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A DIOPHANTINE INEQUALITY WITH ONE PRIME OF THE FORM $p = m^2 + n^2 + 1$

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Abstract. Let $1 < c < \frac{300s+31}{200s+64}$ be fixed. Assume that $N > 0$ is a large enough number and $\varepsilon > 0$ is an arbitrarily small constant. This paper establish that the Diophantine inequality

$$|p_1^c + \cdots + p_s^c - N| < \varepsilon$$

has a solution in prime numbers $p_1, \cdots, p_s$, where $s \geq 7$ and s is a natural number and p_1 can be expressed as $p_1 = m^2 + n^2 + 1$ for some integers m, n.

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1. Introduction

The study of the Diophantine equation of prime solution is an old and important subject in number theory. It involves the methods of dealing with the Diophantine equation and the exponential sum of prime variables, which is of great significance to deepen the study of number theory. At all times and in all over the world, many mathematicians have studied the related problems of Diophantine equation and achieved good results ([25–27]). The bourgain Gamburd Sarnak conjecture (BGS conjecture) asserts that a class of Diophantine equations have enough prime solutions if and only if they have appropriate local solutions. Some problems of Diophantine equation can be solved by exponential sum method (see [7]), which is also widely used in the study of mathematical problems.

For the purposes of this analysis, we may assume that c is a non-integer greater than 1, and N is an arbitrarily large integer. The Diophantine inequality

$$|p_1^c + \cdots + p_s^c - N| < \varepsilon \tag{1.1}$$

is solvable in primes $p_1, \cdots, p_s$, where $\varepsilon > 0$ and p_1 satisfies $p_1 = m^2 + n^2 + 1$ with integers m and n. In 2022, Dimitrov [8] showed that the problem of the Diophantine equation

$$|p_1^c + p_2^c + p_3^c - N| < \varepsilon$$

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with three prime numbers is solvable for $1 < c < \frac{427}{400}$. Furthermore, he investigated the cases of $s = 4$ and 5 for the Diophantine equations under the same set of conditions. After that,Han and Zhang [9] proved that for $1 < c < \frac{1831}{1264}$,then

$$|p_1^c + p_2^c + \cdots + p_6^c - N| < \varepsilon.$$

The above are respectively the cases when $s = 3, 4, 5, 6$ in Diophantine equation

$$|p_1^c + \cdots + p_s^c - N| < \varepsilon,$$

where $p_1 = m^2 + n^2 + 1$. We extend s to the general case, filling the gap where no research has been conducted for the situation when $s \geq 7$.

Moreover, Linnik [10] deduced the asymptotic formula as follows:

$$\sum_{p \leq Y} r(p-1) = \pi \prod_{p>2} \left(1 + \frac{\chi_4(p)}{(p-1)p}\right) \frac{Y}{\log Y} + O\left(\frac{Y(\log\log Y)^7}{(\log Y)^{1+\vartheta_0}}\right),$$

where $r(k)$ is a symbol of the count of integer solutions to the equation $k = m^2 + n^2$, $\chi_4(k)$ stands for the non-principal Dirichlet character modulo 4, and the value of

$$\vartheta_0 = \frac{1}{2} - \frac{1}{4} e \log 2 = 0.0289 \cdots . \tag{1.2}$$

In this work, we extend the study to the Diophantine inequality (1.1) involving primes of a specific type, and the main theorem is as follows.

Theorem 1.1. *For a natural number $s \geq 7$ and c satisfying $1 < c < \frac{300s+31}{200s+64}$, the inequality*

$$|p_1^c + \cdots + p_s^c - N| < \varepsilon$$

admits a solution in primes $p_1, \ldots, p_s$ for all large enough real N, where the prime p_1 can be represented in the form $m^2 + n^2 + 1$. The parameters are given by $\varepsilon = \frac{(\log\log N)^6}{(\log N)^{\vartheta_0}}$, with ϑ_0 defined in (1.2).

Furthermore, we propose the following conjecture.

Conjecture 1.2. *Suppose that $c \in (1, c')$ is fixed with $c' > 1$. If $N > 0$ is sufficiently large and $\varepsilon > 0$ is small enough, then the following inequality*

$$|p_1^c + \cdots + p_s^c - N| < \varepsilon$$

is solvable in primes $p_1, \ldots, p_s$ of the form $p_i = m_i^2 + n_i^2 + 1$ $(1 \leq i \leq s)$.

To better understand this article, we have established the following symbol explanations.

