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2k-TH POWER MEAN VALUE OF THE GENERALIZED CUBIC GAUSS SUMS

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Abstract. This paper establishes explicit evaluations of the $2k$-th power mean for generalized cubic Gauss sums. By exploiting analytic techniques and fundamental properties of classical Gauss sums, we derive closed-form expressions for these means. Furthermore, we develop a computationally efficient framework for analyzing higher-order moments of such sums.

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1. Introduction

1.1. Background

Let $q \geq 3$ be a positive integer and any Dirichlet character ψ mod q, the generalized Gauss sums $\mathcal{G}\left(n,s,\psi;q\right)$ are described as follows:

$$\mathcal{G}\left(n,s,\psi;q\right)=\sum_{u=1}^{q}\psi\left(u\right)e\left(\frac{nu^{s}}{q}\right),$$

defining $e\left(y\right)=\mathrm{e}^{2\pi \mathrm{i} y}$, $\mathrm{i}^2=-1$, s is any positive integer and n is an integer with $(n,q)=1$.

Clearly, these sums generalize the classical Gauss sums $\mathcal{G}\left(1,1,\psi;q\right)$. The classical Gauss sums and their properties are greatly significant, as they are closely related to many problems in number theory. It is therefore necessary to explore the properties of the sum $\mathcal{G}\left(1,1,\psi;q\right)$ and other closely related sums.

The study of such sums has extensively continued for a considerable length of time, and many scholars have contributed significant results in this area. Exponential sums, particularly Gauss sums, paly a important role in analytic number theory, with their fundamental importance rooted in the estimates for individual sums as well as the weighted sum of them.

For example, Weil [1] proved that when $p \geq 3$ is a prime, then

$$\left|\mathcal{G}\left(n,s,\psi;p\right)\right| \leq (s-1)p^{\frac{1}{2}}.$$

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In fact, Cochrane and Zheng [2] generalized this result and proved that when $q \geq 3$ is any integer, then

$$|\mathcal{G}(n, s, \psi; q)| \leq (s-1)^{\omega(q)} \cdot q^{\frac{1}{2}},$$

where $\omega(q)$ represents the number of distinct primes dividing q.

For the value s=2, Zhang [3] gave the following two results

$$\frac{1}{l-1} \sum_{\psi \bmod l} |\mathcal{G}(n, 2, \psi; l)|^4 = \begin{cases} 3l^2 - 6l + 4\left(\frac{n}{l}\right)\sqrt{l} - 1, & \text{if } l \equiv 1 \bmod 4, \\ 3l^2 - 6l - 1, & \text{if } l \equiv 3 \bmod 4, \end{cases}$$

and

$$\frac{1}{l-1} \sum_{\psi \bmod l} |\mathcal{G}(n, 2, \psi; l)|^6 = 10l^3 - 25l^2 - 4l - 1, l \equiv 3 \bmod 4,$$

where $l \in \mathbb{P}$, $\mathbb{P}$ stand for the set of all primes and $\left(\frac{*}{l}\right)$ stands for the Legendre symbol modulo l. In the following sections, the letter l always represents a prime number.

In 2021, Bag and Barman [4] proved the asymptotic formulae

$$\sum_{\psi \bmod l} |\mathcal{G}(n, 2, \psi; l)|^6 = 10l^4 + O\left(l^{\frac{7}{2}}\right)$$

and

$$\sum_{\psi \bmod l} |\mathcal{G}(n, 2, \psi; l)|^8 = 35l^5 + O\left(l^{\frac{9}{2}}\right).$$

During the following two years, Bag, Rojas-Léon and Zhang [5, 6] proved

$$\sum_{\psi \bmod l} |\mathcal{G}(n, 2, \psi; l)|^{10} = 126l^6 + O\left(l^{\frac{11}{2}}\right)$$

and

$$\sum_{\psi \bmod l} |\mathcal{G}(n, 2, \psi; l)|^{2k} = \left(\begin{array}{c} 2k-1 \\ k \end{array}\right) l^{k+1} + O\left(l^{\frac{2k+1}{2}}\right).$$

For s=3, Zhang and Liu [7] yielded

$$\sum_{\psi \bmod l} |\mathcal{G}(1, 3, \psi; l)|^4 = 5l^3 - 18l^2 + 20l + 1 + \frac{V^5}{l} + 5lV - 5V^3 - 4V^2 + 4V,$$

where $l \equiv 1 \bmod 3$ and $V = \sum_{u=0}^{l-1} e\left(\frac{u^3}{l}\right)$ is real-valued.

