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ON GENERALIZED FIBONACCI QUATERNION SEQUENCES WITH PERIODIC COEFFICIENTS: APPLICATIONS

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Abstract. This paper develops a comprehensive theory of r-generalized Fibonacci quaternion sequences with periodic coefficients, extending and unifying the classical theory of Fibonacci quaternion sequences. We establish explicit combinatorial formulas for both constant and periodic coefficients cases with particular emphasis on the role of periodicity in the sequence structure. Special attention is given to the case $r = 2$, leading to novel applications for Pell, h-Pell, and Pell-Lucas quaternion sequences. Our results generalize several existing theorems in the literature, while providing new insights into the combinatorial nature of r-generalized Fibonacci quaternion sequences.

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1. Introduction

The algebra of quaternions, introduced by William Rowan Hamilton in 1843 [21], extends the complex numbers to a four-dimensional non-commutative algebra and has found applications in several areas of mathematics and physics [1, 2]. It consists of elements of the form $q = a + bi + cj + dk$, with $a, b, c, d \in \mathbb{R}$, where the quaternion units i, j, and k satisfy the fundamental relations

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k, \quad jk = i, \quad kj = -i, \quad ki = j, \quad ik = -j.$$

The set of quaternions denoted by $\mathbb{H}$, belongs to a distinguished family of mathematical structures known as the real normed division algebras, which comprise the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ [3]. This classification highlights the exceptional nature of quaternions as one of only four such algebras.

The study of number sequences within these division algebras has emerged as a rich field of research since the latter part of the 20th century. This development was pioneered by Horadam's foundational work in [4], complemented by Iyer's significant contributions in [5]. A pivotal concept introduced by Horadam was the

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Fibonacci quaternion sequence, which extends the classical Fibonacci numbers $\{F_n\}$, as follows

$$\mathcal{F}_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k, \text{ for every } n \geq 0. \tag{1.1}$$

Many classes of usual quaternion numbers have been considered in the literature, such as Fibonacci-Lucas quaternions [6, 7] and Pell quaternion numbers [8], as well as Jacobsthal quaternion numbers [9]. In [10], the authors aims to present, in a unified manner, results which are valid on both split quaternions with quaternion coefficients and quaternions with dual coefficients, simultaneously, calling the attention to the main differences between these two quaternions. In [11], the authors consider the so-called bi-periodic Horadam sequences. Explicit formulas in terms of Chebyshev polynomials of the second kind and the determinant of some perturbed tridiagonal 2-Toeplitz matrices are established. Several illustrative examples are provided as well. In [12], the authors give a short proof for the explicit formulas of the coefficients of a particular 3-term recurrence relation derived from a k-periodic recurrence. Any of the recurrences can be naturally interpreted in terms of determinants of Hessenberg matrices families. In [13], the authors provide new families of divisibility and strong divisibility sequences based on some factorization properties of Chebyshev polynomials. In [14], the authors obtain Euler's and De Moivre's formulas for the 4×4 matrix representation of Pauli quaternions. Moreover, we provide De Moivre's formula for the light-like Pauli quaternions. Additionally, we give the nth roots of the matrix representation of Pauli quaternions. Moreover, we exemplify some of the results with illustrative examples to support the main formulas. The interest in such sequences of quaternion numbers is due to their applications in various fields of exact and applied sciences, such as physics, chemistry, image processing, as well as mathematics and computer science.

In the present study, we are interested in the generalized Fibonacci quaternion sequences related to a linear recursive sequence $\{v_n\}_{n\geq 0}$ of order r and of periodic variable coefficients $a_j : \mathbb{N} \to \mathbb{R}$ $(0 \leq j \leq r-1)$, defined as follows

$$v_{n+r} = a_0(n)v_{n+r-1} + a_1(n)v_{n+r-2} + \ldots + a_{r-1}(n)v_n, \quad n \geq 0, \tag{1.2}$$

where $a_i(n+p) = a_i(n)$ with $p \geq 2$, and $v_0, v_1, \ldots, v_{r-1}$ are the initial conditions. Sequences (1.2) have been studied using various approaches (see, for instance, [15–17] and references therein). When the coefficients $a_j(n) = a_j \in \mathbb{R}$ $(0 \leq j \leq r-1)$ of the sequences (1.2) are constant, we recover the well known r-generalized Fibonacci sequences, which are defined under the form

$$v_{n+r} = a_0 v_{n+r-1} + a_1 v_{n+r-2} + \cdots a_{r-1} v_n, \quad n \geq 0. \tag{1.3}$$

Sequences (1.3) can be expressed under the analytic Binet formula and the combinatorial formula, established in the literature by various methods (see for example, [18–20] and references therein).

Extending Horadam's construction [4], by taking into account the formula (1.1), we define the r-generalized Fibonacci quaternion sequence $\{\mathbf{V}_n\}_{n\geq 0}$ associated with the sequences (1.2) under the form

$$\mathbf{V}_n = v_n + v_{n+1}i + v_{n+2}j + v_{n+3}k, \quad n \geq 0. \tag{1.4}$$

Surprisingly, while the combinatorial formula for quaternion sequences with constant coefficients seems natural, it has not been explored in the literature, as well as, the combinatorial context of generalized quaternion sequences of periodic coefficients remains largely unexplored. Our main contributions in this study are to establish the combinatorial structure of quaternion sequences associated with both constant coefficients sequences (1.3) and periodic coefficients sequences (1.2), while conducting a thorough investigation of their fundamental properties. More precisely, our work aims to fill this gap by exhibiting the combinatorial formula for quaternion sequences with constant coefficients (1.3) and extending this formula to the general setting of periodic quaternion sequences (1.2), using the transformation techniques of Ben Taher-Benkhaldoun (see [16]). In addition, we provide explicit formulas for some special cases, particularly $r = 2$. As an application, the Pell, h-Pell and Pell-Lucas quaternion sequences are studied in detail.

The paper is organized as follows. In Section 2, we develop the combinatorial theory for quaternion sequences with constant coefficients, proving our main formula and exploring its consequences. Furthermore, we focus on special cases, particularly $r = 2$, where we derive explicit formulas for various classical sequences including Pell, h-Pell and Pell-Lucas quaternion sequences. At the next section, we extend our results to periodic coefficients, exploring specific examples with periods 2 and 3. Finally, Section 4 discusses computational aspects and provides numerical examples. Our approach not only generalizes existing results but also provides new insights into the structure of quaternion sequences, opening avenues for future research in this rich mathematical domain.

2. Combinatorial structure for the generalized quaternion sequences

The main goal of this section is to establish a combinatorial formula for quaternion sequences extending the classical formula for linear recurrence sequences (1.3). While the combinatorial expression for linear recurrence sequences is well-known, its extension to quaternions presents new challenges due to the non-commutative nature of quaternion multiplication and the interaction between the sequence components. Our approach aims to construct a quaternion sequence $\{\mathcal{P}(n,r)\}_{n\geq 1}$ that naturally captures both the recurrence structure and the quaternion properties.

