RAIRO-Theor. Inf. Appl. 60 (2026) 12
https://doi.org/10.1051/ita/2026013

RAIRO - Theoretical Informatics and Applications
www.rairo-ita.org

DIOPHANTINE APPROXIMATION WITH TWO SQUARES AND THREE BIQUADRATES OF PRIMES

Yu Fu, Liqun Hu*, Siqi Liu and Li Yang

Abstract. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be nonzero real numbers, neither all positive nor all negative and λ_1/λ_2 be an irrational number. Let $\mathcal{V}$ be a well-spaced sequence and $\delta > 0$. For arbitrary $\varepsilon > 0$, we show that the quantity of $\mathbf{v} \in \mathcal{V}$ with $v \leq N$ making the inequality

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^4 + \lambda_4 p_4^4 + \lambda_5 p_5^4 - v\right| < v^{-\delta}$$

unsolvable in primes p_1, p_2, p_3, p_4, p_5 does not exceed $O(N^{\frac{6}{7}+2\delta+\varepsilon})$.

Mathematics Subject Classification. 11D75, 11P32.

Received January 14, 2026. Accepted February 26, 2026.

1. Introduction

Let $R_s(N)$ stand for the number of representations of the positive integer n as the sum of two squares and s biquadrates. In 1981, Hooley [1] established the standard expected asymptotic formula for $R_s(N)$ when $s \geq 5$. For the cases $s \leq 4$, however, suitable tools or approaches to derive the anticipated asymptotic formula for $R_s(N)$ are still lacking. We define $E_s(N)$ as the number of positive integers $n \leq N$ for which the expected asymptotic formula for $R_s(N)$ breaks down. In 2014, Friedlander and Wooley [2] showed

$$E_3(N) \ll N^{\frac{1}{2}+\varepsilon} \quad \text{and} \quad E_4(N) \ll N^{\frac{1}{4}+\varepsilon}.$$

Before long, Zhao [3] enhanced the results by

$$E_3(N) \ll N^{\frac{3}{8}+\varepsilon} \quad \text{and} \quad E_4(N) \ll N^{\frac{1}{8}+\varepsilon}.$$

In 2020, Zhu [4] considered the corresponding Waring-Goldbach problem, *i.e.*, every large enough integer n satisfying some necessary conditions can be represented as

$$n = p_1^2 + p_2^2 + p_3^4 + p_4^4 + p_5^4. \tag{1.1}$$

Keywords and phrases: Exceptional set, Sieve method, Diophantine inequality, prime.

Nanchang, Jiangxi 330031, P.R. China.

* Corresponding author: huliqun@ncu.edu.cn

Nevertheless, the resolution of this problem remains beyond current reach. Therefore, Zhu [4] established the exceptional set of this problem. He proved $E(N) \ll N^{\frac{15}{32}+\varepsilon}$, where $E(N)$ counts the positive integers $n \leq N$ that satisfy the necessary conditions yet fail to satisfy (1.1).

In this paper, we consider the Diophantine inequality based on such types of problem. We begin by introducing the definition of a well-spaced sequence: an increasing sequence of positive real numbers $v_1 < v_2 < \cdots$ is well-spaced if there exist constants $\mathfrak{C} > \mathfrak{c} > 0$ with

$$0 < \mathfrak{c} < v_{i+1} - v_i < \mathfrak{C}, \ \ i = 1, 2, \ldots .$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be nonzero real numbers, neither all positive nor all negative and the ratio of at least two numbers be irrational. In 2025, Liu, Zhang and Li [5] proved for arbitrary $\varepsilon > 0$, the quantity of $v \in \mathcal{V}$ with $v \leq N$ making the inequality

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^4 + \lambda_4 p_4^4 + \lambda_5 p_5^4 - v\right| < v^{-\delta} \tag{1.2}$$

unsolvable in prime variables p_1, p_2, p_3, p_4, p_5 does not exceed $O(N^{\frac{29}{32}+2\delta+\varepsilon})$. At present, we consider the inequality (1.2) again by sieve method and refine the result. We prove the following theorem.

Theorem 1.1 Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be nonzero real numbers, neither all positive nor all negative and the ratio of λ_1/λ_2 be irrational. Let $\delta > 0$ and $\mathcal{V}$ be a well-spaced sequence. Then, for arbitrary $\varepsilon > 0$, the quantity $E(\mathcal{V}, N, \delta)$ of $v \in \mathcal{V}$ with $v \leq N$ making the inequality

$$\left|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^4 + \lambda_4 p_4^4 + \lambda_5 p_5^4 - v\right| < v^{-\delta}$$

unsolvable in primes $p_1, \ldots, p_5$ does not exceed $O(N^{\frac{6}{7}+2\delta+\varepsilon})$.

Remark. In contrast to the results of Liu, Zhang and Li [5], we find that $\frac{6}{7} < \frac{29}{32}$. As an exceptional set, we improve the previous result.

In order to demonstrate Theorem 1.1, we merge the circle approach from Davenport-Heilbronn [6] with the sieve technique. We pay attention to the exceptional scenarios, employing a refined approximation of the integral based on the efficient utilization of Hölder's inequality, similarly in [7].

The paper is structured into several sections: Section 2 describes the preliminary knowledge for the demonstration of Theorem 1.1. Sections 3-5 partition the entire interval into three parts and respectively analyze the major arc, the minor arc and the trivial arc. Ultimately, Section 6 concludes the demonstration of Theorem 1.1.

Notations. To keep the notation uniform, the letter p (regardless of whether there is a subscript or not) denotes a prime number, $\varepsilon > 0$ stands for a small enough number and its value may change in different situations. We write $\mathrm{e}^{2\pi i x} = e(x)$ and substitute L to $\log X$.