In 2023, Liu and Meng [8] proved the k-th power mean value of one kind generalized cubic Gauss sums $\mathcal{G}(n,3,\psi;l)$

$$\sum_{\psi \bmod l}\sum_{n=0}^{l-1}\left(\mathcal{G}(n,3,\psi;l)\right)^k=(l-1)^k+(l-1)\,S_{k-1}(l),$$

where

$$S_{k-1}(l)=\sum_{n=1}^{l-1}\left(A(n)-1\right)^{k-1}e\left(\frac{n}{l}\right),\,A(n)=\sum_{u=1}^{l-1}e\left(\frac{nu^3}{l}\right)$$

and satisfies

$$S_{k-1}(l)=-3S_{k-2}(l)+3(l-1)\,S_{k-3}(l)+(rl+3l-1)\,S_{k-4}(l)\,.$$

In 2024, Bag [9] investigated generalized cubic Gauss sums and proved the following results:

$$\sum_{\psi \bmod l}\left|\mathcal{G}(n,3,\psi;l)\right|^2=(l-1)^2$$

and

$$\sum_{\psi \bmod l}\left|\mathcal{G}(n,3,\psi;l)\right|^4=5l^3+O\left(l^{\frac{5}{2}}\right),$$

where $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$.

There **are a number of results about** Gauss sums and their recursive properties, as shown in [10–12], [13], [17]. In this paper, we will focus on the $2k$-th power mean value of a certain type of generalized cubic Gauss sums

$$\sum_{n=1}^{l-1}\left|\sum_{u=1}^{l-1}\psi_2(u)\,e\left(\frac{nu^3}{l}\right)\right|^{2k},$$

where $\psi_2=\left(\frac{*}{l}\right)$.

As far as we know, the problem described above has not been studied in the existing literature. Thus, applying the analytic approaches, the properties of Gauss sums, and the theory about trigonometric or exponential sums, we derive a set of identities. That is, the results of our analysis are summarized below:

1.2. Our results

Theorem 1.1. *Let $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$. Then*

$$\sum_{n=1}^{l-1}\left|\sum_{u=1}^{l-1}\psi_2(u)\,e\left(\frac{nu^3}{l}\right)\right|^{2}=3l^2-3l,$$

where $\psi_2(*)=\left(\frac{*}{l}\right)$ denotes the Legendre symbol modulo l.

Theorem 1.2. *Let $l \in \mathbb{P}$, $l \equiv 1 \bmod 3$. Then*

$$\sum_{n=1}^{l-1} \left| \sum_{u=1}^{l-1} \psi_2(u)\, e\left(\frac{nu^3}{l}\right) \right|^4 = 11l^3 + \left(4r^2 - 11\right) l^2 - 4r^2 l,$$

where r is uniquely determined by $4l = r^2 + 27t^2$ and $r \equiv 1 \bmod 3$.

Theorem 1.3. *Let $l \in \mathbb{P}$, $l \equiv 1 \bmod 3$. Then*

$$\sum_{n=1}^{l-1} \left| \sum_{u=1}^{l-1} \psi_2(u)\, e\left(\frac{nu^3}{l}\right) \right|^6 = 43l^4 + \left(46r^2 - 43\right) l^3 + \left(r^4 - 46r^2\right) l^2 - r^4 l.$$

Theorem 1.4. *Let $l \in \mathbb{P}$, $l \equiv 1 \bmod 3$. Then*

$$\sum_{n=1}^{l-1} \left| \sum_{u=1}^{l-1} \psi_2(u)\, e\left(\frac{nu^3}{l}\right) \right|^8 = 171l^5 + \left(360r^2 - 171\right) l^4 + \left(36r^4 - 360r^2\right) l^3 - 36r^4 l^2.$$