Notations. To clarify the proof of Theorem 1.1, we first fix some notations. We reserve p (with or without subscripts) for primes and η for an arbitrarily small positive constant that may vary by occurrence. Additionally, let $1 < c < \frac{300s+31}{200s+64}$ be fixed. We shall employ $\varphi(n)$ to denote Euler's totient function and $\Lambda(n)$ for the von Mangoldt function. The notation $m \sim \mathcal{M}$ means $(\frac{\mathcal{M}}{2}, \mathcal{M}]$. Congruences $m \equiv n \pmod d$ are abbreviated as $m \equiv n\,(d)$.

Let $e(u) = \exp(2\pi i u)$,

$$Y = \left(\frac{N}{2}\right)^{\frac{1}{c}},\ D = \frac{Y^{\frac{1}{2}}}{(\log Y)^{\frac{6A+34}{3}}},\ (A > 10),$$

$$\Delta = Y^{\frac{1}{4}-c},\ \varepsilon = \frac{(\log\log Y)^6}{(\log Y)^{\vartheta_0}},\ \mathcal{H} = \frac{\log^2 Y}{\varepsilon},$$

$$T_{l,d;J}(u) = \sum_{\substack{p\in J\\ p\equiv l(d)}} e(up^c)\log p,$$

$$T(u) = T_{1,1;(\mu Y,Y]}(u),\ P_J = \int_J e(uy^c)\mathrm{d}y,\ P(u) = P_{(\frac{Y}{2},Y]}(u),$$

$$\psi(y,\chi,u) = \sum_{\mu y<n\leq y} \chi(n)\Lambda(n)e(un^c),$$

$$E(y,u,d,a) = \sum_{\substack{\mu y<n\leq y\\ n\equiv a(d)}} \Lambda(n)e(un^c) - \frac{1}{\varphi(d)}\int_{\mu y}^{y} e(ux^c)\mathrm{d}x,\ 0<\mu<1.$$

2. Sketch of proof

The proof of Theorem 1 combines mathematical methods such as sieve mathod and exponential sum. The general idea is as follows:

Define the weighted sum

$$\Gamma(Y) = \sum_{\substack{\frac{Y}{2}<p_1,\cdots,p_s\leq Y\\ |p_1^c+\cdots+p_s^c-N|<\varepsilon}} r(p_1-1)\log p_1\cdots\log p_s.$$

where $r(p_1-1)$ counts representations of p_1-1 as x^2+y^2. By Linnik's equation $r(n) = 4\sum_{d|n}\chi_4(d)$, decompose $\Gamma(Y)$ into three sums: $\Gamma_1(Y)$, $\Gamma_2(Y)$, $\Gamma_3(Y)$. For integrals involving exponential sums, we estimate $P_{l,d;J}(Y)$. The Fourier transform of a smooth approximant $\vartheta(y)$ to the interval $[-\varepsilon,\varepsilon]$ is used to handle the inequality condition. The associated exponential sums, like $T_{l,d;J}(u)$ and their integrals $P_J(u)$, are analyzed by partitioning the frequency domain into major arcs ($|u| < \Delta = Y^{\frac{1}{4}-c}$), minor arcs ($\Delta \leq |u| \leq \mathcal{H} = \frac{\log^2 Y}{\varepsilon}$) and trivial arcs ($|u| > \mathcal{H}$).

On the major arcs $T(u)$ is effectively approximated by $P(u)$ (Lem. 3.3), yielding the crucial lower bound. Bounds on the minor arcs rely on sophisticated estimates for Type I and Type II exponential sums(Lems. 6.2, 6.3) and fourth-moment estimates(Lem. 6.5), combined with the Cauchy-Schwartz inequality.

$\Gamma_3(Y)$ is reduced to sums over small divisors via the substitution $d \longmapsto \frac{(p_1-1)}{m}(m < D)$ and controlled using sieve result related to the distribution of divisors (Lem. 3.13, 3.14). $\Gamma_2(Y)$ is bounded using a counting argument involving an auxiliary Diophantine inequality(Lem. 8.1) and Cauchy-Schwartz inequality. These results confirm the existence of solutions as $Y \mapsto \infty$. The specific threshold $c < \frac{300s+31}{200s+64}$ emerges as the condition ensuring the minor arc contributions are sufficiently dominated by the main term from the major arcs.

In this paper, we deal with the solvability problem of Diophantine inequality with s($s \geq 7$) primes under restricted conditions for the first time. Key innovations include the refined application of exponent pairs for bounding exponential sums in the minor arcs and the optimized treatment of the divisor constraint within the circle method framework.