2. PRELIMINARIES

Let $0 < \sigma < 1$. In the following sections' calculation, we need to take $X = q^{\frac{7}{3}}$ and $\sigma = X^{-\delta}$ for $\delta > 0$, where q is the denominator of a/q satisfying a/q converges to λ_1/λ_2 as $q \to \infty$. Denote

$$K_\sigma(\alpha) = \left(\frac{\sin \pi\sigma\alpha}{\pi\alpha}\right)^2, \ \alpha \neq 0.$$

For continuity, we let $K_\sigma(0) = \sigma^2$. Trivially, we get

$$K_\sigma(\alpha) \ll \min(\sigma^2, |\alpha|^{-2}) \tag{2.1}$$

and

$$\hat{K}_\sigma(t) = \int_{\mathbb{R}} K_\sigma(\alpha) e(t\alpha) \mathrm{d}\alpha = \max(0, \sigma - |t|). \tag{2.2}$$

Let $\eta > 0$ be a fixed constant, taken sufficiently small. Define

$$I_1 = \left[\left(\frac{\eta X}{\lambda_1} \right)^{\frac{1}{2}}, \left(\frac{2\eta X}{\lambda_1} \right)^{\frac{1}{2}} \right], \ I_2 = \left[\left(\frac{\eta X}{\lambda_2} \right)^{\frac{1}{2}}, \left(\frac{2\eta X}{\lambda_2} \right)^{\frac{1}{2}} \right], \ I_j = \left[\left(\frac{\eta X}{\lambda_j} \right)^{\frac{1}{4}}, \left(\frac{2\eta X}{\lambda_j} \right)^{\frac{1}{4}} \right],$$

for $j = 3, 4, 5$ and $\Omega = I_1 \times \cdots \times I_5$. Next, we introduce the sieve function $\varrho^-(m)$ and $\varrho^+(m)$ from Harman and Kumchev [8] (also in [9], Sect. 8). Write

$$\phi(m, z) = \begin{cases} 1, & \text{if } p \mid m \Rightarrow p \geq z, \\ 0, & \text{otherwise.} \end{cases}$$

For the function $\varrho^+(m)$, the relevant form is given by

$$\varrho^+(m) = \phi(m, X^{5/42}) - \sum_{X^{1/7} \leq p \leq X^{3/14}} \phi(m/p, \ p),$$

and the function $\varrho^-(m)$ is represented by the form

$$\varrho^-(m) = \phi(m, X^{5/42}) - \sum_{X^{5/42} \leq p \leq X^{1/4}} \phi(m/p, \ z(p)),$$

$$z(p) = \begin{cases} X^{5/28} p^{-1/2}, & \text{if } p < X^{1/7}, \\ p, & \text{if } X^{1/7} \leq p \leq X^{3/14}, \\ X^{5/14} p^{-1}, & \text{if } p > X^{3/14}. \end{cases}$$

The characteristic function of primes, $\varrho_0(m)$, satisfies

$$\varrho^-(m) \leq \varrho_0(m) \leq \varrho^+(m)$$

by the construction of $\varrho^\pm(m)$. Let I' be an interval satisfying $I' \subseteq I_i$, for $i = 1, 2$. Therefore, we have

$$\sum_{m \in I'} \varrho^\pm(m) = \kappa \, |I'| \, L^{-1} + O(X^{\frac{1}{2}} L^{-2}), \tag{2.3}$$

where

$$\kappa^- > 0.9, \ \kappa^+ < 1.7$$

are both absolute constants. Therefore, the vector sieve provides

$$\varrho_0(m_1)\varrho_0(m_2) \geq \varrho^-(m_1)\varrho^+(m_2) + \varrho^+(m_1)\varrho^-(m_2) - \varrho^+(m_1)\varrho^+(m_2). \tag{2.4}$$

Moreover, we write

$$\begin{aligned} S_1(\alpha,\varrho) &= \sum_{m\in I_1} \varrho(m)e(m^2\alpha), \\ S_2(\alpha,\varrho) &= \sum_{m\in I_2} \varrho(m)e(m^2\alpha), \\ S_j(\alpha) &= \sum_{p\in I_j} \log p e(p^4\alpha), j=3,4,5. \end{aligned}$$

Define $\wp \subseteq \mathbb{R}$ be any measurable subset, let

$$I(\sigma,v,\wp,\varrho_1,\varrho_2) = \int_{\wp} S_1(\lambda_1\alpha,\varrho_1)S_2(\lambda_2\alpha,\varrho_2)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha.$$

Therefore, we set $i=1,2$ and $j=3,4,5$,

$$\begin{aligned} &I(\sigma,v,\mathbb{R},\varrho_0,\varrho_0) \\ =&\sum_{m_i\in I_i,\ p_j\in I_j} \varrho_0(m_1)\varrho_0(m_2)\log p_3\log p_4\log p_5 \\ &\times \int_{\mathbb{R}} K_\sigma(\alpha)e((\lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^4+\lambda_4p_4^4+\lambda_5p_5^4-v)\alpha)\mathrm{d}\alpha \\ =&\sum_{m_i\in I_i,\ p_j\in I_j} \varrho_0(m_1)\varrho_0(m_2)\log p_3\log p_4\log p_5 \\ &\times \max(0,\sigma-\left|\lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^4+\lambda_4p_4^4+\lambda_5p_5^4-v\right|) \\ \le&\sigma L^5\mathfrak{N}(x), \end{aligned}$$

where $\mathfrak{N}(x)$ represents the quantity of solutions to

$$\left|\lambda_1p_1^2+\lambda_2p_2^2+\lambda_3p_3^4+\lambda_4p_4^4+\lambda_5p_5^4-v\right| < \sigma$$

with $(p_1,p_2,p_3,p_4,p_5)\in\Omega$. Recalling (2.4), we can obtain that

$$I(\sigma,v,\mathbb{R},\varrho_0,\varrho_0) \ge I(\sigma,v,\mathbb{R},\varrho^-,\varrho^+) + I(\sigma,v,\mathbb{R},\varrho^+,\varrho^-) - I(\sigma,v,\mathbb{R},\varrho^+,\varrho^+). \tag{2.5}$$

Hence, an estimation of the right-hand side of 2.5 is required. By employing a dichotomy of discussion, we concentrate on those v for $\frac{1}{2}X\le v\le X$. Generally, we may consider the dyadic intervals $2^{-j}X\le v\le 2^{1-j}X$ for some natural number j and derive a suitable estimate for the exceptional set. Now, we partition the entire interval into major arc $\mathfrak{M}$, minor arc $\mathfrak{m}$ and trivial arc $\mathfrak{t}$. Let

$$\mathfrak{M} = \left[-\frac{M}{X},\frac{M}{X}\right],\ \ \mathfrak{m} = \left(-G,-\frac{M}{X}\right)\cup\left(\frac{M}{X},G\right),\ \ \mathfrak{t} = \mathbb{R}\backslash(\mathfrak{M}\cup\mathfrak{m}),$$

where $M = X^{\frac{1}{8}-\varepsilon}$, $G = \sigma^{-2}X^{\frac{1}{4}}L^3$. Therefore,

$$I(\sigma,v,\mathbb{R},\varrho_1,\varrho_2) = I(\sigma,v,\mathfrak{M},\varrho_1,\varrho_2) + I(\sigma,v,\mathfrak{m},\varrho_1,\varrho_2) + I(\sigma,v,\mathfrak{t},\varrho_1,\varrho_2).$$