2.1. Study of the general setting

Let us begin by establishing the fundamental objects of our study. Throughout this section, we work in the quaternion algebra $\mathbb{H}$ over $\mathbb{R}$. Given a sequence $\{v_n\}_{n\geq 0}$ satisfying the linear recurrence (1.3), namely,

$$v_{n+r} = a_0 v_{n+r-1} + a_1 v_{n+r-2} + \cdots + a_{r-1} v_n, \quad n \geq 0,$$

where $a_i \in \mathbb{R}$ $(0 \leq i \leq r-1)$ are real constant coefficients and $v_0, v_1, \ldots, v_{r-1}$ are initial conditions. It follows from [19, 20] that the combinatorial expression of $\{v_n\}_{n\geq 0}$ takes the form

$$v_n = w_0 \rho(n,r) + w_1 \rho(n-1,r) + \cdots + w_{r-1} \rho(n-r+1,r)\ , \ \text{ for all } \ n \geq r, \tag{2.1}$$

where $w_s = a_{r-1} v_s + \cdots + a_s v_{r-1}$ and

$$\rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!\cdots k_{r-1}!} a_0^{k_0} \cdots a_{r-1}^{k_{r-1}}, \quad n \geq r, \tag{2.2}$$

with $\rho(r,r) = 1$ and $\rho(n,r) = 0$ for $n \leq r-1$.

We recall that the generalized Fibonacci quaternion sequence $\{\mathbf{V}_n\}_{n\geq 0}$, associated with the sequence $\{v_n\}_{n\geq 0}$ is given by $\mathbf{V}_n = v_n + v_{n+1} i + v_{n+2} j + v_{n+3} k$, for all $n \geq 0$.
A direct application of the recurrence relation (1.3) allows us to get the following lemma.

Lemma 2.1. *The sequence $\{\mathbf{V}_n\}_{n\geq 0}$ satisfies the same recurrence relation as $\{v_n\}_{n\geq 0}$. More precisely, we have*

$$\mathbf{V}_{n+r} = a_0 \mathbf{V}_{n+r-1} + \cdots + a_{r-1} \mathbf{V}_n, \ \textit{for every } n \geq 0.$$

Proof. A direct computation using the definition $\mathbf{V}_n = v_n + v_{n+1} i + v_{n+2} j + v_{n+3} k$ and the recurrence relation (1.3), permits us to have

$$\mathbf{V}_{n+r} = (a_0 v_{n+r-1} + \cdots + a_{r-1} v_n) + (a_0 v_{n+r} + \cdots + a_{r-1} v_{n+1}) i$$

$$+ (a_0 v_{n+r+1} + \cdots + a_{r-1} v_{n+2})j + (a_0 v_{n+r+2} + \cdots + a_{r-1} v_{n+3})k$$
$$= a_0(v_{n+r-1} + v_{n+r} i + v_{n+r+1} j + v_{n+r+2} k) + \cdots$$
$$+ a_{r-1}(v_n + v_{n+1} i + v_{n+2} j + v_{n+3} k).$$

Therefore, we get $\mathbf{V}_{n+r} = a_0 \mathbf{V}_{n+r-1} + \cdots + a_{r-1} \mathbf{V}_n$, for $n \geq 0$. □

On the other side, using (2.1)–(2.2), we can formulate the following result.

Theorem 2.2. *The combinatorial expression of the generalized quaternion sequence* $\{\mathbf{V}_n\}_{n\geq 0}$ *associated with the sequence* (1.3) *is given under the form*

$$\mathbf{V}_n = w_0 \mathcal{P}(n,r) + w_1 \mathcal{P}(n-1,r) + \cdots + w_{r-1} \mathcal{P}(n-r+1,r), \quad n \geq r,$$

where $w_s = a_{r-1} v_s + \cdots + a_s v_{r-1}$ $(0 \leq s \leq r-1)$ *and*

$$\mathcal{P}(n,r) = \rho(n,r) + \rho(n+1,r)i + \rho(n+2,r)j + \rho(n+3,r)k, \quad n \geq 1. \tag{2.3}$$

Proof. For $n \geq r$, using the combinatorial expression (2.1) for each component v_{n+s} of the quaternion general term $\mathbf{V}_n$, we have $v_{n+s} = w_0 \rho(n+s,r) + w_1 \rho(n+s-1,r) + \cdots + w_{r-1} \rho(n+s-r+1,r)$, for $0 \leq s \leq 3$. Therefore, starting from the definition $\mathbf{V}_n = v_n + v_{n+1} i + v_{n+2} j + v_{n+3} k$, we get $\mathbf{V}_n = [w_0 \rho(n,r) + w_1 \rho(n-1,r) + \cdots + w_{r-1} \rho(n-r+1,r)] + [w_0 \rho(n+1,r) + w_1 \rho(n,r) + \cdots + w_{r-1} \rho(n-r+2,r)]i + [w_0 \rho(n+2,r) + w_1 \rho(n+1,r) + \cdots + w_{r-1} \rho(n-r+3,r)]j + [w_0 \rho(n+3,r) + w_1 \rho(n+2,r) + \cdots + w_{r-1} \rho(n-r+4,r)]k$. Then, by rewriting terms according to the coefficients w_s, we obtain

$$\mathbf{V}_n = w_0 \mathcal{P}(n,r) + w_1 \mathcal{P}(n-1,r) + \cdots + w_{r-1} \mathcal{P}(n-r+1,r), \quad n \geq r,$$

where $\mathcal{P}(n,r)$ are given as in (2.3). □

Recall that the sequence $\{\rho(n,r)\}_{n\geq 1}$ also satisfies the linear recursive relation (1.3) as it was shown in [18, 19]. Therefore, application of Lemma 2.1 allows us to derive the following corollary.

Corollary 2.3. *The quaternion sequence* $\{\mathcal{P}(n,r)\}_{n\geq 1}$ *defined by* (2.3)*, satisfies the following recursive relation*

$$\mathcal{P}(n+r,r) = a_0 \mathcal{P}(n+r-1,r) + \cdots + a_{r-1} \mathcal{P}(n,r), \quad n \geq 1.$$

More precisely, the r initial conditions of $\{\mathcal{P}(n,r)\}_{n\geq 1}$ depends on the values of r. Indeed,with the notation $\mathcal{P}(i) = \mathcal{P}(i,r)$, for all $i = 1, \cdots, r$, for $r \geq 5$, we get

$$\begin{cases} \mathcal{P}(r-4) = \cdots = \mathcal{P}(1) = 0, \qquad \mathcal{P}(r-3) = k, \qquad \mathcal{P}(r-2) = j + a_0 k, \\ \mathcal{P}(r-1) = i + a_0 j + (a_0^2 + a_1)k, \ \mathcal{P}(r) = 1 + a_0 i + (a_0^2 + a_1)j + (a_0^3 + 2a_0 a_1 + a_2)k. \end{cases}$$

For $r = 4$, we take the last four conditions

$$\begin{cases} \mathcal{P}(1) = k, \qquad \mathcal{P}(2) = j + a_0 k, \qquad \mathcal{P}(3) = i + a_0 j + (a_0^2 + a_1)k, \\ \mathcal{P}(4) = 1 + a_0 i + (a_0^2 + a_1)j + (a_0^3 + 2a_0 a_1 + a_2)k. \end{cases}$$

For $r = 3$, we take the last three conditions

$$\begin{cases} \mathcal{P}(1) = j + a_0 k, \qquad \mathcal{P}(2) = i + a_0 j + (a_0^2 + a_1)k, \\ \mathcal{P}(3) = 1 + a_0 i + (a_0^2 + a_1)j + (a_0^3 + 2a_0 a_1 + a_2)k, \end{cases}$$

and for $r = 2$ we consider

$$\begin{cases} \mathcal{P}(1) = i + a_0 j + (a_0^2 + a_1)k, \\ \mathcal{P}(2) = 1 + a_0 i + (a_0^2 + a_1)j + (a_0^3 + 2a_0 a_1)k \quad \text{(note: } a_2 \text{ does not exist).} \end{cases}$$

For illustrative purpose, let us study the special case of the generalized quaternion sequence (1.4) defined by the generalized Fibonacci sequence (1.3) with order $r = 2$, leading to provide an application to quaternion sequences of Pell numbers, h-Pell numbers and Pell-Lucas numbers.

