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ON THE NUMBERS OF RAMANUJAN

Mehmet Cenkci[1,*] and Takao Komatsu[2,3]

Abstract. In Chapter 3 of his second notebook, Ramanujan defined numbers $a(n,k)$ such that $a(2,0)=1$ and for $n\geq 2$, $a(n+1,k)=(n-1)\,a(n,k-1)+(2n-1-k)\,a(n,k)$, where $a(n,k)=0$ when $k<0$ or $k>n-2$. These numbers are expressed in terms of Stirling numbers of the first kind and associated Stirling numbers of the second kind, and satisfy certain divisibility properties. In this paper, we obtain further properties for $a(n,k)$, including a new characterization of prime numbers. We also derive several congruences for the numbers of Ramanujan modulo p^2, some of which lead to new conditions for a prime number to be a Wilson prime.

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1. Introduction and statements of the results

In 1983, Berndt *et al.* [1] proved and explained all of the results set by Ramanujan in Chapter 3 of his second notebook. They stated that the contents of this chapter could be interpreted combinatorially, although no combinatorial problems were mentioned in it. One of the subjects in this discussion is the sequence of integers $a(n,k)$, where n and k are integers, which are called the numbers of Ramanujan by Howard [2]. These numbers satisfy the initial condition $a(2,0)=1$ and the recursive formula

$$a(n+1,k)=(n-1)\,a(n,k-1)+(2n-1-k)\,a(n,k)\,, \tag{1.1}$$

where $n\geq 2$. Moreover, $a(n,k)=0$ when $k<0$ or $k>n-2$. These numbers occur as follows: Let a be a fixed real number satisfying $a>1/e$, and for real h, define $x>0$ by $x^x=a^a\mathrm{e}^h$. Then the differential equation

$$(x+h+a\log a)\,\frac{\mathrm{d}x}{\mathrm{d}h}=x \tag{1.2}$$

produces

$$\left.\frac{\mathrm{d}x}{\mathrm{d}h}\right|_{h=0}=\frac{1}{1+\log a}=r,$$

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[1] Akdeniz University, Department of Mathematics, Antalya 07058, Türkiye.
[2] Institute of Mathematics, Henan Academy of Sciences, Zhengzhou 450046, China.
[3] Department of Mathematics, Institute of Science Tokyo, Meguro-ku, Tokyo 152-8551 Japan.

* Corresponding author: cenkci@akdeniz.edu.tr

since $x=a$ when $h=0$. Expanding $x=x(h)$ about the origin gives

$$\frac{x-a}{a}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{A_n(r)}{n!}\left(\frac{h}{a}\right)^n, \tag{1.3}$$

where h is sufficiently small. Using the differential equation (1.2), a formula for the coefficients $A_n(r)$ in (1.3) is

$$A_n(r)=r(n-2)A_{n-1}(r)+r\sum_{k=1}^{n-1}\binom{n-1}{k}A_k(r)A_{n-k}(r), \tag{1.4}$$

where $n\geq 2$. This formula is Entry 17 of Chapter 3 of Ramanujan's second notebook. Now,

$$A_1(r)=\left.\frac{\mathrm{d}x}{\mathrm{d}h}\right|_{h=0}=r,$$

and in general, it follows from (1.3) that

$$A_n(r)=(-a)^{n-1}\left.\frac{\mathrm{d}^n x}{\mathrm{d}h^n}\right|_{h=0} \tag{1.5}$$

for $n\geq 1$. Inducting on n, the numbers $a(n,k)$ occur as the coefficients in the sum

$$\sum_{k=0}^{n-2}a(n,k)R^{2n-k-1}=(-x)^{n-1}\frac{\mathrm{d}^n x}{\mathrm{d}h^n} \tag{1.6}$$

for $n\geq 2$, where $R=1/(1+\log x)$. Differentiating both sides of (1.6) with respect to h and comparing the coefficients of R^{2n-k+1}, Ramanujan obtained the recursive formula (1.1). Setting $h=0$ in (1.6) and using (1.5), it follows that

$$A_n(r)=\sum_{k=0}^{n-2}a(n,k)r^{2n-k-1}$$

for $n\geq 2$. It is also shown in [1] that

$$\sum_{k=0}^{n-2}a(n,k)=A_n(1)=(n-1)^{n-1}. \tag{1.7}$$

Some values of $a(n,k)$ are given in the following table.