3. The major arc

Now, we deal with the major arc. Firstly, we introduce some functions that will be used in the subsequent proof. Write

$$\begin{aligned} T_i(\alpha) &= \int_{I_i} e(t^2\alpha)\mathrm{d}t, i=1,2, \\ T_j(\alpha) &= \int_{I_j} e(t^4\alpha)\mathrm{d}t,\ j=3,4,5. \end{aligned}$$

Based on the estimate for trigonometric integrals (see [10]), we get

$$\begin{aligned} S_1(\alpha,\varrho^-) &\ll X^{\frac{1}{2}},\ T_1(\alpha) \ll X^{\frac{1}{2}-1}\min(X,|\alpha|^{-1}), \\ S_2(\alpha,\varrho^+) &\ll X^{\frac{1}{2}},\ T_2(\alpha) \ll X^{\frac{1}{2}-1}\min(X,|\alpha|^{-1}), \\ S_j(\alpha) &\ll X^{\frac{1}{4}},\ T_j(\alpha) \ll X^{\frac{1}{4}-1}\min(X,|\alpha|^{-1}),\ j=3,4,5. \end{aligned} \tag{3.1}$$

Then, we will pay attention to $I(\sigma, v, \mathfrak{M}, \varrho^-, \varrho^+)$ because the approach to estimate $I(\sigma, v, \mathfrak{M}, \varrho^+, \varrho^-)$ and $I(\sigma, v, \mathfrak{M}, \varrho^+, \varrho^+)$ is similar. Let $\Upsilon = X^{-1+\frac{5}{48}-\varepsilon}$. Firstly, We examine the region $\mathfrak{M}^{\Xi} \subseteq \mathfrak{M}$ defined by $\mathfrak{M}^{\Xi} = \{\alpha : |\alpha| \le \Upsilon\}$. Therefore,

$$\begin{aligned} & I(\sigma, v, \mathfrak{M}^{\Xi}, \varrho^-, \varrho^+) \\ =& \int_{\mathfrak{M}^{\Xi}} S_1(\lambda_1\alpha,\varrho^-)S_2(\lambda_2\alpha,\varrho^+)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ =& \frac{\kappa^-\kappa^+}{L^2}\int_{\mathfrak{M}^{\Xi}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ &+ \frac{\kappa^+}{L}\int_{\mathfrak{M}^{\Xi}} (S_1(\lambda_1\alpha,\varrho^-) - \frac{\kappa^-}{L}T_1(\lambda_1\alpha))T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ &+ \int_{\mathfrak{M}^{\Xi}} S_1(\lambda_1\alpha,\varrho^-)(S_2(\lambda_2\alpha,\varrho^+) - \frac{\kappa^+}{L}T_2(\lambda_2\alpha))T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ &+ \int_{\mathfrak{M}^{\Xi}} S_1(\lambda_1\alpha,\varrho^-)S_2(\lambda_2\alpha,\varrho^+)(S_3(\lambda_3\alpha) - T_3(\lambda_3\alpha))S_4(\lambda_4\ alpha)S_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ &+ \int_{\mathfrak{M}^{\Xi}} S_1(\lambda_1\alpha,\varrho^-)S_2(\lambda_2\alpha,\varrho^+)S_3(\lambda_3\alpha)(S_4(\lambda_4\alpha) - T_4(\lambda_4\alpha))S_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ &+ \int_{\mathfrak{M}^{\Xi}} S_1(\lambda_1\alpha,\varrho^-)S_2(\lambda_2\alpha,\varrho^+)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)(S_5(\lambda_5\alpha) - T_5(\lambda_5\alpha))K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ =& J_1 + J_2 + \cdots + J_6. \end{aligned} \tag{3.2}$$

Subsequently, we will demonstrate that $J_1 \gg \sigma^2 L^{-2} X^{\frac{3}{4}}$ and $J_i = o(\sigma^2 L^{-2} X^{\frac{3}{4}})$ for $i = 2, \ldots, 6$. Since the estimates for J_5, J_6 are similar to J_4, so we restrict our attention to estimate $J_1, \ldots, J_4$.

We first compute the lower bound for J_1.

$$\begin{aligned} J_1 &= \kappa^-\kappa^+L^{-2}\int_{\mathfrak{M}^{\Xi}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ &= \kappa^-\kappa^+L^{-2}\int_{\mathbb{R}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \\ &\quad + O\bigg(L^{-2}\int_{\Upsilon}^{\infty} |T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_5(\lambda_5\alpha)|\, K_\sigma(\alpha)\mathrm{d}\alpha\bigg). \end{aligned} \tag{3.3}$$

Combining (1.2), (3.1) with the foregoing result yields the satisfaction of the error term in (3.3) by

$$\ll \sigma^2 L^{-2} X^{-\frac{13}{4}} \int_{\Upsilon}^{+\infty} \frac{\mathrm{d}\alpha}{\alpha^5} \ll \sigma^2 L^{-2} X^{\frac{3}{4}} X^{-\frac{5}{12}} = o(\sigma^2 L^{-2} X^{\frac{3}{4}}). \tag{3.4}$$

We write

$$\begin{aligned} f(v) &= \int_{\mathbb{R}} T_1(\lambda_1\alpha) T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha) K_\sigma(\alpha) e(-v\alpha) \mathrm{d}\alpha \\ &= \int \cdots \int_{\Omega} \max(0, \sigma - \left|\lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^4 + \lambda_4 t_4^4 + \lambda_5 t_5^4 - v\right|) \mathrm{d}t_1 \ldots \mathrm{d}t_5. \end{aligned} \tag{3.5}$$

Thus, we can get

$$J_1 = \kappa^- \kappa^+ L^{-2} f(v) + o(\sigma^2 L^{-2} X^{\frac{3}{4}}).$$

Then by (2.3), we also have

$$S_1(\lambda_1\alpha, \varrho^-) - \kappa^- L^{-1} T_1(\lambda_1\alpha) \ll X^{\frac{1}{2}} L^{-2} \left(1 + |\alpha|\, X\right).$$