Remark. Firstly, primes where $l \equiv 2 \bmod 3$ were omitted from consideration, as the result in this scenario is trivial. In this case, the variable u passes through a complete set of residues modulo l, u^3 does likewise. Thus, given this situation, we can obtain the identities

$$\sum_{u=1}^{l-1} \psi_2(u)\, e\left(\frac{nu^3}{l}\right) = \sum_{u=1}^{l-1} \psi_2\left(u^3\right) e\left(\frac{nu^3}{l}\right) = \sum_{u=1}^{l-1} \psi_2(u)\, e\left(\frac{nu}{l}\right) = \psi_2(n)\, \mathcal{B}(\psi_2)$$

and

$$\sum_{n=1}^{l-1} \left| \sum_{u=1}^{l-1} \psi_2(u)\, e\left(\frac{nu^3}{l}\right) \right|^{2k} = \sum_{n=1}^{l-1} \left|\psi_2(n)\, \mathcal{B}(\psi_2)\right|^{2k} = l^{k+1} - l^k,$$

where $\mathcal{B}(\psi) = \sum_{u=1}^{l-1} \psi(u)\, e\left(\frac{u}{l}\right)$ stands for the classical Gauss sums.

Moreover, in fact, for any positive integer k, we can obtain corresponding results by our methods.

Example. To illustrate the validity of our formulas, we provide a numerical example with the prime $l = 1009$ (satisfying $l \equiv 1 \bmod 3$). The corresponding parameter r, determined by the relation $4l = r^2 + 27t^2$ and $r \equiv 1 \bmod 3$, is found to be $r = 1849$. Substituting into the expressions in Theorems 1.1–1.4, we obtain:

$$\sum_{n=1}^{1008} \left| \sum_{u=1}^{1008} \psi_2(u)\, e\left(\frac{nu^3}{1009}\right) \right|^2 = 3.051216 \times 10^6,$$

$$\sum_{n=1}^{1008} \left| \sum_{u=1}^{1008} \psi_2(u)\, e\left(\frac{nu^3}{1009}\right) \right|^4 = 1.881074664 \times 10^{10},$$

$$\sum_{n=1}^{1008} \left| \sum_{u=1}^{1008} \psi_2(u)\, e\left(\frac{nu^3}{1009}\right) \right|^6 = 1.352866152 \times 10^{14},$$

$$\sum_{n=1}^{1008}\left|\sum_{u=1}^{1008}\psi_2(u)\, e\left(\frac{nu^3}{1009}\right)\right|^8 = 9.942068403\times 10^{17}.$$

These numerical results are in excellent agreement with the theoretical formulas, further validating the correctness of the theorems established in this paper.

2. Some Lemmas

This section provides some fundamental lemmas and their proofs. In addition, specific attributes of the Gauss sums and theory of Dirichlet character sums are summarized in [14, 15].

Lemma 2.1. *Let $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$. Then, for an arbitrary third-order Dirichlet character ψ_3 modulo l, it follows that:*

$$\mathcal{B}^3(\psi_3)+\mathcal{B}^3\left(\overline{\psi}_3\right)=rl,$$

where $\mathcal{B}(\psi)=\sum\limits_{u=1}^{l-1}\psi(u)\, e\left(\frac{u}{l}\right)$ stands for the classical Gauss sums, and we adopt the same definition of r as in Theorem 1.1.

Proof. See [12] or [16]. □

Lemma 2.2. *Let $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$. Then, for an arbitrary third-order Dirichlet character ψ_3 modulo l, it follows that:*

$$\mathcal{B}^6(\psi_3)+\mathcal{B}^6\left(\overline{\psi}_3\right)=r^2l^2-2l^3.$$

Proof. Based on Lemma 2.1 and $\mathcal{B}(\psi_3)\,\mathcal{B}\left(\overline{\psi}_3\right)=l$, we can obtain

$$\mathcal{B}^6(\psi_3)+\mathcal{B}^6\left(\overline{\psi}_3\right)=\left(\mathcal{B}^3(\psi_3)+\mathcal{B}^3\left(\overline{\psi}_3\right)\right)^2-2\mathcal{B}^3(\psi_3)\,\mathcal{B}^3\left(\overline{\psi}_3\right)=r^2l^2-2l^3.$$

This proves Lemma 2.2. □

Lemma 2.3. *Let $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$. Then, for an arbitrary third-order Dirichlet character ψ_3 modulo l, it follows that:*

$$\mathcal{B}^{12}(\psi_3)+\mathcal{B}^{12}\left(\overline{\psi}_3\right)=2l^6-4r^2l^5+r^4l^4.$$