$a(n,k)$ for $2 \leq n \leq 10$ and $0 \leq k \leq 8$

n \ k	0	1	2	3	4	5	6	7	8
2	1								
3	3	1							
4	15	10	2						
5	105	105	40	6					
6	945	1260	700	196	24				
7	10 395	17 325	12 600	5068	1148	120			
8	135 135	270 270	242 550	126 280	40 740	7848	720		
9	2 027 025	4 729 725	5 045 040	3 213 210	1 332 100	363 660	61 416	5040	
10	34 459 425	91 891 800	113 513 400	85 345 260	43 022 980	15 020 720	3 584 856	541 728	40 320

In [3], Howard and Hayashi gave congruences for $a\,(n,k)\,(\mathrm{mod}\,p)$, where p is a prime number. In particular, they proved that

$$a\,(p,k) \equiv 0\,(\mathrm{mod}\,p) \qquad (k = 0,1,\ldots,p-3)\,. \tag{1.8}$$

This result and (1.7) surprisingly give a characterization of prime numbers based on the numbers of Ramanujan.

Theorem 1.1. *An odd integer $n \geq 3$ is a prime number if and only if $a\,(n,k) \equiv 0\,(\mathrm{mod}\,n)$ for $k = 0,1,\ldots,n-3$.*

This characterization of prime numbers might not be so practical when n is large. This is because of the fact that the numbers of Ramanujan rapidly grow as the above table suggests. But, just like Wilson's theorem, this result is meaningful at least from a theoretical point of view. For instance, a condition for twin primes can be stated as follows:

Corollary 1.2. *For an odd integer $n \geq 5$, if a pair $(n-2,n)$ consists of twin primes, then $a\,(n,k) \equiv 0\,(\mathrm{mod}\,n\,(n-2))$ for $k = 0,1,\ldots,n-5$.*

The divisibility property (1.8) can also be taken with the context of Fermat, Wilson, and Lerch quotients, defined respectively by

$$q_p\,(a) = \frac{a^{p-1}-1}{p},$$

$$W_p = \frac{(p-1)!+1}{p},$$

(see [4], pp. 216–217) and

$$L_p = \frac{1}{p}\left(\sum_{k=1}^{p-1} q_p\,(k) - W_p\right),$$

([5], Def. 1), where the latter follows from the celebrated Lerch's congruence (*cf.* [6])

$$W_p \equiv \sum_{k=1}^{p-1} q_p\,(k)\,(\mathrm{mod}\,p)\,.$$

We define the integer

$$r_p(k) = \frac{a(p,k)}{p},$$

where p is an odd prime and $k = 0, 1, \ldots, p-3$, as the Ramanujan quotient. The Ramanujan quotient is closely related to Fermat and Wilson quotients.

Theorem 1.3. *For an odd prime p,*

$$W_p \equiv \sum_{k=1}^{p-2} (r_p(k) + q_p(k)) \pmod{p}.$$

A prime p is called a Wilson prime if $W_p \equiv 0 \pmod{p}$. Thus, Theorem 1.3 implies that p is a Wilson prime if

$$\sum_{k=1}^{p-2} r_p(k) \equiv -\sum_{k=1}^{p-2} q_p(k) \pmod{p}.$$

Sondow [5] called the prime numbers satisfying $L_p \equiv 0 \pmod{p}$ Lerch primes, that is, p is a Lerch prime if

$$W_p \equiv \sum_{k=1}^{p-1} q_p(k) \pmod{p^2}.$$

In analogy, we define the Lerch-Ramanujan quotient to be the integer

$$R_p = \frac{1}{p}\left(\sum_{k=1}^{p-2} (r_p(k) + q_p(k)) - W_p\right).$$

We then call a prime number a Lerch-Ramanujan prime if $R_p \equiv 0 \pmod{p}$. Hence, by Theorem 1.3, p is a Lerch-Ramanujan prime if

$$W_p \equiv \sum_{k=1}^{p-2} (r_p(k) + q_p(k)) \pmod{p^2}.$$

Among several conjectures proposed by Howard and Hayashi [3], we confirm the validity of the following two congruence statements concerning the coefficients $a(n,0)$ and $a(n,1)$.