Combining this, we have

$$\begin{aligned} J_2 \,&=\, \kappa^+ L^{-1} \int_{\mathfrak{M}^{\Xi}} (S_1(\lambda_1\alpha, \varrho^-) - \kappa^- L^{-1} T_1(\lambda_1\alpha)) T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha) K_\sigma(\alpha) e(-v\alpha) \mathrm{d}\alpha \\ &\ll \sigma^2 L^{-1} \int_{\mathfrak{M}^{\Xi}} \left|(S_1(\lambda_1\alpha, \varrho^-) - \kappa L^{-1} T_1(\lambda_1\alpha))\right| \left|T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha)\right| \mathrm{d}\alpha \\ &\ll \sigma^2 L^{-1} \int_{\mathfrak{M}^{\Xi}} \left| X^{\frac{1}{2}} L^{-2} \left(1 + |\alpha|\, X\right) \right| \left|T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha)\right| \mathrm{d}\alpha \\ &\ll \sigma^2 L^{-3} X^{\frac{1}{2}} \int_0^{\frac{1}{X}} \left|T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha)\right| \mathrm{d}\alpha \\ &\quad + \sigma^2 L^{-3} X^{\frac{1}{2}} \int_{\frac{1}{X}}^{\frac{L}{X}} \left|T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha)\right| \mathrm{d}\alpha \\ &\quad + \sigma^2 L^{-3} X^{\frac{3}{2}} \int_{\frac{1}{X}}^{\frac{L}{X}} |\alpha| \left|T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha)\right| \mathrm{d}\alpha \\ &\quad + \sigma^2 L^{-3} X^{\frac{1}{2}} \int_{\frac{L}{X}}^{\Upsilon} \left|T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha)\right| \mathrm{d}\alpha \\ &\quad + \sigma^2 L^{-3} X^{\frac{3}{2}} \int_{\frac{L}{X}}^{\Upsilon} |\alpha| \left|T_2(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_5(\lambda_5\alpha)\right| \mathrm{d}\alpha \\ &\ll \sigma^2 L^{-3} X^{\frac{3}{4}} \\ &= o(\sigma^2 L^{-2} X^{\frac{3}{4}}). \end{aligned}$$

Therefore, we get

$$J_2 = o(\sigma^2 L^{-2} X^{\frac{3}{4}}). \tag{3.6}$$

In the same way, we can easily get

$$J_3 = o(\sigma^2 L^{-2} X^{\frac{3}{4}}). \tag{3.7}$$

Following the similar argument in [5], we obtain

$$J_i = o(\sigma^2 L^{-2} X^{\frac{3}{4}}), i = 4, 5, 6. \tag{3.8}$$

Combining (3.2)–(3.8), we get

$$I(\sigma, v, \mathfrak{M}^{\Xi}, \varrho^-, \varrho^+) = \kappa^- \kappa^+ L^{-2} f(v) + o(\sigma^2 L^{-2} X^{\frac{3}{4}}). \tag{3.9}$$

Moreover, in a similar way, we can obtain

$$I(\sigma, v, \mathfrak{M}^{\Xi}, \varrho^+, \varrho^-) = \kappa^- \kappa^+ L^{-2} f(v) + o(\sigma^2 L^{-2} X^{\frac{3}{4}}) \tag{3.10}$$

and

$$I(\sigma, v, \mathfrak{M}^{\Xi}, \varrho^+, \varrho^+) = \kappa^- \kappa^+ L^{-2} f(v) + o(\sigma^2 L^{-2} X^{\frac{3}{4}}). \tag{3.11}$$

By [5], Lemma 3.2, for any $v \in [\frac{X}{2}, X]$, we obtain

$$f(v) \gg \sigma^2 X^{\frac{3}{4}}. \tag{3.12}$$

We write

$$\varDelta(\alpha) = S_1(\lambda_1\alpha, \varrho^-) S_2(\lambda_2\alpha, \varrho^+) + S_1(\lambda_1\alpha, \varrho^+) S_2(\lambda_2\alpha, \varrho^-) - S_1(\lambda_1\alpha, \varrho^+) S_2(\lambda_2\alpha, \varrho^+) \tag{3.13}$$

Thus from (3.9)–(3.13), we obtain

$$\begin{aligned} &\int_{\mathfrak{M}^{\Xi}} \varDelta(\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) S_5(\lambda_5\alpha) K_\sigma(\alpha) e(-v\alpha) \mathrm{d}\alpha \\ &= (\kappa^-\kappa^+ + \kappa^+\kappa^- - \kappa^+\kappa^+) f(v) + o(\sigma^2 L^{-2} X^{\frac{3}{4}}) \\ &\gg \sigma^2 L^{-2} X^{\frac{3}{4}}. \end{aligned} \tag{3.14}$$

According to (2.4) and noting that $\kappa^+\kappa^- + \kappa^-\kappa^+ - \kappa^+\kappa^+ > 0$, we confirm that the coefficient of $f(v)$ in (3.14) is positive. Moreover, what we still need to address is $\mathfrak{M} \setminus \mathfrak{M}^{\Xi}$, before that, we should introduce some Lemmas.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$ and exist $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, \ q) = 1$ and $|\alpha - a/q| < q^{-2}$. Let $k \geq 2$ be an integer. Thus, for arbitrary $\varepsilon > 0$, we have*

$$\sum_{1 \leq p \leq N} (\log p) e(\alpha p^k) \ll N^{1+\varepsilon} (\frac{1}{q} + \frac{1}{N^{1/2}} + \frac{q}{N^k})^{4^{1-k}}. \tag{3.15}$$

Proof. The Lemma is from Harman [11]. □

Corollary 3.2. *Assume that $X^{-1+\frac{1}{2k}-\varepsilon} < \alpha \leq X^{-\frac{3}{4}}$ and $k \geq 4$. Then*

$$S_k(\lambda\alpha) \ll X^{\frac{1}{k} - \frac{4^{1-k}}{2k}}. \tag{3.16}$$

Proof. Taking $q = [\left|\lambda\alpha^{-1}\right|],\ a = 1$ in (3.15), so that (3.16) can be deduced from (3.15). □

Lemma 3.3. *We have*

$$\int_{|\alpha|\le X^{-\frac{2}{3}}} |S_2(\lambda\alpha)|^2 \,\mathrm{d}\alpha \ll 1.$$