Proof. Based on Lemma 2.2 and $\mathcal{B}(\psi_3)\,\mathcal{B}\left(\overline{\psi}_3\right)=l$, we can obtain

$$\begin{aligned}\mathcal{B}^{12}(\psi_3)+\mathcal{B}^{12}\left(\overline{\psi}_3\right)&=\left(\mathcal{B}^6(\psi_3)+\mathcal{B}^6\left(\overline{\psi}_3\right)\right)^2-2\mathcal{B}^6(\psi_3)\,\mathcal{B}^6\left(\overline{\psi}_3\right)\\&=\left(r^2l^2-2l^3\right)^2-2l^6=2l^6-4r^2l^5+r^4l^4.\end{aligned}$$

The proof of Lemma 2.3 is completed. □

Lemma 2.4. *Let $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$, then*

$$\mathcal{B}(\psi_2\psi_3)=\frac{\psi_3(2)\,\mathcal{B}(\psi_2)\,\mathcal{B}^2\left(\overline{\psi}_3\right)}{l}.$$

Proof. On one hand, we get

$$\begin{aligned}\sum_{u=0}^{l-1}\overline{\psi}_3\left(u^2-1\right)&=\sum_{u=0}^{l-1}\overline{\psi}_3\left(\left(u+1\right)^2-1\right)=\sum_{u=0}^{l-1}\overline{\psi}_3\left(u\right)\overline{\psi}_3\left(u+2\right)\\&=\frac{1}{\mathcal{B}\left(\psi_3\right)}\sum_{v=1}^{l-1}\psi_3\left(v\right)\sum_{u=1}^{l-1}\overline{\psi}_3\left(u\right)e\left(\frac{v\left(u+2\right)}{l}\right)\\&=\frac{\mathcal{B}\left(\overline{\psi}_3\right)}{\mathcal{B}\left(\psi_3\right)}\sum_{v=1}^{l-1}\overline{\psi}_3\left(v\right)e\left(\frac{2v}{l}\right)\\&=\frac{\psi_3\left(2\right)\mathcal{B}^2\left(\overline{\psi}_3\right)}{\mathcal{B}\left(\psi_3\right)}=\frac{\psi_3\left(2\right)\mathcal{B}^3\left(\overline{\psi}_3\right)}{l}.\end{aligned}\tag{2.1}$$

Alternatively, we have

$$\begin{aligned}\sum_{u=0}^{l-1}\overline{\psi}_3\left(u^2-1\right)&=\frac{1}{\mathcal{B}\left(\psi_3\right)}\sum_{v=1}^{l-1}\psi_3\left(v\right)\sum_{u=0}^{l-1}e\left(\frac{v\left(u^2-1\right)}{l}\right)\\&=\frac{1}{\mathcal{B}\left(\psi_3\right)}\sum_{v=1}^{l-1}\psi_3\left(v\right)e\left(\frac{-v}{l}\right)\sum_{u=0}^{l-1}e\left(\frac{u^2v}{l}\right)\\&=\frac{1}{\mathcal{B}\left(\psi_3\right)}\sum_{v=1}^{l-1}\psi_3\left(v\right)e\left(\frac{-v}{l}\right)\left(1+\sum_{u=1}^{l-1}\left(1+\psi_2\left(u\right)\right)e\left(\frac{uv}{l}\right)\right)\\&=\frac{1}{\mathcal{B}\left(\psi_3\right)}\sum_{v=1}^{l-1}\psi_3\left(v\right)e\left(\frac{-v}{l}\right)\sum_{u=1}^{l-1}\psi_2\left(u\right)e\left(\frac{uv}{l}\right)\\&=\frac{\mathcal{B}\left(\psi_2\right)}{\mathcal{B}\left(\psi_3\right)}\sum_{v=1}^{l-1}\psi_2\psi_3\left(v\right)e\left(\frac{-v}{l}\right)\\&=\frac{\psi_2\left(-1\right)\mathcal{B}\left(\psi_2\right)\mathcal{B}\left(\psi_2\psi_3\right)}{\mathcal{B}\left(\psi_3\right)}\\&=\frac{\psi_2\left(-1\right)\mathcal{B}\left(\psi_2\right)\mathcal{B}\left(\psi_2\psi_3\right)\mathcal{B}\left(\overline{\psi}_3\right)}{l}.\end{aligned}\tag{2.2}$$

Note that $\mathcal{B}^2\left(\psi_2\right)=\psi_2\left(-1\right)p$, combining (1) and (2) we obtain

$$\mathcal{B}\left(\psi_2\psi_3\right)=\frac{\psi_3\left(2\right)\mathcal{B}\left(\psi_2\right)\mathcal{B}^2\left(\overline{\psi}_3\right)}{l}.$$