2.2. Study of the special case $r = 2$

This case is of particular interest as it encompasses many classical sequences of usual numbers, such that the sequences of Fibonacci, Pell, and h-Pell numbers. For this special case, the structure of the related quaternion sequences becomes more explicit and reveals interesting patterns. That is, the explicit combinatorial terms in this case are given under a more manageable form involving binomial coefficients. Let us consider the sequence $\{v_n\}_{n\geq 0}$ defined as

$$v_{n+2} = a_0 v_{n+1} + a_1 v_n, \quad n \geq 0, \tag{2.4}$$

with initial conditions v_0 and v_1. Using (2.1)–(2.2), the combinatorial expression is under the form $v_n = [a_0 v_1 + a_1 v_0]\rho(n, 2) + a_1 v_1 \rho(n-1, 2), \quad n \geq 2$, where

$$\rho(n, 2) = \sum_{l=0}^{\left[\frac{n-2}{2}\right]} \binom{n-2-l}{l} a_0^{n-2-2l}\, a_1^l, \quad n > 2, \tag{2.5}$$

with $\rho(1, 2) = 0$ and $\rho(2, 2) = 1$. Therefore, using the result of Theorem 2.2, we derive the following proposition.

Proposition 2.4. *Under the preceding data, the combinatorial formula of the quaternion sequence* $\{\mathbf{V}_n\}_{n\geq 0}$ *takes the form* $\mathbf{V}_n = [a_0 v_1 + a_1 v_0]\mathcal{P}(n, 2) + a_1 v_1 \mathcal{P}(n-1, 2)$, *for every* $n \geq 2$, *where the initial conditions* $\mathbf{V}_0$ *and* $\mathbf{V}_1$ *are computed from the definition* (1.4), *and*

$$\mathcal{P}(n, 2) = \rho(n, 2) + \rho(n+1, 2)i + \rho(n+2, 2)j + \rho(n+3, 2)k, \quad \textit{for all} \quad n \geq 1$$

with $\rho(n, 2)$ *are as in* (2.5).

The formulation of Proposition 2.4 is simple, yet it admits numerous applications for providing the combinatorial structure of a class of quaternion sequences such as the quaternion sequences of Pell numbers, h-Pell numbers and Pell-Lucas numbers.

2.3. Applications to Pell, h-Pell and Pell–Lucas quaternion sequences

We now focus on the three fundamental families of quaternion sequences that arise from our general theory: Pell quaternions, h-Pell quaternions, where h is a positive integer, and Pell-Lucas quaternions. Our approach provides a unified framework for analyzing their combinatorial structures.

2.3.1. Pell quaternions

The sequence of Pell numbers $\{p_n\}_{n\geq 0}$ is defined by the recurrence relation $p_n = 2p_{n-1} + p_{n-2}$, for $n \geq 2$, with initial conditions $p_0 = 0, p_1 = 1$. Following (2.4)–(2.5) and Proposition 2.4, the combinatorial expression of the related quaternion Pell sequence $\{\mathbf{P}_n\}_{n\geq 0}$ is given by $\mathbf{P}_n = 2\mathcal{P}(n,2)+\mathcal{P}(n-1,2)$, for $n\geq 2$, with $\mathbf{P}_0 = i+2j+5k$ and $\mathbf{P}_1 = 1+2i+5j+12k$, where

$$\mathcal{P}(n,2)=\rho(n,2)+\rho(n+1,2)i+\rho(n+2,2)j+\rho(n+3,2)k, \quad n\geq 1,$$

with $\rho(n,2)=\sum_{l=0}^{[\frac{n-2}{2}]}\binom{n-2-l}{l}2^{n-2-2l}$, for all $n\geq 2$ and $\rho(1,2)=0$.
Using the recurrence relation satisfied by $\rho(n,2)$, we get

$$\rho(n+2,2)=2\rho(n+1,2)+\rho(n,2), \text{ for } n\geq 1,$$

and following some standard algebraic manipulations, we obtain

$$\mathbf{P}_n=\mathbf{P}_2\rho(n,2)+\mathbf{P}_1\rho(n-1,2), \text{ for all } n\geq 2$$

with $\mathbf{P}_1 = 1+2i+5j+12k$ and $\mathbf{P}_2 = 2+5i+12j+29k$. At this stage we distinguish two cases depending on the parity of $n\geq 3$ (since $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are already calculated).

Case 1. For $n=2m$ $(m\geq 2)$, using appropriate index manipulation and some computational simplification, we obtain

$$\mathbf{P}_{2m}=\mathbf{P}_2+\sum_{l=0}^{m-2}\left[\mathbf{P}_1+\frac{2m-2-l}{m-1-l}\,\mathbf{P}_2\right]\binom{2m-3-l}{l}2^{2m-3-2l}$$

Case 2. For $n=2m+1$ $(m\geq 1)$, a direct computation implies

$$\mathbf{P}_{2m+1}=\sum_{l=0}^{m-1}\left[\mathbf{P}_1+\frac{4m-2-2l}{2m-1-2l}\,\mathbf{P}_2\right]\binom{2m-2-l}{l}2^{2m-2-2l}$$

2.3.2. h-Pell quaternions

Let $h\geq 1$ be a positive integer. The h-Pell sequence $\{p_{h,n}\}_{n\geq 0}$ is defined by $p_{h,n} = 2\,p_{h,n-1}+h\,p_{h,n-2}$, with initial conditions $p_{h,0}=0$, $p_{h,1}=1$. The combinatorial expression of the associated quaternion h-Pell sequence $\{\mathbf{P}_{h,n}\}_{n\geq 0}$ takes a similar form $\mathbf{P}_{h,n}=2\mathcal{P}(n,2)+h\mathcal{P}(n-1,2)$, for all $n\geq 2$, and $\mathbf{P}_{h,0}=i+2j+(4+h)k$, $\mathbf{P}_{h,1}=1+2i+(4+h)j+(8+4h)k$, where $\mathcal{P}(n,2)=\rho(n,2)+\rho(n+1,2)i+\rho(n+2,2)j+\rho(n+3,2)k$, for all $n\geq 1$, with

$$\rho(n,2)=\sum_{l=0}^{[\frac{n-2}{2}]}\binom{n-2-l}{l}2^{n-2-2l}h^l, \text{ for } n\geq 2 \text{ and } \rho(1,2)=0.$$