Proof. This Lemma is from Ge and Zhao [12]. Although the definition of $S_1(\lambda_1\alpha)$ and $S_2(\lambda_2\alpha)$ exists sieve function $\varrho(m)$, we can also use the similar argument in [12] to get this Lemma. □

Then by Cauchy's inequality, we conclude that

$$\begin{aligned}
&\int_{\mathfrak{M}\setminus\mathfrak{M}^{\Xi}} \left|S_1(\lambda_1\alpha,\varrho^-)S_2(\lambda_2\alpha,\varrho^+)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right| K_\sigma(\alpha)\mathrm{d}\alpha \\
&\ll \sigma^2 |S_3(\lambda_3\alpha)|\,|S_4(\lambda_4\alpha)|\,|S_5(\lambda_5\alpha)| \left(\int_{|\alpha|\le X^{-\frac{2}{3}}} \left|S_1(\lambda_1\alpha,\varrho^-)\right|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{|\alpha|\le X^{-\frac{2}{3}}} \left|S_2(\lambda_2\alpha,\varrho^+)\right|^2 \mathrm{d}\alpha\right)^{\frac{1}{2}} \\
&\ll \sigma^2 X^{\frac{3}{4}-\frac{4-3}{8}\times 3} \\
&= o(\sigma^2 L^{-2} X^{\frac{3}{4}}).
\end{aligned}$$

Therefore, by similar argument, we can get

$$\int_{\mathfrak{M}\setminus\mathfrak{M}^{\Xi}} |\mathit{\Delta}(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)|\, K_\sigma(\alpha)\mathrm{d}\alpha \ll \sigma^2 X^{\frac{3}{4}-\frac{4-3}{8}\times 3} = o(\sigma^2 L^{-2} X^{\frac{3}{4}}). \tag{3.17}$$

Combining (3.14) and (3.17), we have

$$\int_{\mathfrak{M}} \mathit{\Delta}(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha \gg \sigma^2 X^{\frac{3}{4}} L^{-2}. \tag{3.18}$$

4. The minor arc

We now turn to calculate the minor arc. Note that

$$\begin{aligned}
&\int_{\mathfrak{m}} |\mathit{\Delta}(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)|^2\, K_\sigma(\alpha)\mathrm{d}\alpha \\
&\ll \int_{\mathfrak{m}} \left|S_1(\lambda_1\alpha,\varrho^-)S_2(\lambda_2\alpha,\varrho^+)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha \\
&\quad + \int_{\mathfrak{m}} \left|S_1(\lambda_1\alpha,\varrho^+)S_2(\lambda_2\alpha,\varrho^-)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha \\
&\quad + \int_{\mathfrak{m}} \left|S_1(\lambda_1\alpha,\varrho^+)S_2(\lambda_2\alpha,\varrho^+)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha.
\end{aligned}$$

From Li and Wang [13], the sieve functions can be explicitly written as a finite collection of summations of the form

$$\sum_{m=rs} f_r g_s,$$

where either $X^{\frac{1}{7}} \leq r \leq X^{\frac{3}{14}}$ or $f_r \equiv 1$ and $r \geq X^{\frac{1}{7}}$. Otherwise, f_r, g_s are dominated by the divisor function. Generally, we write

$$S_i(\alpha) = \sum_{m \in I_i} a_m e(m^2\alpha),\ i = 1, 2,$$

where a_m is one or the other: $\varrho^+(m)$ or $\varrho^-(m)$. Evidently, it suffices to consider the integral

$$\int_{\mathfrak{m}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)|^2 K_\sigma(\alpha)\mathrm{d}\alpha.$$

We write

$$\mathfrak{m}_1 = \left\{\alpha \in \mathfrak{m} : |S_1(\lambda_1\alpha)| \leq X^{\frac{3}{7}+\frac{\varepsilon}{2}}\right\}$$

and

$$\mathfrak{m}_2 = \left\{\alpha \in \mathfrak{m} : |S_2(\lambda_2\alpha)| \leq X^{\frac{3}{7}+\frac{\varepsilon}{2}}\right\}.$$

Lemma 4.1. *For $j = 1, 2$, we have*

$$\int_{\mathbb{R}} |S_j(\lambda_j\alpha)|^2 |S_3(\lambda_3\alpha)|^4 K_\sigma(\alpha)\mathrm{d}\alpha \ll \sigma X^{1+\varepsilon}.$$

Proof. The Lemma is essentially Ge and Zhao [12], Lemma 2.8 or Qu and Zeng [7], Lemma 6.3. □

Now we estimate the intervals over $\mathfrak{m}_1, \mathfrak{m}_2$. Combining Hua's Lemma, Hölder's inequality with Lemma 4.1, we obtain

$$\begin{aligned}
&\int_{\mathfrak{m}_1} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)|^2 K_\sigma(\alpha)\mathrm{d}\alpha \\
&\ll \left(\sup_{\alpha\in\mathfrak{m}_1} |S_1(\lambda_1\alpha)|\right)^2 \left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)|^2 |S_3(\lambda_3\alpha)|^4 K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)|^4 K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} |S_4(\lambda_4\alpha)|^{16} K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{8}} \\
&\quad \times \left(\int_{\mathbb{R}} |S_5(\lambda_5\alpha)|^{16} K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{8}} \\
&\ll \sigma X^{\frac{6}{7}+\frac{1}{2}+\frac{1}{4}+\frac{3}{8}+\frac{3}{8}+\varepsilon} \\
&\ll \sigma X^{\frac{33}{14}+\varepsilon}.
\end{aligned} \tag{4.1}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathfrak{m}_2} \left|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha \\
&\ll \left(\sup_{\alpha\in\mathfrak{m}_2}|S_2(\lambda_2\alpha)|\right)^2\left(\int_{\mathbb{R}}|S_1(\lambda_1\alpha)|^2|S_3(\lambda_3\alpha)|^4K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{\mathbb{R}}|S_1(\lambda_1\alpha)|^4K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}}|S_4(\lambda_4\alpha)|^{16}K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{8}} \\
&\quad\times\left(\int_{\mathbb{R}}|S_5(\lambda_5\alpha)|^{16}K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{8}} \\
&\ll \sigma X^{\frac{6}{7}+\frac{1}{2}+\frac{1}{4}+\frac{3}{8}+\frac{3}{8}+\varepsilon} \\
&\ll \sigma X^{\frac{33}{14}+\varepsilon}. \qquad (4.2)
\end{aligned}$$

Thus, combining (4.1) and (4.2), we obtain

$$\int_{\mathfrak{m}_i} \left|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha \ll \sigma X^{\frac{33}{14}+\varepsilon}, \qquad (4.3)$$

where $i = 1, 2$.