Theorem 1.4. *For an odd prime p and $n \geq \frac{p+3}{2}$, $a(n,0) \equiv p \pmod{2p}$.*

Theorem 1.5. *For an odd prime p and $n \geq \frac{p+3}{2}$,*

$$a(n,1) \equiv \begin{cases} 0 \pmod{2p}, & \text{if } n \text{ is even,} \\ p \pmod{2p}, & \text{if } n \text{ is odd,} \end{cases}$$

except when $p = 3$ and $n = 3, 4$.

In [2] Howard obtained expressions for $a(n,k)$ in terms of the Stirling numbers of the first kind and associated Stirling numbers of the second kind. In particular, Theorems 1.4 and 1.5 follow from the relationships between the numbers of Ramanujan and the associated Stirling numbers of the second kind.

Howard and Hayashi obtained numerous results for the numbers of Ramanujan in [3] using (1.1) and (1.4). For example, for an odd prime p, they proved modulo p that

$$\begin{array}{ll} a(p+1,p-1) \equiv -1 & a(p+2,p) \equiv 0 \\ a(p+1,p-2) \equiv 1 & a(p+2,p-1) \equiv -2 \\ a(p+1,p-3) \equiv 0 & a(p+2,p-2) \equiv 3 \\ a(p+1,p-4) \equiv 0 & a(p+2,p-3) \equiv 0 \end{array}$$

Using the expression involving Stirling numbers of the first kind, we extend these congruences to modulo p^2.

Theorem 1.6. *For an odd prime p,*

$$a(p+1,p-1) \equiv -p + pB_{p-1} \pmod{p^2},$$

where B_n is the nth Bernoulli number.

We note that the famous congruence by Glaisher [7]

$$W_p \equiv B_{p-1} - 1 + \frac{1}{p} \pmod{p}$$

implies that

$$a(p+1,p-1) \equiv pW_p - 1 \pmod{p^2}.$$

Thus, p is a Wilson prime if and only if $a(p+1,p-1) \equiv -1 \pmod{p^2}$.

Theorem 1.7. *For an odd prime p,*

$$a(p+1,p-2) \equiv -pB_{p-1} \pmod{p^2}.$$

Equivalently,

$$a(p+1,p-2) \equiv 1 - p - pW_p \pmod{p^2},$$

and p is a Wilson prime if and only if $a(p+1,p-2) \equiv 1 - p \pmod{p^2}$.

Theorem 1.8. *For an odd prime $p > 3$,*

$$a(p+1,p-3) \equiv \frac{p}{2} - 3pH_{p-1}^{(2)}B_{p-1} \pmod{p^2},$$

where $H_n^{(m)}$ is the generalized harmonic number defined by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

Since

$$H_{p-1}^{(m)} \equiv 0 \pmod{p} \quad \text{when} \quad (p-1) \nmid m \tag{1.9}$$

(see [8]),

$$a\left(p+1,p-3\right)\equiv\frac{p}{2}-3pH_{p-1}^{(2)}B_{p-1}\equiv\frac{p}{2}-3H_{p-1}^{(2)}\left(pW_p+p-1\right)\equiv\frac{p}{2}+3H_{p-1}^{(2)}\left(\mathrm{mod}\,p^2\right).$$

Theorem 1.9. *For a prime $3<p\neq 13$,*

$$a\left(p+1,p-4\right)\equiv\frac{p}{6}+p\left(13H_{p-1}^{(2)}-8H_{p-1}^{(3)}\right)B_{p-1}\left(\mathrm{mod}\,p^2\right).$$

By the similar argument as above,

$$a\left(p+1,p-4\right)\equiv\frac{p}{6}-13H_{p-1}^{(2)}+8H_{p-1}^{(3)}\left(\mathrm{mod}\,p^2\right).$$

Theorem 1.10. *For an odd prime p,*

$$a\left(p+2,p\right)\equiv-p\left(\mathrm{mod}\,p^2\right).$$

Theorem 1.11. *For an odd prime p,*

$$a\left(p+2,p-1\right)\equiv-2p+2pB_{p-1}\left(\mathrm{mod}\,p^2\right).$$

Equivalently,

$$a\left(p+2,p-1\right)\equiv2pW_p-2\left(\mathrm{mod}\,p^2\right),$$

and p is a Wilson prime if and only if $a\left(p+1,p-2\right)\equiv-2\left(\mathrm{mod}\,p^2\right)$.