This verifies Lemma 2.4. □

Lemma 2.5. *Let $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$, this yields*

$$\mathcal{B}^3\left(\psi_2\psi_3\right)+\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)=\begin{cases} l^{\frac{1}{2}}\left(r^2-2l\right), & \text{if } l\equiv 1 \bmod 12,\\ -\mathrm{i}l^{\frac{1}{2}}\left(r^2-2l\right), & \text{if } l\equiv 7 \bmod 12,\end{cases}$$

where the symbol r here is identical to that introduced in Theorem 1.1.

Proof. An application of Lemma 2.4 shows that

$$\begin{aligned}
&\mathcal{B}^3\left(\psi_2\psi_3\right)+\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)\\
=&\frac{\mathcal{B}^3\left(\psi_2\right)\mathcal{B}^6\left(\overline{\psi}_3\right)}{l^3}+\frac{\mathcal{B}^3\left(\psi_2\right)\mathcal{B}^6\left(\psi_3\right)}{l^3}\\
=&\frac{\mathcal{B}^3\left(\psi_2\right)\left(\mathcal{B}^6\left(\overline{\psi}_3\right)+\mathcal{B}^6\left(\psi_3\right)\right)}{l^3}\\
=&\frac{\mathcal{B}^3\left(\psi_2\right)\left[\left(\mathcal{B}^3\left(\psi_3\right)+\mathcal{B}^3\left(\overline{\psi}_3\right)\right)^2-2\mathcal{B}^3\left(\psi_3\right)\mathcal{B}^3\left(\overline{\psi}_3\right)\right]}{l^3}\\
=&\frac{\mathcal{B}^3\left(\psi_2\right)\left(r^2l^2-2l^3\right)}{l^3}\\
=&\begin{cases} l^{\frac{1}{2}}\left(r^2-2l\right), & \text{if } l\equiv 1 \bmod 12,\\ -\mathrm{i}l^{\frac{1}{2}}\left(r^2-2l\right), & \text{if } l\equiv 7 \bmod 12.\end{cases}
\end{aligned}$$

Thus, we have demonstrated Lemma 2.5. □

3. Proof of Theorems

We now finish the proof of Theorems 1.1–1.4 by applying Lemmas 2.1–2.5. Let $l\in\mathbb{P}$, $l\equiv 1 \bmod 3$, and we write

$$A\left(n\right)=\sum_{u=1}^{l-1}\psi_2\left(u\right)e\left(\frac{nu^3}{l}\right).$$

Combining the following trigonometric identity

$$\sum_{n=0}^{l-1}e\left(\frac{nu}{l}\right)=\begin{cases} l, & \text{if } l\mid u,\\ 0, & \text{if } l\nmid u,\end{cases} \tag{3.1}$$

with the characteristics of the Gauss sums, we can get

$$\begin{aligned}
A\left(n\right)&=\sum_{u=1}^{l-1}\psi_2\left(u\right)e\left(\frac{nu^3}{l}\right)=\sum_{u=1}^{l-1}\psi_2\left(u^3\right)e\left(\frac{nu^3}{l}\right)\\
&=\sum_{u=1}^{l-1}\psi_2\left(u\right)\left(1+\psi_3\left(u\right)+\overline{\psi}_3\left(u\right)\right)e\left(\frac{nu}{l}\right)\\
&=\psi_2\left(n\right)\left(\mathcal{B}\left(\psi_2\right)+\overline{\psi}_3\left(n\right)\mathcal{B}\left(\psi_2\psi_3\right)+\psi_3\left(n\right)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right).
\end{aligned} \tag{3.2}$$

First, we start with the proof of Theorem 1.1. For an arbitrary nonprincipal Dirichlet character ψ modulo l,

$$\sum_{n=1}^{l-1}\psi\left(n\right)=0. \tag{3.3}$$

Combining (3.2), (3.3) and Lemma 2.4, we may immediately get

$$\begin{aligned}\sum_{n=1}^{l-1}|A(n)|^2 &= \sum_{n=1}^{l-1}\left|\mathcal{B}(\psi_2)+\overline{\psi}_3(n)\mathcal{B}(\psi_2\psi_3)+\psi_3(n)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &= (l-1)|\mathcal{B}(\psi_2)|^2+(l-1)|\mathcal{B}(\psi_2\psi_3)|^2+(l-1)\left|\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &= 3l(l-1)=3l^2-3l.\end{aligned}$$

This proves Theorem 1.1.