Write $\mathfrak{m}^* = \mathfrak{m}\setminus(\mathfrak{m}_1\cup\mathfrak{m}_2)$. We will use some specific techniques from Harman [9] to derive the upper bound of the integral $\mathfrak{m}^*$. Obviously, for any $\alpha\in\mathfrak{m}^*$,

$$|S_1(\lambda_1\alpha)| > X^{\frac{3}{7}+\frac{\varepsilon}{2}},\ \ |S_2(\lambda_2\alpha)| > X^{\frac{3}{7}+\frac{\varepsilon}{2}}.$$

We divide $\mathfrak{m}^*$ into disjoint set $S(\mathfrak{T}_1,\mathfrak{T}_2,y)$, such that for $\alpha\in S(\mathfrak{T}_1,\mathfrak{T}_2,y)$, we have

$$\mathfrak{T}_1 < |S_1(\lambda_1\alpha)| \le 2\mathfrak{T}_1,\ \ \mathfrak{T}_2 < |S_2(\lambda_2\alpha)| \le 2\mathfrak{T}_2,\ \ y < |\alpha| \le 2y,$$

where $\mathfrak{T}_1 = 2^{k_1}X^{\frac{3}{7}+\frac{\varepsilon}{2}}$, $\mathfrak{T}_2 = 2^{k_2}X^{\frac{3}{7}+\frac{\varepsilon}{2}}$ and $y = 2^rX^{-\frac{7}{8}-\varepsilon}$, k_1, k_2, r are non-negative integers. Then we should introduce a crucial Lemma:

Lemma 4.2. *Suppose that* $X^{\frac{1}{2}} \ge \mathfrak{T}_i \ge X^{\frac{1}{2}-\frac{1}{14}+\frac{\varepsilon}{2}}$, $|S_i(\lambda_i\alpha)| > \mathfrak{T}_i$, $i = 1, 2$. *Then there exist integers* a_i, q_i *satisfying* $(a_i, q_i) = 1$ *and*

$$1 \le q_i \ll \left(\frac{X^{\frac{1}{2}+\frac{\varepsilon}{4}}}{\mathfrak{T}_i}\right)^4,\ \ |q_i\lambda_i\alpha - a_i| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\frac{\varepsilon}{4}}}{\mathfrak{T}_i}\right)^4.$$

Proof. The proof is similar as [14], Lemma 1. □

We define $a_1a_2\neq 0$, otherwise we assume that there exist $\alpha\in\mathfrak{M}$. In addition, we partition $S(\mathfrak{T}_1,\mathfrak{T}_2,y)$ into sets $S(\mathfrak{T}_1,\mathfrak{T}_2,y,\mathcal{Q}_1,\mathcal{Q}_2)$ with $\mathcal{Q}_j < q_j \le 2\mathcal{Q}_j$. Since

$$a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2 = (q_1\lambda_1\alpha - a_1)\frac{a_2}{\lambda_2\alpha} - (q_2\lambda_2\alpha - a_2)\frac{a_1}{\lambda_2\alpha},$$

then it follows from Lemma 4.2 that

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \ll X^{3+\frac{9}{4}\varepsilon}\mathfrak{T}_1^{-4}\mathfrak{T}_2^{-4} \ll X^{-\frac{3}{7}-\frac{23}{4}\varepsilon}.$$

Recalling that $q = X^{\frac{3}{7}}$. Thus

$$\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| = o(q^{-1}).$$

We also can get

$$|a_2 q_1| \ll y \mathcal{Q}_1 \mathcal{Q}_2.$$

Therefore, if $|a_2 q_1|$ took W different values, we could prove that there exist some n satisfying

$$\left\| n \frac{\lambda_1}{\lambda_2} \right\| \ll X^{-\frac{3}{7} - \frac{23}{4}\varepsilon}, \ \ n \ll \frac{y \mathcal{Q}_1 \mathcal{Q}_2}{W}.$$

If q is taken adequately large, this conflicts with the condition that a/q is a convergent to λ_1/λ_2, unless

$$W \ll \frac{y \mathcal{Q}_1 \mathcal{Q}_2}{q}.$$

Since $d(n) \ll n^{\varepsilon}$ where $d(n)$ is the divisor function, any fixed value of $|a_2 q_1|$ can arise from at most $O(X^{\varepsilon})$ distinct choices of a_2 and q_1. Therefore, each set in $S(\mathfrak{T}_1, \mathfrak{T}_2, y, \mathcal{Q}_1, \mathcal{Q}_2)$ consists of $O(WX^{\varepsilon})$ length intervals

$$\begin{aligned}
&\ll \min\left(\mathcal{Q}_1^{-1} X^{-1+\varepsilon} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{\mathfrak{T}_1}\right)^4, \ \ \mathcal{Q}_2^{-1} X^{-1+\varepsilon} \left(\frac{X^{\frac{1}{2}+\varepsilon}}{\mathfrak{T}_2}\right)^4\right) \\
&\ll \frac{X^{1+\frac{5}{4}\varepsilon}}{\mathfrak{T}_1^2 \mathfrak{T}_2^2 \mathcal{Q}_1^{\frac{1}{2}} \mathcal{Q}_2^{\frac{1}{2}}}.
\end{aligned}$$

Let $\mathfrak{L}$ represent the set $S(\mathfrak{T}_1, \mathfrak{T}_2, y, \mathcal{Q}_1, \mathcal{Q}_2)$. Note that

$$\begin{aligned}
\int_{\mathfrak{L}} \mathrm{d}\alpha &\ll y q^{-1} \mathcal{Q}_1 \mathcal{Q}_2 \frac{X^{1+\frac{5}{4}\varepsilon}}{\mathfrak{T}_1^2 \mathfrak{T}_2^2 \mathcal{Q}_1^{\frac{1}{2}} \mathcal{Q}_2^{\frac{1}{2}}} \\
&\ll \frac{y q^{-1} X^{1+\frac{5}{4}\varepsilon} \mathcal{Q}_1^{\frac{1}{2}} \mathcal{Q}_2^{\frac{1}{2}}}{\mathfrak{T}_1^2 \mathfrak{T}_2^2}.
\end{aligned}$$