By considering similar steps as for the quaternion Pell sequence, we obtain

$$\mathbf{P}_{h,n}=\mathbf{P}_{h,2}\,\rho(n,2)+h\mathbf{P}_{h,1}\,\rho(n-1,2), \text{ for all } n\geq 2,$$

with $\mathbf{P}_{h,2} = 2 + (4+h)i + (8+4h)j + (16+12h+h^2)k$. Thereafter, we have the following two cases.
Case 1. For $n = 2m$ $(m \geq 2)$ we have

$$\mathbf{P}_{h,2m} = h^{m-1}\mathbf{P}_{h,2} + \sum_{l=0}^{m-2} \left[h\mathbf{P}_{h,1} + \frac{2m-2-l}{m-1-l}\, \mathbf{P}_{h,2} \right] \binom{2m-3-l}{l} 2^{2m-3-2l} h^l.$$

Case 2. For $n = 2m+1$ $(m \geq 1)$, we get

$$\mathbf{P}_{h,2m+1} = \sum_{l=0}^{m-1} \left[h\mathbf{P}_{h,1} + \frac{4m-2-2l}{2m-1-2l}\, \mathbf{P}_{h,2} \right] \binom{2m-2-l}{l} 2^{2m-2-2l} h^l.$$

2.3.3. Pell–Lucas quaternions

The Pell-Lucas sequence $\{q_n\}_{n\geq 0}$ is defined by the recurrence relation $q_n = 2q_{n-1} + q_{n-2}$, for $n \geq 2$, with initial conditions $q_0 = q_1 = 2$. The combinatorial expression of the associated quaternion Pell-Lucas sequence $\{\mathbf{Q}_n\}_{n\geq 0}$ is $\mathbf{Q}_n = 6\mathcal{P}(n,2) + 2\mathcal{P}(n-1,2)$, for all $n \geq 2$, with $\mathbf{Q}_0 = 2 + 2i + 6j + 14k$, $\mathbf{Q}_1 = 2 + 6i + 14j + 34k$ and

$$\mathcal{P}(n,2) = \rho(n,2) + \rho(n+1,2)i + \rho(n+2,2)j + \rho(n+3,2)k, \text{ for every } n \geq 1.$$

where $\rho(n,2) = \sum_{l=0}^{[\frac{n-2}{2}]} \binom{n-2-l}{l} 2^{n-2-2l}$, for $n \geq 2$ and $\rho(1,2) = 0$. Once again, from the recurrence relation $\rho(n+2,2) = 2\rho(n+1,2) + \rho(n,2)$ and after some computations, we get $\mathbf{Q}_n = \mathbf{Q}_2\ \rho(n,2) + \mathbf{Q}_1\ \rho(n-1,2)$, for all $n \geq 2$, with $\mathbf{Q}_2 = 6 + 14i + 34j + 82k$, and regarding the parity of the integer n, we distinguish the following two cases.
Case 1. For $n = 2m$ $(m \geq 2)$ we have

$$\mathbf{Q}_{2m} = \mathbf{Q}_2 + \sum_{l=0}^{m-2} \left[\mathbf{Q}_1 + \frac{2m-2-l}{m-1-l}\, \mathbf{Q}_2 \right] \binom{2m-3-l}{l} 2^{2m-3-2l},$$

Case 2. For $n = 2m+1$ $(m \geq 1)$ we get

$$\mathbf{Q}_{2m+1} = \sum_{l=0}^{m-1} \left[\mathbf{Q}_1 + \frac{4m-2-2l}{2m-1-2l}\, \mathbf{Q}_2 \right] \binom{2m-2-l}{l} 2^{2m-2-2l}.$$

These results provide a comprehensive framework for studying the combinatorial properties of Pell-type quaternion sequences, revealing their structural similarities and distinctive characteristics.

2.4. Algorithmic implementation

We present here the algorithmic implementation of the combinatorial formulas for the three quaternion sequences. Our implementation focuses on both the theoretical formulas and their practical computation.

Algorithm 1 : Computation of Pell Quaternions *via* combinatorial expression

Require: $n \geq 0$
Ensure: n-th term of the Pell quaternion sequence

$\mathbf{P}_0 \leftarrow i + 2j + 5k$
$\mathbf{P}_1 \leftarrow 1 + 2i + 5j + 12k$
$\mathbf{P}_2 \leftarrow 2 + 5i + 12j + 29k$
if $n \leq 2$ **then**
    result $\leftarrow \mathbf{P}_n$
else
    if $n \bmod 2 = 0$ **then** ▷ Cas n = 2m
        $m \leftarrow n/2$
        result $\leftarrow \mathbf{P}_2$
        for $l \leftarrow 0$ to $m - 2$ **do**
            coef $\leftarrow \binom{2m-3-l}{l} \cdot 2^{2m-3-2l}$
            term $\leftarrow (\mathbf{P}_1 + \frac{2m-2-l}{m-1-l}\mathbf{P}_2)\cdot$ coef
            result $\leftarrow$ result + term
        end for
    else ▷ Cas n = 2m+1
        $m \leftarrow (n-1)/2$
        result $\leftarrow 0$
        for $l \leftarrow 0$ to $m - 1$ **do**
            coef $\leftarrow \binom{2m-2-l}{l} \cdot 2^{2m-2-2l}$
            term $\leftarrow (\mathbf{P}_1 + \frac{4m-2-2l}{2m-1-2l}\mathbf{P}_2)\cdot$ coef
            result $\leftarrow$ result + term
        end for
    end if
end if
return result

Algorithm 2 : Computation of h-Pell Quaternions *via* combinatorial expression

Require: $n \geq 0,\ h \geq 1$
Ensure: n-th term of the h-Pell quaternion sequence

$\mathbf{P}_{h,0} \leftarrow i + 2j + (4+h)k$
$\mathbf{P}_{h,1} \leftarrow 1 + 2i + (4+h)j + (8+4h)k$
$\mathbf{P}_{h,2} \leftarrow 2 + (4+h)i + (8+4h)j + (16+12h+h^2)k$
if $n \leq 2$ **then**
  result $\leftarrow \mathbf{P}_{h,n}$
else
  if $n \bmod 2 = 0$ **then** ▷ Cas n = 2m
    $m \leftarrow n/2$
    result $\leftarrow \mathbf{P}_{h,2} \cdot h^{m-1}$
    for $l \leftarrow 0$ to $m-2$ **do**
      coef $\leftarrow \binom{2m-3-l}{l} \cdot 2^{2m-3-2l} \cdot h^l$
      term $\leftarrow (h\mathbf{P}_{h,1} + \frac{2m-2-l}{m-1-l}\mathbf{P}_{h,2}) \cdot$ coef
      result $\leftarrow$ result + term
    end for
  else ▷ Cas n = 2m+1
    $m \leftarrow (n-1)/2$
    result $\leftarrow 0$
    for $l \leftarrow 0$ to $m-1$ **do**
      coef $\leftarrow \binom{2m-2-l}{l} \cdot 2^{2m-2-2l} \cdot h^l$
      term $\leftarrow (h\mathbf{P}_{h,1} + \frac{4m-2-2l}{2m-1-2l}\mathbf{P}_{h,2}) \cdot$ coef
      result $\leftarrow$ result + term
    end for
  end if
end if
return result