Theorem 1.12. *For an odd prime p,*

$$a\left(p+2,p-2\right)\equiv p-3pB_{p-1}\left(\mathrm{mod}\,p^2\right).$$

Thus,

$$a\left(p+2,p-2\right)\equiv3-2p-3pW_p\left(\mathrm{mod}\,p^2\right),$$

and p is a Wilson prime if and only if $a\left(p+2,p-2\right)\equiv3-2p\left(\mathrm{mod}\,p^2\right)$.

Theorem 1.13. *For a prime $p>3$,*

$$a\left(p+2,p-3\right)\equiv-2p-12pH_{p-1}^{(2)}B_{p-1}\left(\mathrm{mod}\,p^2\right).$$

Equivalently,

$$a\left(p+2,p-3\right)\equiv-2p+12H_{p-1}^{(2)}\left(\mathrm{mod}\,p^2\right).$$

In Section 2 we provide proofs of the results stated in this section.

2. Proofs of the results

This section covers the proofs of the results stated in the previous section and some matters needed in the proofs.

Proof of Theorem 1.1. If $n = p$ is a prime number, then by Howard's result (1.8), $a(p,k) \equiv 0 \pmod p$ when $k = 0, 1, \ldots, p-3$. Inversely, let $a(n,k) \equiv 0 \pmod n$ for $k = 0, 1, \ldots, n-3$ and odd $n \geq 3$. Thus,

$$\sum_{k=0}^{n-2} a(n,k) = a(n,n-2) + \sum_{k=0}^{n-3} a(n,k) = (n-2)! + \sum_{k=0}^{n-3} a(n,k) \equiv (n-2)! \pmod n$$

and by (1.7),

$$\sum_{k=0}^{n-2} a(n,k) = (n-1)^{n-1} \equiv (-1)^{n-1} = 1 \pmod n.$$

Now, by Wilson's theorem, the result follows. Here we use the identity $a(n,n-2) = (n-2)!$, which follows from repeated application of the recurrence (1.1) and is proved explicitly later in this section. □

Proof of Corollary 1.2. By (1.1),

$$a(n,k) = (n-2)\,a(n-1,k-1) + (2n-k-3)\,a(n-1,k)$$

and

$$a(n-1,k) = (n-3)\,a(n-2,k-1) + (2n-k-7)\,a(n-2,k).$$

Inserting second equality into the first,

$$\begin{aligned} a(n,k) = {} & (n-2)(n-3)\,a(n-2,k-2) \\ & + ((n-2)(2n-k-6) + (n-3)(2n-k-3))\,a(n-2,k-1) \\ & + (2n-k-3)(2n-k-7)\,a(n-2,k). \end{aligned} \tag{2.1}$$

For an odd integer $n \geq 5$, let n and $n-2$ be twin primes. Then, by Theorem 1.1,

$$a(n,k) \equiv 0 \pmod n, \quad k = 0, 1, \ldots, n-3$$

and

$$a(n-2,k) \equiv 0 \pmod{n-2}, \quad k = 0, 1, \ldots, n-5,$$

so

$$a(n-2,k-1) \equiv 0 \pmod{n-2}, \quad k = 0, 1, \ldots, n-4. \tag{2.2}$$

It then follows from (2.1) for $k = 0, 1, \ldots, n-5$ that

$$a(n,k) \equiv (n-3)(2n-k-3)\,a(n-2,k-1) \equiv (1-k)\,a(n-2,k-1) \pmod{n-2}.$$

Since $a(n,k)=0$ for $k<0$, we obtain from (2.2) that

$$a(n,k)\equiv 0 \pmod{n-2}, \quad k=0,1,\dots,n-5.$$

Therefore, by the Chinese Remainder Theorem,

$$a(n,k)\equiv 0 \pmod{n(n-2)}, \quad k=0,1,\dots,n-5,$$

provided that n and $n-2$ are primes. □

Proof of Theorem 1.3. By (1.7),

$$\sum_{k=0}^{p-2} r_p(k) = \frac{1}{p}\sum_{k=0}^{p-2} a(p,k) = \frac{1}{p}(p-1)^{p-1} = \frac{(p-1)^{p-1}-1}{p} + \frac{1}{p} = q_p(p-1) + \frac{1}{p}. \tag{2.3}$$

Lerch's congruence implies that

$$W_p \equiv \sum_{k=1}^{p-1} q_p(k) = q_p(p-1) + \sum_{k=1}^{p-2} q_p(k) \pmod{p}.$$