Integrating over the set $\mathfrak{L}$, we obtain

$$\begin{aligned}
&\int_{\mathfrak{L}} \left|S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) S_5(\lambda_5\alpha)\right|^2 K_{\sigma}(\alpha) \mathrm{d}\alpha \\
&\ll \min(\sigma^2, y^{-2}) \mathfrak{T}_1^2 \mathfrak{T}_2^2 X^{\frac{3}{2}} \left(\int_{\mathfrak{L}} \mathrm{d}\alpha\right) \\
&\ll \sigma y^{-1} \mathfrak{T}_1^2 \mathfrak{T}_2^2 X^{\frac{3}{2}} \frac{y q^{-1} X^{1+\frac{5}{4}\varepsilon} \mathcal{Q}_1^{\frac{1}{2}} \mathcal{Q}_2^{\frac{1}{2}}}{\mathfrak{T}_1^2 \mathfrak{T}_2^2}.
\end{aligned}$$

Recalling that

$$q = X^{\frac{3}{7}}, \ \ \mathcal{Q}_1 \ll \left(\frac{X^{\frac{1}{2}+\frac{\varepsilon}{4}}}{\mathfrak{T}_1}\right)^4, \ \ \mathcal{Q}_2 \ll \left(\frac{X^{\frac{1}{2}+\frac{\varepsilon}{4}}}{\mathfrak{T}_2}\right)^4,$$

we can confirm that

$$\int_{\mathfrak{L}} \left|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha \ll \sigma X^{3+\frac{3}{2}-\frac{15}{7}+\frac{\varepsilon}{4}}.$$

Hence, we can get

$$\begin{aligned}\int_{\mathfrak{m}^*} \left|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha &\ll (\log X)^5\sigma X^{3+\frac{3}{2}-\frac{15}{7}+\frac{\varepsilon}{4}} \\ &\ll \sigma X^{\frac{33}{14}+\varepsilon}. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\int_{\mathfrak{m}} \left|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha \ll \sigma X^{\frac{33}{14}+\varepsilon}.$$

Finally, we can get

$$\int_{\mathfrak{m}} \left|\varDelta(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right|^2 K_\sigma(\alpha)\mathrm{d}\alpha \ll \sigma X^{\frac{33}{14}+\varepsilon}. \tag{4.5}$$

5. The trivial arc

For convenience and based on the argument in Sec. 4, we still substitute $S_i(\alpha)$ for $S_i(\alpha, \varrho^{\pm})$, $i = 1, 2$. Combining Hölder's inequality with (1.2), we can get

$$\begin{aligned}&\int_{\mathfrak{t}} \left|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right| K_\sigma(\alpha)\mathrm{d}\alpha \\ &\ll \left(\int_{\mathfrak{t}} \left|S_1(\lambda_1\alpha)\right|^4 K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\int_{\mathfrak{t}} \left|S_2(\lambda_2\alpha)\right|^4 K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{4}} \\ &\quad \prod_{j=3}^{5}\left(\int_{\mathfrak{t}} \left|S_j(\lambda_j\alpha)\right|^6 K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{6}} \\ &\ll \left(\sum_{n=[G]}^{\infty}\int_n^{n+1} \frac{\left|S_1(\lambda_1\alpha)\right|^4}{|\alpha|^2}\mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\sum_{n=[G]}^{\infty}\int_n^{n+1} \frac{\left|S_2(\lambda_2\alpha)\right|^4}{|\alpha|^2}\mathrm{d}\alpha\right)^{\frac{1}{4}} \\ &\quad \prod_{j=3}^{5}\left(\sum_{n=[G]}^{\infty}\int_n^{n+1} \frac{\left|S_j(\lambda_j\alpha)\right|^6}{|\alpha|^2}\mathrm{d}\alpha\right)^{\frac{1}{6}}. \end{aligned}$$

Applying Hua's Lemma in the above formula, we obtain

$$\begin{aligned}&\int_{\mathfrak{t}} \left|S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)\right| K_\sigma(\alpha)\mathrm{d}\alpha \\ &\ll \left(\sum_{n=[G]}^{\infty}\frac{1}{n^2}\right)\left(\int_0^1 \left|S_1(\lambda_1\alpha)\right|^4 \mathrm{d}\alpha\right)^{\frac{1}{4}} \left(\int_0^1 \left|S_2(\lambda_2\alpha)\right|^4 \mathrm{d}\alpha\right)^{\frac{1}{4}} \end{aligned}$$

$$\prod_{j=3}^{5}\left(\int_0^1 |S_j(\lambda_j\alpha)|^6\,\mathrm{d}\alpha\right)^{\frac{1}{6}}$$
$$\ll G^{-1}X^{\frac{1}{2}+\frac{1}{2}+\varepsilon}$$
$$\ll \sigma^2 X^{\frac{3}{4}+\varepsilon}.$$

Therefore, we can get

$$G=\sigma^{-2}X^{\frac{1}{4}}L^3,$$

and we obtain

$$\int_{\mathfrak{t}}|\varDelta(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)|\,K_\sigma(\alpha)\mathrm{d}\alpha\ll\sigma X^{\frac{33}{14}-\varepsilon}. \tag{5.1}$$

6. Completion of the demonstration

Finally, we finalize the demonstration of Theorem 1.1. Let $\mathcal{E}(\mathcal{V},N,\delta)$ be the set of $v\in\mathcal{V}$ with $v\le N$ making the inequality (1.2) unsolvable in primes $p_1,\dots,p_5$. For any set $\mathfrak{B}\subseteq[0,N]$, define

$$\mathcal{E}(\mathfrak{B})=\mathcal{E}(\mathcal{V},N,\delta)\cap\mathfrak{B},\quad E(\mathfrak{B})=|\mathcal{E}(\mathfrak{B})|\,. \tag{6.1}$$

We denote by $\varPsi$ the closed interval $[N^{\frac{6}{7}},N]$. Trivially,

$$|\mathcal{E}(\mathcal{V},N,\delta)|\ll E(\varPsi)+E([0,N^{\frac{6}{7}}])\ll E(\varPsi)+N^{\frac{6}{7}},$$

so it is crucial to get the upper bound estimate of $E(\varPsi)$. Let $Y=\left[\frac{1}{2}X,X\right]$, so we have $\mathcal{E}(Y)=\mathcal{E}(\mathcal{V},N,\delta)\cap Y$ and $E(Y)=|\mathcal{E}(Y)|$. By a splitting argument, we focus on the estimate of $E(Y)$ for any $X\in\varPsi$. Consequently, we only need to consider those $v\in Y\cap\mathcal{V}$. From (2.6), we can conclude that $I(\sigma,v,\mathbb{R},\varrho_0,\varrho_0)=0$. Then by (2.7), we can get

$$I(\sigma,v,\mathbb{R},\varrho^-,\varrho^+)+I(\sigma,v,\mathbb{R},\varrho^+,\varrho^-)-I(\sigma,v,\mathbb{R},\varrho^+,\varrho^+)\le 0.$$