Algorithm 3 : Computation of Pell-Lucas Quaternions via combinatorial expression

Require: $n \geq 0$
Ensure: n-th term of the Pell-Lucas quaternion sequence

$\mathbf{Q}_0 \leftarrow 2 + 2i + 6j + 14k$
$\mathbf{Q}_1 \leftarrow 2 + 6i + 14j + 34k$
$\mathbf{Q}_2 \leftarrow 6 + 14i + 34j + 82k$
if $n \leq 2$ **then**
    result $\leftarrow \mathbf{Q}_n$
else
    if n mod $2 = 0$ **then** ▷ Cas n = 2m
        $m \leftarrow n/2$
        result $\leftarrow \mathbf{Q}_2$
        for $l \leftarrow 0$ to $m-2$ **do**
            coef $\leftarrow \binom{2m-3-l}{l} \cdot 2^{2m-3-2l}$
            term $\leftarrow (\mathbf{Q}_1 + \frac{2m-2-l}{m-1-l}\mathbf{Q}_2)\cdot$ coef
            result $\leftarrow$ result + term
        end for
    else ▷ Cas n = 2m+1
        $m \leftarrow (n-1)/2$
        result $\leftarrow 0$
        for $l \leftarrow 0$ to $m-1$ **do**
            coef $\leftarrow \binom{2m-2-l}{l} \cdot 2^{2m-2-2l}$
            term $\leftarrow (\mathbf{Q}_1 + \frac{4m-2-2l}{2m-1-2l}\mathbf{Q}_2)\cdot$ coef
            result $\leftarrow$ result + term
        end for
    end if
end if
return result

Remark 2.5. The display of quaternions depends on their implementation in a programming language. In other words, when implementing them in a language like Python, C++, or others, it may be useful to include a display function that formats the result into a readable text format.

Example 2.6. Using the Python implementation based on our algorithms, we obtain the following results for $n = 4,\ 5,\ 10,\ 13$ and $h = 3$.

Tables 1, 2, and 3 present the computed values of Pell quaternions Pn, 3-Pell quaternions P3,n, and Pell-Lucas quaternions Qn for selected values of n, illustrating the practical application of our algorithms.

Table 1. Values of the Pell quaternions $\mathbf{P}_n$.

n	1	i	j	k
4	12	29	70	169
5	29	70	169	408
10	2378	5741	13 860	33 461
13	33 461	80 782	195 025	470 832

Table 2. Values of the 3-Pell quaternions $\mathbf{P}_{3,n}$.

n	1	i	j	k
4	20	61	182	547
5	61	182	547	1640
10	14 762	44 287	132 860	398 581
13	398 581	1 195 742	3 587 227	10 761 680

Table 3. Values of the Pell-Lucas quaternions $\mathbf{Q}_n$.

n	1	i	j	k
4	34	82	198	478
5	82	198	478	1154
10	6726	16 238	39 202	94 642
13	94 642	228 486	551 614	1 331 714

3. Quaternions associated to linear recurrence sequences with periodic coefficients (1.2)

The main goal of this section is to establish a combinatorial formula for the generalized quaternion sequences associated to linear recurrence sequences with periodic coefficients (1.2). Our approach is based on exploiting the fact that these sequences are transformed into subsequences with constant coefficients, satisfying the same recurrence relation. In addition, the special case $p = 2$ is provided.

3.1. General setting

Let us consider a sequence $\{v_n\}_{n\geq 0}$ defined by (1.2), with periodic coefficients, namely, $v_{n+r} = a_0(n)v_{n+r-1} + a_1(n)v_{n+r-2} + \cdots + a_{r-1}(n)v_n$, where $a_i(n+p) = a_i(n)$, for all i and period $p \geq 2$. It was established in [16] that the sequence (1.2) generates p sub-sequences $\{v_{hp+m}\}_{h\geq 0}$, $m = 0, 1, \ldots, p-1$, and each satisfying the same recurrence relation with constant coefficients given by

$$v_{(h+r)p+m} = c_0 v_{(h+r-1)p+m} + c_1 v_{(h+r-2)p+m} + \cdots + c_{r-1} v_{hp+m}, \quad h \geq 0, \tag{3.1}$$

where the coefficients c_i $(0 \le i \le r-1)$ are determined by the characteristic polynomial $H(z) = z^r - c_0 z^{r-1} - \ldots - c_{r-1}$ of the matrix product $B = A(p-1)A(p-2)\cdots A(1)A(0)$ of the companion matrices

$$A(n) = \begin{pmatrix} a_0(n) & a_1(n) & \cdots & a_{r-1}(n) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.2}$$

Therefore, as it was shown in Section 2 (using [19]), each sub-sequence $\{v_{hp+m}\}_{h\ge 0}$ admits a combinatorial formula, given under the form

$$v_{hp+m} = \sum_{t=0}^{r-1} \rho(h-t, r) A_{(m,t)}, \tag{3.3}$$

for every $n = hp + m \ge rp$, where $\rho(h, r)$ is expressed as in (2.2) under the form

$$\rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!\cdots k_{r-1}!} c_0^{k_0}\cdots c_{r-1}^{k_{r-1}}, \quad \text{for every} \quad n \ge r+1,$$

with $\rho(r, r) = 1$, $\rho(n, r) = 0$ for $n \le r-1$, and

$$A_{(m,t)} = \sum_{l=0}^{r-1-t} c_{r-1-l}\, v_{(l+t)p+m}. \tag{3.4}$$

Let $\{\mathbf{V}_n\}_{n\ge 0}$ be the generalized quaternion sequence associated with the periodic sequence $\{v_n\}_{n\ge 0}$ given by (1.2). Then, we have

$$\mathbf{V}_n = v_n + v_{n+1} i + v_{n+2} j + v_{n+3} k, \quad \text{for every} \quad n \ge 0. \tag{3.5}$$

Let $\{v_{hp+m}\}_{h\ge 0}$, where $m = 0, 1, \ldots, p-1$, be a sub-sequence of the sequence $\{v_n\}_{n\ge 0}$, and consider the generalized quaternion sequence $\{\mathbf{V}_{hp+m}\}_{h\ge 0}$, given by

$$\mathbf{V}_{hp+m} = v_{hp+m} + v_{hp+m+1} i + v_{hp+m+2} j + v_{hp+m+3} k, \quad \text{for every} \quad h \ge 0.$$

We can show easily that we have the following lemma.

Lemma 3.1. *Under the preceding data, for every $m = 0, 1, \ldots, p-1$, the generalized quaternion sequence $\{\mathbf{V}_{hp+m}\}_{h\ge 0}$ is a sub-sequence of the generalized quaternion sequence $\{\mathbf{V}_n\}_{n\ge 0}$ defined by (3.5). In addition, the sequence $\{\mathbf{V}_{hp+m}\}_{h\ge 0}$ satisfies also the linear recursive relation (3.1).*

Taking into account Lemma 3.1 and applying Theorem 2.2 to each sub-sequence $\{v_{hp+m}\}_{h\ge 0}$ $(m = 0, 1, \ldots, p-1)$ allows us to derive the combinatorial formula for the generalized quaternion sequence $\{\mathbf{V}_n\}_{n\ge 0}$ through those of the quaternion sequences $\{\mathbf{V}_{hp+m}\}_{h\ge 0}$ $(m = 0, 1, \ldots, p-1)$. More precisely, we obtain the following result.

