrm{d}\alpha\right)^{\frac14}\left(\int_0^1|S_3^8(\alpha)|\mathrm{d}\alpha\right)^{\frac34}\ll N^{\frac{11}{6}+\varepsilon}.\tag{3.4}$$

Moreover, Lemmas 2.3, 2.4 together with Hua's Lemma and the Cauchy-Schwarz inequality yield

$$\int_{\mathfrak{m}_2}|S_3^{12}(\alpha)S_4^4(\alpha)|\mathrm{d}\alpha\ll\sup_{\alpha\in\mathfrak{m}_2}|S_3(\alpha)|^6\left(\int_0^1|S_3^4(\alpha)S_4^8(\alpha)|\mathrm{d}\alpha\right)^{\frac12}\left(\int_0^1|S_3^8(\alpha)|\mathrm{d}\alpha\right)^{\frac12}\ll N^{\frac{23}{6}+\varepsilon}.\tag{3.5}$$

Inserting (3.3)–(3.5) into (3.2) gives

$$\int_{\mathfrak{m}_2}|S_3^8(\alpha)S_4^2(\alpha)|\mathrm{d}\alpha\ll N^{\frac53+\frac24-\frac1{24}+\varepsilon}.\tag{3.6}$$

Case 2 ($k\ge5$). By Lemmas 2.3, 2.4, Hua's Lemma and Hölder's inequality,

$$\begin{aligned}\int_{\mathfrak{m}_2}|S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha&\ll\max_{\alpha\in\mathfrak{m}_2}|S_3(\alpha)|^{\frac{8}{m(k)}}\left(\int_0^1|S_3^8(\alpha)|\mathrm{d}\alpha\right)^{1-\frac{2}{m(k)}}\left(\int_0^1\left|S_3^4(\alpha)S_k^{m(k)}(\alpha)\right|\mathrm{d}\alpha\right)^{\frac{2}{m(k)}}\\&\ll N^{\frac53+\frac2k-\frac{2}{9m(k)}+\varepsilon}\ll N^{\frac53+\frac2k-\vartheta(k)+\varepsilon}.\end{aligned}\tag{3.7}$$

Combining (3.1), (3.6) and (3.7) completes the proof of Proposition 3.2.

Proposition 3.3. *For $k\geqslant4$, define $\mathscr{E}_1=\{n\in\mathbb{N}:n\equiv1\,(\mathrm{mod}\,2)\,,\,n\ne p_1^3+p_2^3+p_3^3+p_4^3+p_5^k\}$, and $E_1(N)=|\mathscr{E}_1|_0^N$. Then we have*

$$E_1(N)\ll N^{1-\vartheta_k+\varepsilon},$$

where ϑ_k is defined in (1.3).

Proof. On recalling Proposition 3.1, we introduce $Z(N)$ as the collection of odd integers n in the range $(N/2, N]$ for which

$$\left|\int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)e(-\alpha n)\mathrm{d}\alpha\right| \geqslant n^{\frac{1}{k}+\frac{1}{3}}L^{-A}. \tag{3.8}$$

Write $Z = |Z(N)|$. Define $\xi(n) = 0$ for $n \notin Z(N)$, and for $n \in Z(N)$ *via*

$$\left|\int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)e(-\alpha n)\mathrm{d}\alpha\right| = \xi(n)\int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)e(-\alpha n)\mathrm{d}\alpha. \tag{3.9}$$

Plainly $|\xi(n)| = 1$ when $\xi(n) \neq 0$. Thus

$$\sum_{n\in Z(N)} \xi(n)\int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)e(-\alpha n)\mathrm{d}\alpha = \int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)K(\alpha)\mathrm{d}\alpha, \tag{3.10}$$

where

$$K(\alpha) = \sum_{n\in Z(N)} \xi(n)e(-\alpha n).$$

By (3.8)–(3.10),

$$\int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)K(\alpha)\mathrm{d}\alpha \geqslant \sum_{n\in Z(N)} n^{\frac{1}{k}+\frac{1}{3}}L^{-A} \gg ZN^{\frac{1}{k}+\frac{1}{3}}L^{-A}. \tag{3.11}$$

Moreover, the Cauchy-Schwarz inequality and Proposition 3.2 give

$$\begin{aligned}\int_{\mathfrak{m}} S_3^4(\alpha)S_k(\alpha)K(\alpha)\mathrm{d}\alpha &\ll \left(\int_{\mathfrak{m}} |S_3^8(\alpha)S_k^2(\alpha)|\mathrm{d}\alpha\right)^{1/2}\left(\int_{\mathfrak{m}} |K(\alpha)|^2\mathrm{d}\alpha\right)^{1/2}\\ &\ll N^{\frac{5}{6}+\frac{1}{k}-\frac{\vartheta(k)}{2}+\varepsilon}Z^{1/2}.\end{aligned} \tag{3.12}$$

From (3.11) and (3.12),

$$Z \ll N^{1-\vartheta(k)+\varepsilon}.$$

A dyadic argument yields

$$E_1(N) \ll N^{1-\vartheta(k)+\varepsilon}.$$

□

4. Proof of the theorem

Generate a decreasing sequence $N_0, N_1, \ldots$ recursively *via*

$$N_0 = \left\lceil \frac{N}{2} \right\rceil, \quad N_{j+1} = \left\lceil \frac{2N_j}{3} \right\rceil. \tag{4.1}$$

Denote by T the minimal index satisfying $N_T \leqslant 10$, whence $T = O(L)$. Consider the auxiliary sets

$$\mathcal{A} = \{p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^k : k \geqslant 4\}, \quad \mathcal{S} = \{p^2\}, \quad \mathcal{N} = \{n \in \mathbb{N} : n \equiv 0 \,(\mathrm{mod}\, 2)\},$$
$$\mathcal{M} = \mathcal{L} = \{n \in \mathbb{N} : n \equiv 1 \,(\mathrm{mod}\, 2)\}.$$

Evidently $\mathcal{N}$ constitutes a union of arithmetic progressions modulo 2, and $\mathcal{M}, \mathcal{L}$ share this property, with the inclusion $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$ holding trivially. The Prime Number Theorem for arithmetic progressions ensures

$$\langle \mathcal{S} \wedge \mathcal{L} \rangle_0^N \gg N^{\frac{1}{2}} L^{-1},$$

confirming that $\mathcal{S}$ is a high-density subset of squares relative to $\mathcal{L}$. Proposition 3.3 supplies the bound

$$|\overline{\mathcal{A}} \cap \mathcal{M}|_0^N = E_1(N) \ll N^{1-\vartheta(k)+\varepsilon}. \tag{4.2}$$

Consequently $\mathcal{A}$ exhibits $\mathcal{M}$-complementary density growth exponent strictly below 1. Bringing together (1.2), Lemma 2.1 and the estimate (4.2), we derive the chain of inequalities

$$\begin{aligned} |\mathscr{E}|_{2N}^{3N} &= |\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}|_{2N}^{3N} \\ &\ll N^{\varepsilon - \frac{1}{2}} |\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} \ll N^{\varepsilon - \frac{1}{2}} E_1(3N) \ll N^{\frac{1}{2} - \vartheta(k) + \varepsilon}. \end{aligned}$$

Summing over the dyadic intervals *via* (4.1) produces

$$E(N) \leqslant 10 + \sum_{j=1}^{T} |\mathscr{E}|_{2N_j}^{3N_j} \ll N^{\frac{1}{2} - \vartheta(k) + \varepsilon}.$$

This establishes the Theorem.

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