Thus, (2.3) gives

$$W_p \equiv -\frac{1}{p} + \sum_{k=0}^{p-2} r_p(k) + \sum_{k=1}^{p-2} q_p(k) \pmod{p}.$$

Since

$$q_p(0) = -\frac{1}{p},$$

the result follows. □

The (unsigned) Stirling numbers of the first kind, $s(n,k)$, which count the permutation of n elements that are product of k disjoint cycles, may be defined by means of the generating function

$$[-\log(1-x)]^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}.$$

It follows from this definition that

$$s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1)$$

with

$$\begin{aligned} &s(n,0) = 0 \quad \text{if } n>0 \\ &s(n,n) = 1 \\ &s(n,1) = (n-1)! \quad \text{if } n>0 \\ &s(n,k) = 0 \quad \text{if } k>n \quad \text{or} \quad k<0. \end{aligned}$$

We also note that

$$s(n,2) = (n-1)!H_{n-1} \quad \text{and} \quad s(n,3) = \frac{1}{2}(n-1)!\left(H_{n-1}^2 - H_{n-1}^{(2)}\right),$$

where H_n is the nth harmonic number and $H_n^{(m)}$ is the generalized harmonic number defined by

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}.$$

Stirling numbers of the second kind, $S(n,k)$, which are the number of ways to partition a set of n elements into k non-empty subsets, may be defined by means of the generating function

$$\left(\mathrm{e}^x - 1\right)^k = k! \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}.$$

It follows from this definition that

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

It is known that $S(n,k)$ is a polynomial in n of degree $2k$, which can be written as

$$S(n,n-k) = \sum_{j=0}^{k} b(2k-j,k-j) \binom{n}{2k-j},$$

where $b(n,k)$ is the associated Stirling number of the second kind ([9], pp. 221–222, [10], pp. 76–78). A generating function for these numbers is

$$\left(\mathrm{e}^x - x - 1\right)^k = k! \sum_{n=k}^{\infty} b(n,k) \frac{x^n}{n!}.$$

This equation implies

$$b(n,k) = kb(n-1,k) + (n-1)\,b(n-2,k-1) \tag{2.4}$$

with

$$\begin{aligned} &b(0,0) = 1 \\ &b(n,0) = 0 \quad \text{if } n > 0 \\ &b(n,1) = 1 \quad \text{if } n > 1 \\ &b(n,k) = 0 \quad \text{if } 2k > n \quad \text{or} \quad k < 0. \end{aligned}$$

In [11] Howard obtained that if $n - k = (p-1)w$, then

$$b(n,k) \equiv \binom{w-1}{k-1} \pmod{p},$$

and if $n-k=(p-1)w+v$, where $1\leq v\leq p-2$, then

$$b(n,k)\equiv\sum_{m=0}^{p-1-v}\binom{w}{k-m}b(m+v,m)\pmod{p}. \tag{2.5}$$

Using (2.5) and special values of $b(n,k)$, it is easy to deduce that

$$b(n,k)\equiv\binom{w}{k-1}\pmod{p},\quad (n-k=(p-1)w+1).$$

In particular,

$$b(n,k)\equiv\binom{n-k-1}{k-1}\pmod{2}. \tag{2.6}$$

The congruence (2.6) follows directly from Howard's congruences for the associated Stirling numbers [11].

It follows from (2.4) that

$$b(2n,n)=1\cdot3\cdot5\cdot\cdots\cdot(2n-1).$$

Since

$$\begin{aligned}a(n,0)&=(2n-3)\,a(n-1,0)=(2n-3)(2n-5)\,a(n-2,0)\\&=\cdots\\&=1\cdot3\cdot5\cdot\cdots\cdot(2n-3)\end{aligned}$$

by (1.1),

$$a(n,0)=b(2n-2,n-1) \tag{2.7}$$

for $n\geq2$. Moreover,

$$\begin{aligned}b(2n-1,n-1)&=(n-1)\,b(2n-2,n-1)+(2n-2)\,b(2n-3,n-2)\\&=(n-1)\,a(n,0)+(2n-2)\,b(2n-3,n-2)\end{aligned}$$

with $b(3,1)=1$. Comparing this with

$$a(n,1)=(n-2)\,a(n-1,0)+(2n-4)\,a(n-1,1),$$

we find that

$$a(n,1)=b(2n-3,n-2) \tag{2.8}$$

for $n\geq3$.