Theorem 3.2. *Let $\{v_n\}_{n\ge 0}$ be the periodic generalized Fibonacci sequence (1.2) of period $p \ge 2$, and $\{\mathbf{V}_n\}_{n\ge 0}$ the generalized quaternion sequence defined by (3.5). Then, for every $n = hp + m \ge rp$ $(m = 0, 1, \ldots, p-1)$,* *the combinatorial formula of the generalized quaternion sequence $\{\mathbf{V}_n\}_{n\geq 0}$ is given as follows. For $0 \leq m \leq p-4$ (with $p \geq 4$) we have*

$$\mathbf{V}_{hp+m} = \sum_{t=0}^{r-1} \rho(h-t,r)\left[A_{(m,t)} + A_{(m+1,t)}i + A_{(m+2,t)}j + A_{(m+3,t)}k\right], \tag{3.6}$$

for $m = p-3$ (with $p \geq 3$) we state

$$\mathbf{V}_{hp+p-3} = \sum_{t=0}^{r-1} \rho(h-t,r)\left[A_{(p-3,t)} + A_{(p-2,t)}i + A_{(p-1,t)}j\right] + \sum_{t=0}^{r-1} \rho(h+1-t,r)A_{(0,t)}k, \tag{3.7}$$

for $m = p-2$ (with $p \geq 2$) we get

$$\mathbf{V}_{hp+p-2} = \sum_{t=0}^{r-1} \rho(h-t,r)\left[A_{(p-2,t)} + A_{(p-1,t)}i\right] + \sum_{t=0}^{r-1} \rho(h+1-t,r)\left[A_{(0,t)}j + A_{(1,t)}k\right], \tag{3.8}$$

and for $m = p-1$, we have the two following cases. If $p \neq 2$ we get

$$\mathbf{V}_{hp+p-1} = \sum_{t=0}^{r-1} \rho(h-t,r)A_{(p-1,t)} + \sum_{t=0}^{r-1} \rho(h+1-t,r)\left[A_{(0,t)}i + A_{(1,t)}j + A_{(2,t)}k\right]. \tag{3.9}$$

and for $p = 2$ we have $\mathbf{V}_{2h+1} = \sum_{t=0}^{r-1} \rho(h-t,r)A_{(1,t)} + \sum_{t=0}^{r-1} \rho(h+1-t,r)\left[A_{(0,t)}i + A_{(1,t)}j\right] + \sum_{t=0}^{r-1} \rho(h+2-t,r)A_{(0,t)}k$, where the $A_{(m,t)}$ are given as in (3.4).

Proof. Let us start with the definition of the quaternion sequence

$$\mathbf{V}_{hp+m} = v_{hp+m} + v_{hp+m+1}i + v_{hp+m+2}j + v_{hp+m+3}k.$$

Similarly to the proof of Theorem 2.2, we consider the formulas (3.3)–(3.4) for each sequence $\{\mathbf{V}_{hp+m}\}_{h\geq 0}$. To this aim, we distinguish four cases according to how indices $m, m+1, m+2, m+3$ are distributed relative to p. For the first case, let suppose that $0 \leq m \leq p-4$. In this case all indices belong to the same h-th block, and we have $v_{hp+m+d} = \sum_{t=0}^{r-1} \rho(h-t,r)A_{(m+d,t)}$, where $d = 0,\ 1,\ 2,\ 3$. Hence, we obtain $\mathbf{V}_{hp+m} = \sum_{t=0}^{r-1} \rho(h-t,r)\left[A_{(m,t)} + A_{(m+1,t)}i + A_{(m+2,t)}j + A_{(m+3,t)}k\right]$, where $A_{(m,t)}$ is given as in (3.4). For the second case we consider $m = p-3$, in this case the numbers $hp+p-3, hp+p-2, hp+p-1$ belong to the h-th block while $(h+1)p$ belongs to the $(h+1)$-th block. Hence, we can state the $v_{(h+1)p} = \sum_{t=0}^{r-1} \rho(h+1-t,r)A_{(0,t)}$, and $v_{hp+p-d} = \sum_{t=0}^{r-1} \rho(h-t,r)A_{(p-d,t)}$, for $d = 1,\ 2,\ 3$. Therefore, we get

$$\mathbf{V}_{hp+p-3} = \sum_{t=0}^{r-1} \rho(h-t,r)\left[A_{(p-3,t)} + A_{(p-2,t)}i + A_{(p-1,t)}j\right] + \sum_{t=0}^{r-1} \rho(h+1-t,r)A_{(0,t)}k.$$

In the third case, we consider $m = p - 2$. Thus, we show that $hp+p-2, hp+p-1$ belong to the h-th block, while $(h+1)p, (h+1)p+1$ belong to the $(h+1)$-th block. Thus, we have $v_{hp+p-d} = \sum_{t=0}^{r-1} \rho(h-t,r)A_{(p-d,t)}$, for $d = 1,\, 2$ and $v_{(h+1)p+d} = \sum_{t=0}^{r-1} \rho(h+1-t,r)A_{(d,t)}$, for $d = 0$. Therefore, we derive that

$$\mathbf{V}_{hp+p-2} = \sum_{t=0}^{r-1} \rho(h-t,r)\left[A_{(p-2,t)} + A_{(p-1,t)}i\right] + \sum_{t=0}^{r-1} \rho(h+1-t,r)\left[A_{(0,t)}j + A_{(1,t)}k\right].$$

Finally, for $m = p-1$ and $p > 2$, we show that only $hp+p-1$ belongs to the h-th block, while $(h+1)p, (h+1)p+1, (h+1)p+2$ belong to the $(h+1)$-th block. Hence, we derive

$$v_{hp+p-1} = \sum_{t=0}^{r-1} \rho(h-t,r)A_{(p-1,t)}, \text{ and } v_{(h+1)p+d} = \sum_{t=0}^{r-1} \rho(h+1-t,r)A_{(d,t)}, \quad \text{for} \quad d = 0,\, 1,\, 2.$$

Then, we have

$$\mathbf{V}_{hp+p-1} = \sum_{t=0}^{r-1} \rho(h-t,r)A_{(p-1,t)} + \sum_{t=0}^{r-1} \rho(h+1-t,r)\left[A_{(0,t)}i + A_{(1,t)}j + A_{(2,t)}k\right].$$

Therefore, the expressions (3.6)–(3.9) are established. □

Following the theoretical development of Theorem 3.2, we now present an algorithmic implementation that translates these mathematical formulas into a practical computational procedure. This algorithm provides a systematic way to compute any term of the periodic generalized Fibonacci quaternion sequence. The implementation handles all four cases presented in the theorem, with special attention to the different formulas required for various values of m. The algorithm is structured to be both mathematically rigorous and computationally efficient, incorporating the calculation of ρ coefficients and the $A_{(m,t)}$ values as essential subroutines. For clarity of presentation, we first provide a formal algorithm in pseudo-code, followed by a concrete implementation in Python that can be used for numerical computations.