3. Auxiliary lemmas

To prove Theorem 1.1, we will introduce the following lemmas as proof tools. First of all, we need to introduce the Fourier transform of function $\vartheta(y)$.

Lemma 3.1. *[11] One can construct a k-times continuously differentiable function $\vartheta(y)$ that attains 1 on $[-h+\gamma, h-\gamma]$, transitions to 0 outside $[-h-\gamma, h+\gamma]$, and is strictly between 0 and 1 in the transition intervals, i.e.*

$$\begin{aligned}&\vartheta(y)=1 \quad \text{for} \quad |y|\le h-\gamma,\\ &0<\vartheta(y)<1 \quad \text{for} \quad h-\gamma<|y|\le h+\gamma,\\ &\vartheta(y)=0 \quad \text{for} \quad |y|\ge h+\gamma.\end{aligned}$$

where $h,\gamma\in\mathbb{R}$ with $0<\gamma<\frac{h}{4}$ and $k\in\mathbb{N}$.Its Fourier transform

$$\Upsilon(x)=\int_{-\infty}^{\infty}\vartheta(y)e(-xy)\mathrm{d}y,$$

satisfies

$$|\Upsilon(x)|\le\min\left(\frac{1}{\pi|x|},2h,\frac{1}{\pi|x|}\left(\frac{k}{2\pi|x|\gamma}\right)^{k}\right).$$

In the sequel, the function $\vartheta(y)$ is fixed as the one given by Lemma 3.1 with the choices $h=\frac{9\varepsilon}{10}$, $\gamma=\frac{\varepsilon}{10}$, and $k=[\log Y]$. Its Fourier transform, following the notation of the lemma, is denoted $\Upsilon(x)$.

The following lemma presents the asymptotic formula for $\sum_{\mu Y<p\le Y}e(up^c)\log p$.

Lemma 3.2 ([12], Lem. 14)**.** *Suppose $1<c<3$, $c\neq 2$. For $|u|\le\Delta$, then we obtain*

$$\sum_{\mu Y<p\le Y}e(up^c)\log p=\int_{\mu Y}^{Y}e(uy^c)\mathrm{d}y+O\left(\frac{Y}{e^{(\log Y)^{1/5}}}\right).$$

To successfully obtain the result we desire, we need to prepare the upper bound estimation for the following functions. According to [12], Lemma 7, [13], Lemma 6(i), [14], p. 178, [8], Lemma 22 and [15], Lemma 2.4, we obtain the lemma as follows.

Lemma 3.3 ([12], Lem. 7[13, 14], Lemma 6 (i)[8], Lem. 22)**.** *Upper bound estimation:*

$$\int_{-\Delta}^{\Delta}|T(u)|^2\mathrm{d}u\ll Y^{2-c}\log^3 Y, \tag{3.1}$$

$$\int_{-\Delta}^{\Delta}|P(u)|^2\mathrm{d}u\ll Y^{2-c}\log Y, \tag{3.2}$$

$$\int_{n}^{n+1}|T(u)|^2\mathrm{d}u\ll Y\log^3 Y, \tag{3.3}$$

$$\int_{\Delta}^{\mathcal{H}}|T(u)|^4|\Upsilon(u)|\mathrm{d}u\ll Y^{4-c+\eta}+Y^{2+\eta}, \tag{3.4}$$

$$\int_{-\Delta}^{\Delta} |T_{l,d;J}(u)|^2 \mathrm{d}u \ll \frac{Y^{2-c}\log^3 Y}{d^2}, \tag{3.5}$$

$$\int_{\Delta \le |u| \le \mathcal{H}} |K(u)|^2 |\Upsilon(u)| \mathrm{d}u \ll Y \log^7 Y. \tag{3.6}$$

where

$$K(u) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm 1} \chi_4(j) T_{1+jm,4m;J_m}(u),$$

and (3.5) *is uniform for l and J.*

Besides, we also need the lower bound estimation in order to deduce our result.

Lemma 3.4. *Lower bound estimation:*

$$\int_{-\infty}^{\infty} P^6(u)\Upsilon(u)e(-Nu)\mathrm{d}u \gg \varepsilon Y^{6-c}, \tag{3.7}$$

In the analysis of T_I, we introduce the following lemma to ensure that corresponding results can be obtained across $Y^{\frac{128s+47}{450s+144}} \ll \mathcal{M} \ll Y^{\frac{13}{25}}$.