Proof of Theorem 1.4. (2.7) and (2.6) give

$$a(n,0)\equiv1\pmod{2}.$$

On the other hand, since $a(n,0) = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)$, $n \geq \frac{p+3}{2}$ implies that $a(n,0) \equiv 0 \pmod{p}$. Thus, by the Chinese Remainder Theorem,

$$a(n,0) \equiv p \pmod{2p}$$

provided that $n \geq \frac{p+3}{2}$. □

Proof of Theorem 1.5. Let $p \geq 5$ and $n \geq \frac{p+3}{2}$. (2.8) and (2.6) give

$$a(n,1) \equiv \binom{n-2}{n-3} = n-2 \equiv n \pmod{2},$$

which is equivalent to

$$a(n,1) \equiv \begin{cases} 0 \pmod{2}, & \text{if } n \text{ is even,} \\ 1 \pmod{2}, & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, $a(n,1) = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)\frac{n-2}{3}$, so $n \geq \frac{p+3}{2}$ implies that $a(n,1) \equiv 0 \pmod{p}$. Thus, by the Chinese Remainder Theorem, the result follows. Now, let $p = 3$ and $n \geq 4$. When $n \geq 6$, $2n-3 \geq 9$; thus

$$a(n,1) = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)\frac{n-2}{3} \equiv 0 \pmod{3},$$

and the result follows in this case. The facts $a(4,1) = 10$ and $a(5,1) = 105 \equiv 3 \pmod{6}$ complete the proof. □

In [2] Howard obtained expressions for the numbers of Ramanujan and the Stirling numbers. By (1.1) and the fact that $a(n,k) = 0$ when $k > n-2$,

$$\begin{aligned} a(n,n-2) &= (n-2)\,a(n-1,n-3) = (n-2)(n-3)\,a(n-2,n-4) \\ &= \cdots = (n-2)!a(2,0) = (n-2)! \end{aligned}$$

since $a(2,0) = 1$. Therefore,

$$a(n,n-2) = s(n-1,1). \tag{2.9}$$

More generally, Howard proved that

$$a(n,n-k) = \sum_{t=1}^{k-1} P_{k,t}(n)\, s(n-1,t) \tag{2.10}$$

for $k \geq 2$, where $P_{k,t}(n)$ is a certain polynomial in n of degree $k-1-t$. Using (2.10), it follows that

$$\begin{aligned} P_{k,k-1}(n) &= (k-1)! \\ P_{k,k-2}(n) &= n(k-2)! + (k-2)(k-2)! - s(k,2), \end{aligned}$$

and

$$P_{2,1}(n) = 1, \quad P_{3,1}(n) = n-2, \quad P_{4,1}(n) = \binom{n-1}{2},$$

$$P_{5,1}(n)=\binom{n}{3},\quad P_{6,1}(n)=\binom{n+1}{4}.$$

The Bernoulli numbers B_n may be defined by means of the generating function

$$\frac{t}{\mathrm{e}^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

First few Bernoulli numbers are $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, $B_4=-\frac{1}{30}$, and $B_{2n+1}=0$ for $n\geq 1$. The Bernoulli numbers are rational numbers whose denominators are explicitly determined by the von Staudt-Clausen theorem, which states that

$$B_{2n}=A_{2n}-\sum_{(p-1)|2n}\frac{1}{p},$$

where A_{2n} is an integer and the sum is taken over all primes with $(p-1)\,|2n$. This result can be equivalently stated as

$$pB_{2n}\equiv\begin{cases}0\,(\mathrm{mod}\,p)\,, & \text{if } (p-1)\nmid 2n,\\ -1\,(\mathrm{mod}\,p)\,, & \text{if } (p-1)\,|2n.\end{cases}$$

In particular, for an odd prime p,

$$p^2B_{p-1}\equiv -p\,(\mathrm{mod}\,p^2)\,. \tag{2.11}$$

Proof of Theorem 1.6. Writing $n=p+1$ in (2.9) gives $a(p+1,p-1)=s(p,1)=(p-1)!$. Since

$$(p-1)!\equiv -p+pB_{p-1}\,(\mathrm{mod}\,p^2)\,, \tag{2.12}$$

(*cf.* [7]), the result follows. □

Proof of Theorem 1.7. Writing $k=3$ in (2.10) gives

$$a(n,n-3)=\sum_{t=1}^{2}P_{3,t}(n)\,s(n-1,t)=P_{3,1}(n)\,s(n-1,1)+P_{3,2}(n)\,s(n-1,2)\,.$$