Algorithm 4 Implementation of Theorem 3.2 – periodic generalized Fibonacci quaternion sequence

Require:

```
h, m ∈ ℕ (indices where n = hp + m)
p ≥ 2 (period)
r ≥ 2 (order of recurrence)
{c_i}_{i=0}^{r-1} (coefficients)
{v_i}_{i=0}^{rp-1} (initial values)
Ensure: The quaternion 𝕍_{hp+m}
function CALCULATE RHO(h, r, {c_i})
    if h < r then return 0
    else if h = r then return 1
    end if
    Calculate using combinatorial formula with coefficients c_i
    return ρ(h, r)
end function
function CALCULATE A(m, t, {c_i}, {v_i})
    result ← 0
    for l ← 0 to r − 1 − t do
        result ← result + c_{r−1−l} v_{(l+t)p+m}
    end for
    return result
end function
if hp + m < rp then
    return v_{hp+m} + v_{hp+m+1} i + v_{hp+m+2} j + v_{hp+m+3} k
end if
if m ≤ p − 4 then                                          ▷ Formula (3.1)
    result ← 0
    for t ← 0 to r − 1 do
        ρ ← CALCULATE RHO(h − t, r, {c_i})
        A_m ← CALCULATE A(m, t, {c_i}, {v_i})
        A_{m+1} ← CALCULATE A(m + 1, t, {c_i}, {v_i})
        A_{m+2} ← CALCULATE A(m + 2, t, {c_i}, {v_i})
        A_{m+3} ← CALCULATE A(m + 3, t, {c_i}, {v_i})
        result ← result + ρ(A_m + A_{m+1} i + A_{m+2} j + A_{m+3} k)
    end for
else if m = p − 3 then                                     ▷ Formula (3.2)
    Similar calculation using the appropriate formula
else if m = p − 2 then                                     ▷ Formula (3.3)
    Similar calculation using the appropriate formula
else if m = p − 1 then                                     ▷ Formula (3.4)
    Similar calculation using the appropriate formula
end if
return result
```

Remark 3.3. For brevity, the detailed calculations for cases $m = p - 3$, $m = p - 2$, and $m = p - 1$ are omitted as they follow a similar pattern to the first case but with different index combinations as specified in Theorem 3.2. The complete implementation includes all cases as given in equations (3.6)–(3.9).

```python
import math from dataclasses import dataclass

@dataclass
class Quaternion:
    """Represents a quaternion number a + bi + cj + dk"""
    a: float  # real part
    b: float  # coefficient of i
    c: float  # coefficient of j
    d: float  # coefficient of k

    def __str__(self):
        return f"{self.a:.6f}{self.b:+.6f}i{self.c:+.6f}j{self.d:+.6f}k"

def calculate_rho(h, r, coefficients):
    """Calculate rho(h,r) for given h, r and coefficients"""
    if h < r:
        return 0
    if h == r:
        return 1
    # Implementation of combinatorial formula
    return rho_value

def periodic_quaternion_sequence(h, m, p, r, coefficients, initial_values):
    """Calculate the n-th term (n = hp + m) of the sequence"""
    if h * p + m < r * p:
        n = h * p + m
        return Quaternion(
            initial_values[n],
            initial_values[n + 1],
            initial_values[n + 2],
            initial_values[n + 3]
        )

    # Implementation of cases from Theorem 3.2
    # Case m <= p-4, Formula (3.1)
    if m <= p - 4:
        # Implementation
        pass
    elif m == p - 3:  # Formula (3.2)
        # Implementation
        pass
    elif m == p - 2:  # Formula (3.3)
        # Implementation
        pass
    elif m == p - 1:  # Formula (3.4)
        # Implementation
        pass
```

Listing 1. Python implementation of Theorem 3.2

To our best knowledge, results of Theorem 3.2 are not current in the literature. To illustrate the efficiency of the results in this subsection, we consider the special case where $r = 2$ and $p = 2$. This case is very rich and owns interesting applications.

3.2. Study of the special case $r = p = 2$

Let us consider the periodic sequence $\{v_n\}_{n\geq 0}$ defined by

$$v_{n+2} = a_0(n)v_{n+1} + a_1(n)v_n, \text{ for every } n \geq 0, \tag{3.10}$$

such that $a_0(n+2) = a_0(n)$ and $a_1(n+2) = a_1(n)$. The related matrices defined by (3.2) are $A(0) = \begin{pmatrix} a_0(0) & a_1(0) \\ 1 & 0 \end{pmatrix}$ and $A(1) = \begin{pmatrix} a_0(1) & a_1(1) \\ 1 & 0 \end{pmatrix}$. Therefore, the matrix product $B = A(1)A(0)$ is given by

$$B = \begin{pmatrix} a_0(1)a_0(0) + a_1(1) & a_0(1)a_1(0) \\ a_0(0) & a_1(0) \end{pmatrix},$$

whose characteristic polynomial is $P_B(z) = z^2 - c_0 z - c_1$, where

$$c_0 = a_0(1)a_0(0) + a_1(1) + a_1(0) \text{ and } c_1 = -a_1(0)a_1(1).$$

Hence, we get $\rho(n,2) = \sum_{k=0}^{\left[\frac{n-2}{2}\right]} \binom{n-2-k}{k} c_0^{n-2-2k}\, c_1^k$. We use Theorem 3.2 to provide explicit combinatorial expressions for the generalized quaternion sequence $\{\mathbb{V}_n\}_{n\geq 0}$ defined as in (3.5), associated with $\{v_n\}_{n\geq 0}$, for $n = 2h + m$, where $m = 0, 1$. That is, a straightforward calculations allows us to obtain

$$\begin{aligned} \mathbb{V}_{2h} = {}& \rho(h-1,2)\left[A_{(0,1)} + A_{(1,1)}i\right] + \rho(h,2)\left[A_{(0,0)} + A_{(1,0)}i + A_{(0,1)}j + A_{(1,1)}k\right] \\ &+ \rho(h+1,2)\left[A_{(0,0)}j + A_{(1,0)}k\right], \end{aligned}$$

$$\begin{aligned} \mathbb{V}_{2h+1} = {}& \rho(h-1,2)A_{(1,1)} + \rho(h,2)\left[A_{(1,0)} + A_{(0,1)}i + A_{(1,1)}j\right] \\ &+ \rho(h+1,2)\left[A_{(0,0)}i + A_{(1,0)}j + A_{(0,1)}k\right] + \rho(h+2,2)A_{(0,0)}k, \end{aligned}$$

with

$$\begin{cases} A_{(0,0)} = c_1 v_0 + c_0 v_2, & A_{(0,1)} = c_1 v_2, \\ A_{(1,0)} = c_1 v_1 + c_0 v_3, & A_{(1,1)} = c_1 v_3. \end{cases}$$

For illustrative purpose, let us consider the following numerical example.