Lemma 3.5 ([16], Thm. 2)**.** *Assume that ϑ and ν are real numbers satisfying*

$$\nu(\nu-1)(\nu-2)\vartheta(\vartheta-1)(\nu+\vartheta-2)(\nu+\vartheta-3)(\nu+2\vartheta-3)(2\nu+\vartheta-4) \neq 0.$$

We have

$$\Sigma_I = \sum_{m \sim \mathcal{M}} a(m) \sum_{l \sim \mathcal{L}} b(l) e\left(F \frac{m^\beta l^\alpha}{\mathcal{M}^\beta \mathcal{L}^\alpha}\right),$$

where

$$F > 0, \quad \mathcal{M} \ge 1, \quad \mathcal{L} \ge 1, \quad |a_m| \le 1, \quad |b(l)| \le 1.$$

Then

$$\begin{aligned}\Sigma_I \ll &\left(\left(F^{12}\mathcal{M}^{41}\mathcal{L}^{29}\right)^{\frac{1}{56}} + \left(F^4\mathcal{M}^{15}\mathcal{L}^{11}\right)^{\frac{1}{20}} + \left(F^2\mathcal{M}^{13}\mathcal{L}^{11}\right)^{\frac{1}{16}} \right.\\ &\left. + \mathcal{M}^{\frac{3}{4}}\mathcal{L} + \mathcal{M}\mathcal{L}^{\frac{3}{4}} + F^{-1}\mathcal{M}\mathcal{L}\right)(\mathcal{M}\mathcal{L})^\eta.\end{aligned}$$

Next, we will establish a number of inequalities[8], Lemma 18, [17], Lemma 8.17 and [18], Lemma 2.3 related to the estimates in the text, and these results play an important role in the conclusions of this study.

Lemma 3.6 ([8], Lem. 18)**.** *For fixed $A > 0$, then*

$$\sum_{d \le \sqrt{Y}/(\log Y)^{A+5}} \max_{y \le Y} \max_{(a,d)=1} |E(y,u,d,a)| \ll \frac{Y}{\log^A Y}, \tag{3.8}$$

where $|u| < \Delta$.

If $a(n)$ are complex numbers, the following inequality [17]

$$\left|\sum_{a<n\le b} a(n)\right|^2 \le \left(1+\frac{b-a}{\mathcal{Q}}\right)\sum_{|q|\le\mathcal{Q}}\left(1-\frac{|q|}{\mathcal{Q}}\right)\sum_{a<n+q,n\le b}\overline{a(n)}a(n+q) \tag{3.9}$$

holds for any integer $\mathcal{Q}>0$.

Suppose that (k,l) is a exponent pair with $0\le k\le\frac{1}{2}\le l\le 1$, then [18]

$$\sum_{\mathcal{M}\le m\le\mathcal{M}'} e\left(Um^c\right) \ll \left(|U|\mathcal{M}^c\right)^k \mathcal{M}^{l-k} + \frac{\mathcal{M}}{|U|\mathcal{M}^c}. \tag{3.10}$$

where $|U|>0$, $\mathcal{M}\le\mathcal{M}'\le 10\mathcal{M}$.

From [19], we can obtain the next lemma.

Lemma 3.7. *Let $\mathcal{F}_\omega(Y)$, for a constant $\omega>0$, be defined as the number of primes $p\le Y$ by the condition that $p-1$ has a divisor in the interval. Then we obtain*

$$\sum_{p\le Y}\left|\sum_{\substack{d|p-1\\ \sqrt{Y}(\log Y)^{-\omega}<d<\sqrt{Y}(\log Y)^{\omega}}}\chi_4(d)\right|^2 \ll_\omega \frac{Y(\log\log Y)^7}{\log Y}. \tag{3.11}$$

$$\mathcal{F}_\omega(Y) \ll_\omega \frac{Y(\log\log Y)^3}{(\log Y)^{1+2\vartheta_0}}, \tag{3.12}$$

where ϑ_0 is defined by (1.2).

4. Preparation for the proof of the Theorem 1.1

Before proving Theorem 1.1, we first consider the following summation

$$\Gamma(Y) = \sum_{\substack{\frac{Y}{2}<p_1,\cdots,p_s\le Y\\ |p_1^c+\cdots+p_s^c-N|<\varepsilon}} r(p_1-1)\log p_1\cdots\log p_s.$$