Since $P_{3,1}(n)=n-2$, $P_{3,2}(n)=(3-1)!=2$, $s(n-1,1)=(n-2)!$, and $s(n-1,2)=(n-2)!H_{n-2}$, we obtain

$$a(n,n-3)=(n-2)(n-2)!+2(n-2)!H_{n-2}. \tag{2.13}$$

Letting $n=p+1$ for an odd prime p in the last equation yields

$$a(p+1,p-2)=(p-1)(p-1)!+2(p-1)!H_{p-1}.$$

Since $H_{p-1}\equiv 0\,(\mathrm{mod}\,p^2)$ by Wolstenholme's theorem, (2.12) implies

$$a(p+1,p-2)\equiv p^2B_{p-1}-pB_{p-1}+p\,(\mathrm{mod}\,p^2)\,,$$

and (2.11) completes the proof. □

Proof of Theorem 1.8. For an odd prime $p>3$, we write $n=p+1$ and $k=4$ in (2.10) to obtain

$$a\left(p+1,p-3\right)=P_{4,1}\left(p+1\right)s\left(p,1\right)+P_{4,2}\left(p+1\right)s\left(p,2\right)+P_{4,3}\left(p+1\right)s\left(p,3\right).$$

Since

$$P_{4,1}\left(p+1\right)=\binom{p}{2},\quad P_{4,2}\left(p+1\right)=2p-5,\quad P_{4,3}\left(p+1\right)=6,$$

and

$$s\left(p,1\right)=(p-1)!,\quad s\left(p,2\right)=(p-1)!H_{p-1},\quad s\left(p,3\right)=\frac{1}{2}\left(p-1\right)!\left(H_{p-1}^{2}-H_{p-1}^{(2)}\right),$$

we find by Wolstenholme's theorem and (2.11) that

$$a\left(p+1,p-3\right)\equiv-\frac{p}{2}\left(pB_{p-1}-p\right)-3H_{p-1}^{(2)}\left(pB_{p-1}-p\right)\equiv\frac{p}{2}-3pH_{p-1}^{(2)}B_{p-1}\left(\mathrm{mod}\,p^{2}\right),$$

as desired. □

Proof of Theorem 1.9. For an odd prime $p>3$, we write $n=p+1$ and $k=5$ in (2.10) to obtain

$$a\left(p+1,p-4\right)=P_{5,1}\left(p+1\right)s\left(p,1\right)+P_{5,2}\left(p+1\right)s\left(p,2\right)+P_{5,3}\left(p+1\right)s\left(p,3\right)+P_{5,4}\left(p+1\right)s\left(p,4\right).$$

Since

$$P_{5,1}\left(p+1\right)=\binom{p+1}{3},\quad P_{5,2}\left(p+1\right)=\left(p-1\right)\left(p-3\right),\quad P_{5,3}\left(p+1\right)=6p-26,\quad P_{5,4}\left(p+1\right)=24,$$

and

$$s\left(p,1\right)=(p-1)!,\quad s\left(p,2\right)=(p-1)!H_{p-1},\quad s\left(p,3\right)=\frac{1}{2}\left(p-1\right)!\left(H_{p-1}^{2}-H_{p-1}^{(2)}\right),$$
$$s\left(p,4\right)=-\frac{(p-1)!}{3!}\left(H_{p-1}^{3}-3H_{p-1}H_{p-1}^{(2)}+2H_{p-1}^{(3)}\right),$$

we find by Wolstenholme's theorem, (1.9), (2.12) and (2.11) that

$$a\left(p+1,p-4\right)\equiv\frac{p}{6}+\left(13H_{p-1}^{(2)}-8H_{p-1}^{(3)}\right)pB_{p-1}\left(\mathrm{mod}\,p^{2}\right),$$

which is the desired result. □

Proof of Theorem 1.10. Writing $n=p+2$ in (2.9) gives

$$a\left(p+2,p\right)=s\left(p+1,1\right)=p!=p\left(p-1\right)!.$$

The result then follows by (2.12) and (2.11). □

Proof of Theorem 1.11. Writing $n=p+2$ for an odd prime p in (2.13) gives

$$a\left(p+2,p-1\right)=p\left(p!\right)+2\left(p!\right)H_{p}=p\left(p!\right)+2\left(p!\right)H_{p-1}+2\left(p-1\right)!.$$