We now verify Theorem 1.2. Combining (3.2) and (3.3), we get

$$\begin{aligned}\sum_{n=1}^{l-1}|A(n)|^4 &= \sum_{n=1}^{l-1}\left|A^2(n)\right|^2\\ &= (l-1)\left|\mathcal{B}^2(\psi_2)+2\mathcal{B}(\psi_2\psi_3)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &\quad+(l-1)\left|\mathcal{B}^2(\psi_2\psi_3)+2\mathcal{B}(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &\quad+(l-1)\left|\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)+2\mathcal{B}(\psi_2)\mathcal{B}(\psi_2\psi_3)\right|^2.\end{aligned}\tag{3.4}$$

The classical Gauss sums' properties immediately imply

$$\left|\mathcal{B}^2(\psi_2)+2\mathcal{B}(\psi_2\psi_3)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2=|\psi_2(-1)l+2\psi_2(-1)l|^2=9l^2.\tag{3.5}$$

Employing Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}&\left|\mathcal{B}^2(\psi_2\psi_3)+2\mathcal{B}(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ =&\left|\frac{\psi_3(4)\mathcal{B}^2(\psi_2)\mathcal{B}^4\left(\overline{\psi}_3\right)}{l^2}+\frac{2\overline{\psi}_3(2)\mathcal{B}^2(\psi_2)\mathcal{B}^2(\psi_3)}{l}\right|^2\\ =&\,5l^2+\frac{2\left(\mathcal{B}^6(\psi_3)+\mathcal{B}^6\left(\overline{\psi}_3\right)\right)}{l}\\ =&\,5l^2+\frac{2\left(r^2l^2-2l^3\right)}{l}\\ =&\,l^2+2r^2l.\end{aligned}\tag{3.6}$$

Similarly,

$$\left|\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)+2\mathcal{B}(\psi_2)\mathcal{B}(\psi_2\psi_3)\right|^2=l^2+2r^2l.\tag{3.7}$$

From (3.4)–(3.7), we can conclude

$$\sum_{n=1}^{l-1}|A(n)|^4=11l^3+\left(4r^2-11\right)l^2-4r^2l.$$

The proof of Theorem 1.2 is completed.

We now turn to verify Theorem 1.3. From (3.2) and (3.3), we have

$$\begin{aligned}\sum_{n=1}^{l-1}\left|A(n)\right|^6=\sum_{n=1}^{l-1}\left|A^3(n)\right|^2=\sum_{n=1}^{l-1}\left|\psi_2(n)A^3(n)\right|^2\\ =(l-1)\left|\mathcal{B}^3(\psi_2)+\mathcal{B}^3(\psi_2\psi_3)+\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)+6\mathcal{B}(\psi_2)\mathcal{B}(\psi_2\psi_3)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ +(l-1)\left|3\mathcal{B}^2(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+3\mathcal{B}(\psi_2)\mathcal{B}^2(\psi_2\psi_3)+3\mathcal{B}(\psi_2\psi_3)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)\right|^2\\ +(l-1)\left|3\mathcal{B}^2(\psi_2)\mathcal{B}(\psi_2\psi_3)+3\mathcal{B}(\psi_2)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)+3\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\mathcal{B}^2(\psi_2\psi_3)\right|^2.\end{aligned}\tag{3.8}$$

Utilizing Lemma 2.5 and the known properties of the Gauss sums, we get

$$\begin{aligned}&\left|\mathcal{B}^3(\psi_2)+\mathcal{B}^3(\psi_2\psi_3)+\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)+6\mathcal{B}(\psi_2)\mathcal{B}(\psi_2\psi_3)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &=\left|\mathcal{B}^3(\psi_2)+6\mathcal{B}(\psi_2)\psi_2(-1)\,l+\mathcal{B}^3(\psi_2\psi_3)+\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &=\left|7\mathcal{B}^3(\psi_2)+\mathcal{B}^3(\psi_2\psi_3)+\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &=\left(7l^{\frac{3}{2}}+r^2l^{\frac{1}{2}}-2l^{\frac{3}{2}}\right)^2\\ &=25l^3+10r^2l^2+r^4l.\end{aligned}\tag{3.9}$$