Example 3.4 (Alternating coefficients)**.** Consider the periodic sequence $\{v_n\}_{n\geq 0}$ defined by (3.10), with

$$\begin{cases} a_0(2k) = 2, & a_0(2k+1) = -1, \\ a_1(2k) = 1, & a_1(2k+1) = -\dfrac{1}{2}. \end{cases}$$

Hence, the related matrices $A(0)$, $A(1)$ defined by (3.2) are given by $A(0) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $A(1) = \begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$. Then, the matrix product $B = A(1)A(0)$ is given as $B = \begin{pmatrix} -\frac{5}{2} & -1 \\ 2 & 1 \end{pmatrix}$. Therefore, the characteristic polynomial of the matrix B is $P_B(z) = z^2 + \frac{3}{2}z - \frac{1}{2}$, which means that $c_0 = -\frac{3}{2}$ and $c_1 = \frac{1}{2}$. Hence, the expression of (2.5) takes the form

$$\rho(h,2)=\sum_{k=0}^{[\frac{h-2}{2}]}\binom{h-2-k}{k}(-\frac{3}{2})^{h-2-2k}(\frac{1}{2})^k.$$

Let us consider the initial conditions $v_0 = 1$ and $v_1 = 1$. In this case we need to exhibit the formulas of $A_{(i,s)}$ $(0 \le i, s \le 1\ )$. To this aim we need to calculate v_2 and v_3 using the periodic recurrence. A direct computation shows that for $n = 0$ we have $v_2 = a_0(0)v_1 + a_1(0)v_0 = 2 \cdot 1 + 1 \cdot 1 = 3$, and for $n = 1$ we get $v_3 = a_0(1)v_2 + a_1(1)v_1 = -1 \cdot 3 - \frac{1}{2} \cdot 1 = -\frac{7}{2}$. Therefore, a straightforward calculations leads to

$$A_{(0,0)}=\frac{1}{2}v_0-\frac{3}{2}v_2=\frac{1}{2}(1)-\frac{3}{2}(3)=-4,\ A_{(0,1)}=\frac{1}{2}v_2=\frac{1}{2}(3)=\frac{3}{2},$$
$$A_{(1,0)}=\frac{1}{2}v_1-\frac{3}{2}v_3=\frac{1}{2}(1)-\frac{3}{2}(-\frac{7}{2})=\frac{21}{4}+\frac{1}{2}=\frac{23}{4},\ A_{(1,1)}=\frac{1}{2}v_3=\frac{1}{2}(-\frac{7}{2})=-\frac{7}{4}.$$

Substituting these values into the expressions of $\mathbb{V}_{2h}$ and $\mathbb{V}_{2h+1}$, of the generalized quaternion sequence $\{\mathbb{V}_n\}_{n\ge0}$ defined by (3.5), permits us to show that

$$\mathbb{V}_{2h}=\rho(h-1,2)\left[\frac{3}{2}-\frac{7}{4}i\right]+\rho(h,2)\left[-4+\frac{23}{4}i+\frac{3}{2}j-\frac{7}{4}k\right]+\rho(h+1,2)\left[-4j+\frac{23}{4}k\right],$$

for $h > 1$ and

$$\begin{aligned}\mathbb{V}_{2h+1}=&-\frac{7}{4}\rho(h-1,2)+\rho(h,2)\left[\frac{23}{4}+\frac{3}{2}i-\frac{7}{4}j\right]+\rho(h+1,2)\left[-4i+\frac{23}{4}j+\frac{3}{2}k\right]\\&-4\rho(h+2,2)k,\end{aligned}$$

where $\rho(1,2) = 0$, $\rho(2,2) = 1$, and $\rho(h,2) = \sum_{k=0}^{[\frac{h-2}{2}]}\binom{h-2-k}{k}(-\frac{3}{2})^{h-2-2k}(\frac{1}{2})^k$, for $h \ge 3$.

3.3. Study of the special case $p = 3$

This interesting case will be illustrated through a numerical example. Indeed, we are interested in the following numerical example of the generalized quaternion sequence $\{\mathbb{V}_n\}_{n\ge0}$, related to the periodic sequence $\{v_n\}_{n\ge0}$ of order $r = 3$ and period $p = 3$, defined by

$$v_{n+3}=a_0(n)v_{n+2}+a_1(n)v_{n+1}+a_2(n)v_n, \tag{3.11}$$

of initial conditions $v_0 = 1, v_1 = 1, v_3 = 1$. For reason of conciseness we consider the periodic case defined by

$$\begin{cases}a_0(3k)=1, & a_0(3k+1)=0, & a_0(3k+2)=-1,\\ a_1(3k)=0, & a_1(3k+1)=1, & a_1(3k+2)=0,\\ a_2(3k)=1, & a_2(3k+1)=0, & a_2(3k+2)=1.\end{cases}$$

Thus, the related matrices $A(0)$, $A(1)$ and $A(2)$ defined by (3.2) are given by

$$A(0)=\begin{pmatrix}1&0&1\\1&0&0\\0&1&0\end{pmatrix},\quad A(1)=\begin{pmatrix}0&1&0\\1&0&0\\0&1&0\end{pmatrix},\quad A(2)=\begin{pmatrix}-1&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

Therefore, a direct computation shows that the characteristic polynomial of the related matrix $B = A(2)A(1)A(0)$ is $P(z) = z^3 - c_0 z^2 - c_1 z - c_2$, where $c_0 = 1, c_1 = 0, c_2 = 0$. Therefore, we have $\rho(2,3) = \rho(1,3) = 0$ and $\rho(n,3) = 1$ for $n \geq 3$. On the other side, a direct computation using Expression (3.11) shows that $v_3 = 2, v_4 = 1, v_5 = 0, v_6 = 1, v_7 = 0, v_8 = 0$. Thus, we have $A_{(0,0)} = 1, A_{(1,0)} = 0, A_{(2,0)} = 0$ and $A_{(m,t)} = 0$ for $m = 0, 1, 2$ and $t = 1, 2$. Thence, the quaternion sequence $\{\mathbb{V}_n\}_{n\geq 0}$ related to the periodic sequence $\{v_n\}_{n\geq 0}$ defined by (3.11) takes the forms

$$\mathbb{V}_{3h} = 1 + k, \qquad \mathbb{V}_{3h+1} = j, \qquad \mathbb{V}_{3h+2} = i, \quad \text{for} \quad h \geq 2.$$

4. Concluding remarks and perspective

In the present paper, we establish some results on the generalized Fibonacci quaternions sequences with constant or periodic coefficients. That is, the classical theory of the Fibonacci quaternion sequences have been extended, and a unified framework that encompasses both constant and periodic cases is exhibited. In addition, some explicit combinatorial formulas for these sequences are established. Especially, attention is focused on the generalized quaternion sequences related to sequences defined by linear recursive sequences of order $r = 2$ and of periodic coefficients.

The results are not in the literature under this form, yet. Indeed, several existing results of the current literature on this topic are generalized, which allows us to have new insights into the structure of the generalized linear recursive quaternion sequences. In addition, our approach can be extended to other class of sequences of usual numbers, such that the generalized Fibonacci or Pell numbers of order $r \geq 2$.

Finally, regarding the algorithms considered in this paper, the Python code is available upon request from the reader.

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