By (3.18) and (5.1), we obtain

$$\left|\sum_{v\in\mathcal{E}(Y)}\int_{\mathfrak{m}}\varDelta(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha\right|\gg\sigma^2L^{-2}X^{\frac{3}{4}}E(Y). \tag{6.2}$$

Combining (2.2), (4.5) and applying Cauchy's inequality, we get

$$\left|\sum_{v\in\mathcal{E}(Y)}\int_{\mathfrak{m}}\varDelta(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)K_\sigma(\alpha)e(-v\alpha)\mathrm{d}\alpha\right|$$
$$\ll\left(\int_{\mathfrak{m}}|\varDelta(\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_5(\lambda_5\alpha)|^2\,K_\sigma(\alpha)\mathrm{d}\alpha\right)^{\frac{1}{2}}$$

$$\times\bigg(\int_{\mathbb{R}}\bigg|\sum_{v\in\mathcal{E}(Y)}e(-\alpha v)\bigg|^2K_\sigma(\alpha)\mathrm{d}\alpha\bigg)^{\frac{1}{2}}$$
$$\ll\Big(\sigma X^{\frac{33}{14}+\varepsilon}\Big)^{\frac{1}{2}}\bigg(\sum_{v_1,v_2\in\mathcal{E}(Y)}\max(0,\sigma-|v_1-v_2|)\bigg)^{\frac{1}{2}}$$
$$\ll\sigma(E(Y))^{\frac{1}{2}}\Big(X^{\frac{33}{14}+\varepsilon}\Big)^{\frac{1}{2}}. \tag{6.3}$$

Then combining (6.2), (6.3) and taking $\sigma=X^{-\delta}$, we get

$$E(Y)\ll\sigma^{-2}X^{\frac{6}{7}+\varepsilon}\ll X^{\frac{6}{7}+2\delta+\varepsilon}. \tag{6.4}$$

Ultimately, we apply the dyadic argument. Let $\mathcal{V}$ represent a well-spaced sequence, $v_i\in\mathcal{V}$ for $i=1,2,\cdots$. From (6.1), (6.4) and the meaning of $\mathcal{E}(\mathcal{V},N,\delta)$, we can infer that $E(\mathcal{V},N,\delta)=|\mathcal{E}(\mathcal{V},N,\delta)|$ and

$$\begin{aligned}E(\mathcal{V},N,\delta)&\ll E(\Psi)+N^{\frac{6}{7}}\\&\ll\sum_{1\le k\ll\log N}E([2^{-k}N,2^{1-k}N])+N^{\frac{6}{7}}\\&\ll\sum_{1\le k\ll\log N}(2^{1-k}N)^{\frac{6}{7}+2\delta+\varepsilon}+N^{\frac{6}{7}}\\&\ll N^{\frac{6}{7}+2\delta+\varepsilon}.\end{aligned}$$

The irrationality of the ratio λ_1/λ_2 implies that there exist infinitely many integer pairs q,a. Then we get $X=q^{\frac{7}{3}}\to+\infty$, as $q\to+\infty$. In conclusion, the demonstration of Theorem 1.1 is concluded.

Funding

This work is supported by Natural Science Foundation of China (Grant No. 12361002), Natural Science Foundation of Jiangxi Province (Grant No. 20224BAB201001), the conference (DART2025Z) and the organizer (Takao Komatsu).

Conflicts of interest

The authors report there are no conflicts of interest pertaining to this study.

Author contribution statement

All authors consented to their individual contributions prior to publication. Conceptualization, Y. Fu and L.Q. Hu; Methodology, L. Yang; Validation, S.Q. Liu and L. Yang; Supervision, S.Q. Liu and L. Yang; Writing Original Draft Preparation, Y. Fu; Writing Review and Editing, Y. Fu and L.Q. Hu.

Data availability statement

There is no new data generation and analysis.

References

[1] C. Hooley, On a new approach to various problems of Waring's type, in *Recent Progress in Analytic Number Theory*, Vol. 1. Academic Press, Durham, London (2023) 127–191.

[2] J.B. Friedlander and T.D. Wooley, On Waring's problem: two squares and three biquadrates. *Mathematics* **60** (2014) 153–165.

[3] L. Zhao, Exceptional sets in Waring's problem: two squares and s biquadrates, *Acta Arith.* **162** (2014) 369–379.

[4] L. Zhu, Waring–Goldbach problem: two squares and three biquadrates. *Rocky Mountain J. Math.* **50** (2020) 355–367.

[5] M.M. Liu, M. Zhang and J.J. Li, Exceptional set for Diophantine approximation with primes of two squares and three biquadrates. *Adv. Math, Chin. Ser.* **54** (2025) 18–36.

[6] H. Davenport and H. Heilbronn, On indefinite quadratic forms in five variables. *J. Lond. Math. Soc.* **21** (1946) 185–193.

[7] Y.Y. Qu and J.W. Zeng, Diophantine approximation with prime variables and mixed powers. *Ramanujan J.* **52** (2020) 625–639.

[8] G. Harman and A.V. Kumchev, On sums of squares of primes. *Math. Proc. Cambridge Philos. Soc.* **140** (2006) 1–13.

[9] G. Harman, The values of ternary quadratic forms at prime arguments. *Mathematics* **51** (2005) 83–96.

[10] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd edn. Oxford University Press, Oxford (1986).

[11] G. Harman, Trigonometric sums over primes I. *Mathematics* **28** (1981) 249–254.

[12] W.X. Ge and F. Zhao, The exceptional set for Diophantine inequality with unlike powers of prime variables. *Czech. Math. J.* **68** (2018) 149–168.

[13] B.Y. Li and Y.C. Wang, Diophantine approximation with prime variables and mixed powers. *Ramanujan J.* **60** (2023) 371–389.

[14] Y.C. Wang and W.L. Yao, Diophantine approximation with one prime ang three squares of primes. *J. Number Theory.* **180** (2017) 234–250.