Employing Lemma 2.2 and Lemma 2.4, then

$$\begin{aligned}&\left|3\mathcal{B}^2(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+3\mathcal{B}(\psi_2)\mathcal{B}^2(\psi_2\psi_3)+3\mathcal{B}(\psi_2\psi_3)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &=\left|3\mathcal{B}^2(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+3\mathcal{B}(\psi_2)\mathcal{B}^2(\psi_2\psi_3)+3\psi_2(-1)\,l\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\right|^2\\ &=\left|6\mathcal{B}^2(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+3\mathcal{B}(\psi_2)\mathcal{B}^2(\psi_2\psi_3)\right|^2\\ &=\left|\frac{6\overline{\psi}_3(2)\mathcal{B}^3(\psi_2)\mathcal{B}^2(\psi_3)}{l}+\frac{3\overline{\psi}_3(2)\mathcal{B}^3(\psi_2)\mathcal{B}^4\left(\overline{\psi}_3\right)}{l^2}\right|^2\\ &=\left|\frac{3\overline{\psi}_3(2)\mathcal{B}^3(\psi_2)}{l}\left(2\mathcal{B}^2(\psi_3)+\frac{\mathcal{B}^4\left(\overline{\psi}_3\right)}{l}\right)\right|^2\\ &=9l\left(5l^2+\frac{2\left(\mathcal{B}^6(\psi_3)+\mathcal{B}^6\left(\overline{\psi}_3\right)\right)}{l}\right)\\ &=9l^3+18r^2l^2.\end{aligned}\tag{3.10}$$

Similarly,

$$\left|3\mathcal{B}^2(\psi_2)\mathcal{B}(\psi_2\psi_3)+3\mathcal{B}(\psi_2)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)+3\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\mathcal{B}^2(\psi_2\psi_3)\right|^2=9l^3+18r^2l^2.\tag{3.11}$$

From (3.8)–(3.11), we obtain

$$\sum_{n=1}^{l-1}\left|A(n)\right|^6=43l^4+\left(46r^2-43\right)l^3+\left(r^4-46r^2\right)l^2-r^4l.$$

With this identity, we have now proven Theorem 1.3.

Theorem 1.4 is proved as follows. Applying the same technique, we arrive at

$$\begin{aligned}
&\sum_{n=1}^{l-1}|A(n)|^8=\sum_{n=1}^{l-1}\left|A^4(n)\right|^2\\
=&(l-1)\,|\mathcal{B}^4(\psi_2)+4\mathcal{B}(\psi_2)\mathcal{B}^3(\psi_2\psi_3)+4\mathcal{B}(\psi_2)\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)\\
&+6\mathcal{B}^2(\psi_2\psi_3)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)+12\mathcal{B}^2(\psi_2)\mathcal{B}(\psi_2\psi_3)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)|^2\\
&+(l-1)\,|\mathcal{B}^4\left(\psi_2\overline{\psi}_3\right)+4\mathcal{B}^3(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+4\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\mathcal{B}^3(\psi_2\psi_3)\\
&+6\mathcal{B}^2(\psi_2)\mathcal{B}^2(\psi_2\psi_3)+12\mathcal{B}(\psi_2)\mathcal{B}(\psi_2\psi_3)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)|^2\\
&+(l-1)\,|\mathcal{B}^4(\psi_2\psi_3)+4\mathcal{B}^3(\psi_2)\mathcal{B}(\psi_2\psi_3)+4\mathcal{B}(\psi_2\psi_3)\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)\\
&+6\mathcal{B}^2(\psi_2)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)+12\mathcal{B}(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\mathcal{B}^2(\psi_2\psi_3)|^2. \qquad (3.12)
\end{aligned}$$