Now, Wolstenholme's theorem and (2.12) yield the result. □

Proof of Theorem 1.12. For an odd prime p, we write $n = p+2$ and $k = 4$ in (2.10) to obtain

$$a(p+2, p-2) = P_{4,1}(p+2)\, s(p+1,1) + P_{4,2}(p+2)\, s(p+1,2) + P_{4,3}(p+2)\, s(p+1,3).$$

Since

$$P_{4,1}(p+2) = \binom{p+1}{2}, \quad P_{4,2}(p+2) = 2p-3, \quad P_{4,3}(p+2) = 6,$$

and

$$s(p+1,1) = p!, \quad s(p+1,2) = p!H_p, \quad s(p+1,3) = \frac{p!}{2}\left(H_p^2 - H_p^{(2)}\right),$$

we find that

$$\begin{aligned} a(p+2, p-2) &= \frac{(p+1)\,p}{2}(p!) + (2p-3)(p!)\left(H_{p-1} + \frac{1}{p}\right) \\ &\quad + 3(p!)\left(\left(H_{p-1} + \frac{1}{p}\right)^2 - \left(H_{p-1}^{(2)} + \frac{1}{p^2}\right)\right) \\ &= \frac{(p+1)\,p}{2}(p!) + (2p-3)(p!)\,H_{p-1} + (2p-3)(p-1)! \\ &\quad + 3(p!)\,H_{p-1}^2 + 6(p-1)!H_{p-1} - 3(p!)\,H_{p-1}^{(2)}. \end{aligned}$$

Now, for $p \geq 3$, $H_{p-1} \equiv 0 \pmod{p}$ and for $p > 3$, $H_{p-1} \equiv H_{p-1}^{(2)} \equiv 0 \pmod{p^2}$. Thus, for $p > 3$,

$$a(p+2, p-2) \equiv (2p-3)(p-1)! \equiv p - 3pB_{p-1} \pmod{p^2}$$

by (2.12) and (2.11). Note that $a(5,1) = 105$, $3 - pB_2 = \frac{3}{2}$, and $105 \equiv 6 \equiv \frac{3}{2} \pmod{9}$. □

Proof of Theorem 1.13. For an odd prime $p > 3$, we let $n = p+2$ and $k = 5$ in (2.10) to obtain

$$\begin{aligned} a(p+2, p-3) &= P_{5,1}(p+2)\, s(p+1,1) + P_{5,2}(p+2)\, s(p+1,2) \\ &\quad + P_{5,3}(p+2)\, s(p+1,3) + P_{5,4}(p+2)\, s(p+1,4). \end{aligned}$$

Since

$$P_{5,1}(p+2) = \binom{p+2}{3}, \quad P_{5,2}(p+2) = p(p-2), \quad P_{5,3}(p+2) = 6p-20, \quad P_{5,4}(p+2) = 24,$$

and

$$\begin{aligned} &s(p+1,1) = p!, \quad s(p+1,2) = p!H_p, \quad s(p+1,3) = \frac{p!}{2}\left(H_p^2 - H_p^{(2)}\right), \\ &s(p+1,4) = -\frac{p!}{3!}\left(H_p^3 - 3H_pH_p^{(2)} + 2H_p^{(3)}\right), \end{aligned}$$

we find that

$$\begin{aligned} a\left(p+2,p-3\right) &= \frac{\left(p+2\right)\left(p+1\right)p}{6}\left(p!\right)+p\left(p-2\right)\left(p!\right)H_{p-1}+p\left(p-2\right)\left(p-1\right)! \\ &\quad -\left(3p-10\right)\left(p!\right)H_{p-1}^{2}-\left(6p-20\right)\left(p-1\right)!H_{p-1}+\left(3p-10\right)\left(p!\right)H_{p-1}^{(2)} \\ &\quad +4\left(p!\right)\left(H_{p-1}^{3}+\frac{3H_{p-1}^{2}}{p}-3H_{p-1}H_{p-1}^{(2)}-\frac{3H_{p-1}^{(2)}}{p}+2H_{p-1}^{(3)}\right). \end{aligned}$$

Now, by Wolstenholme's theorem, (1.9), (2.12) and (2.11) we obtain

$$a\left(p+2,p-3\right)\equiv 2p-12pH_{p-1}^{(2)}B_{p-1}\left(\mathrm{mod}\,p^{2}\right),$$

which is the desired result. □

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