From Lemmas 2.2–2.5, then

$$\begin{aligned}
&|\mathcal{B}^4(\psi_2)+4\mathcal{B}(\psi_2)\mathcal{B}^3(\psi_2\psi_3)+4\mathcal{B}(\psi_2)\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)+6\mathcal{B}^2(\psi_2\psi_3)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)\\
&+12\mathcal{B}^2(\psi_2)\mathcal{B}(\psi_2\psi_3)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)|^2\\
=&\left|19l^2+4\mathcal{B}(\psi_2)\left(\mathcal{B}^3(\psi_2\psi_3)+\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)\right)\right|^2\\
=&\left(19l^2+4l\left(r^2-2l\right)\right)^2\\
=&121l^4+88r^2l^3+16r^4l^2 \qquad (3.13)
\end{aligned}$$

and

$$\begin{aligned}
&|\mathcal{B}^4\left(\psi_2\overline{\psi}_3\right)+4\mathcal{B}^3(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+4\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\mathcal{B}^3(\psi_2\psi_3)+6\mathcal{B}^2(\psi_2)\mathcal{B}^2(\psi_2\psi_3)\\
&+12\mathcal{B}(\psi_2)\mathcal{B}(\psi_2\psi_3)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)|^2\\
=&|\mathcal{B}^4\left(\psi_2\overline{\psi}_3\right)+4\mathcal{B}^3(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+4\psi_2(-1)\,l\mathcal{B}^2(\psi_2\psi_3)+6\mathcal{B}^2(\psi_2)\mathcal{B}^2(\psi_2\psi_3)\\
&+12\mathcal{B}(\psi_2)\psi_2(-1)\,l\mathcal{B}\left(\psi_2\overline{\psi}_3\right)|^2\\
=&|\mathcal{B}^4\left(\psi_2\overline{\psi}_3\right)+4\mathcal{B}^3(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+4\mathcal{B}^2(\psi_2)\mathcal{B}^2(\psi_2\psi_3)+6\mathcal{B}^2(\psi_2)\mathcal{B}^2(\psi_2\psi_3)\\
&+12\mathcal{B}^3(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)|^2\\
=&\left|\mathcal{B}^4\left(\psi_2\overline{\psi}_3\right)+16\mathcal{B}^3(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)+10\mathcal{B}^2(\psi_2)\mathcal{B}^2(\psi_2\psi_3)\right|^2\\
=&\left|\frac{\overline{\psi}_3(2)\mathcal{B}^4(\psi_2)\mathcal{B}^8(\psi_3)}{l^4}+\frac{16\overline{\psi}_3(2)\mathcal{B}^4(\psi_2)\mathcal{B}^2(\psi_3)}{l}+\frac{10\overline{\psi}_3(2)\mathcal{B}^4(\psi_2)\mathcal{B}^4\left(\overline{\psi}_3\right)}{l^2}\right|^2\\
=&\left|\frac{\overline{\psi}_3(2)\mathcal{B}^8(\psi_3)}{l^2}+16l\overline{\psi}_3(2)\mathcal{B}^2(\psi_3)+10\overline{\psi}_3(2)\mathcal{B}^4\left(\overline{\psi}_3\right)\right|^2\\
=&357l^4+176l\left(\mathcal{B}^6(\psi_3)+\mathcal{B}^6\left(\overline{\psi}_3\right)\right)+\frac{10\left(\mathcal{B}^{12}(\psi_3)+\mathcal{B}^{12}\left(\overline{\psi}_3\right)\right)}{l^2}\\
=&25l^4+136r^2l^3+10r^4l^2. \qquad (3.14)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|\mathcal{B}^4(\psi_2\psi_3)+4\mathcal{B}^3(\psi_2)\mathcal{B}(\psi_2\psi_3)+4\mathcal{B}(\psi_2\psi_3)\mathcal{B}^3\left(\psi_2\overline{\psi}_3\right)+6\mathcal{B}^2(\psi_2)\mathcal{B}^2\left(\psi_2\overline{\psi}_3\right)\\
&+12\mathcal{B}(\psi_2)\mathcal{B}\left(\psi_2\overline{\psi}_3\right)\mathcal{B}^2(\psi_2\psi_3)|^2\\
=&25l^4+136r^2l^3+10r^4l^2. \qquad (3.15)
\end{aligned}$$

From (3.12)–(3.15) we can obtain

$$\sum_{n=1}^{l-1} |A(n)|^8 = 171l^5 + \left(360r^2 - 171\right) l^4 + \left(36r^4 - 360r^2\right) l^3 - 36r^4 l^2.$$

This proves Theorem 1.4. Thus, we complete the demonstrations of all our theorems.

Conclusions

This study establishes three main theorems for the $2k$-th power mean value of generalized cubic Gauss sums. These findings provide closed-form evaluations of these sums, advancing the study of exponential sums in analytic number theory.

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Conflicts of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Author contribution statement

All authors have equally contributed to this work. All authors read and approved the final manuscript.

Data availability statement

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