By Lemma 3.1, let

$$\Gamma_0(Y) = \sum_{\frac{Y}{2}<p_1,\cdots,p_s\le Y} r\left(p_1-1\right)\vartheta\left(p_1^c+\cdots+p_s^c-N\right)\log p_1\cdots\log p_s.$$

then, we get

$$\Gamma(Y)\ge\Gamma_0(Y). \tag{4.1}$$

Using the well-known identity

$$r(n) = 4\sum_{d|n} \chi_4(d),$$

we can transform $\Gamma_0(Y)$ into

$$\Gamma_0(Y) = 4\left(\Gamma_1(Y) + \Gamma_2(Y) + \Gamma_3(Y)\right), \tag{4.2}$$

where

$$\Gamma_1(Y) = \sum_{\frac{Y}{2}<p_1,\cdots,p_s\le Y} \left(\sum_{\substack{d|p_1-1\\ d\le D}} \chi_4(d)\right) \vartheta\left(p_1^c + \cdots + p_s^c - N\right)\log p_1 \cdots \log p_s, \tag{4.3}$$

$$\Gamma_2(Y) = \sum_{\frac{Y}{2}<p_1,\cdots,p_s\le Y} \left(\sum_{\substack{d|p_1-1\\ D<d<\frac{Y}{D}}} \chi_4(d)\right) \vartheta\left(p_1^c + \cdots + p_s^c - N\right)\log p_1 \cdots \log p_s, \tag{4.4}$$

$$\Gamma_3(Y) = \sum_{\frac{Y}{2}<p_1,\cdots,p_s\le Y} \left(\sum_{\substack{d|p_1-1\\ d\ge\frac{Y}{D}}} \chi_4(d)\right) \vartheta\left(p_1^c + \cdots + p_s^c - N\right)\log p_1 \cdots \log p_s. \tag{4.5}$$

Assume that l and d are positive integers with $(l,d) = 1$, and let $J \subset \left(\frac{Y}{2}, Y\right]$ be an interval. Define

$$P_{l,d;J}(Y) = \sum_{\substack{\frac{Y}{2}<p_2,\cdots,p_s\le Y\\ p_1\equiv l(d)\\ p_1\in J}} \vartheta\left(p_1^c + \cdots + p_s^c - N\right)\log p_1 \cdots \log p_s. \tag{4.6}$$

When $J = \left(\frac{Y}{2}, Y\right]$, we simply write $P_{l,d}(Y)$.

5. The Estimations of $\Gamma_1(Y)$ and $\Gamma_3(Y)$

We shall estimate the sums $\Gamma_1(Y)$ and $\Gamma_3(Y)$ by studying the sum $P_{l,d;J}(Y)$. Using the inverse Fourier transform, we derive

$$\begin{aligned} P_{l,d;J}(Y) &= \sum_{\substack{\frac{Y}{2}<p_2,\cdots,p_s\le Y\\ p_1=l(d)\\ p_1\in J}} \log p_1 \cdots \log p_s \int_{-\infty}^{\infty} \Upsilon(u) e\left(\left(p_1^c + \cdots + p_s^c - N\right)u\right)\mathrm{d}u \\ &= \int_{-\infty}^{\infty} \Upsilon(u) T^{s-1}(u) T_{l,d;J}(u) e(-Nu)\mathrm{d}u. \end{aligned}$$

The sum $P_{l,d;J}(Y)$ is decomposed by partitioning into major, minor, and trivial arcs.

$$P_{l,d;J}(Y) = P^{(1)}_{l,d;J}(Y) + P^{(2)}_{l,d;J}(Y) + P^{(3)}_{l,d;J}(Y), \tag{5.1}$$

where

$$P^{(1)}_{l,d;J}(Y) = \int_{|u|<\Delta} \Upsilon(u) T^{s-1}(u) T_{l,d;J}(u) e(-Nu) \mathrm{d}u, \tag{5.2}$$

$$P^{(2)}_{l,d;J}(Y) = \int_{\Delta\le|u|\le\mathcal{H}} \Upsilon(u) T^{s-1}(u) T_{l,d;J}(u) e(-Nu) \mathrm{d}u, \tag{5.3}$$

$$P^{(3)}_{l,d;J}(Y) = \int_{|u|>\mathcal{H}} \Upsilon(u) T^{s-1}(u) T_{l,d;J}(u) e(-Nu) \mathrm{d}u. \tag{5.4}$$

Firstly, we consider $P^{(1)}_{l,d;J}(Y)$ and set formulas

$$T_1 = T(u), \quad T_2 = T_{l,d;J}(u), \quad P_1 = P(u), \quad P_2 = \frac{P_J(u)}{\varphi(d)},$$

then we set

$$\Psi_{\Delta,J}(Y,d) = \frac{1}{\varphi(d)} \int_{|u|<\Delta} \Upsilon(u) P^{s-1}(u) P_J(u) e(-Nu) \mathrm{d}u, \tag{5.5}$$

$$\Psi_J(Y,d) = \frac{1}{\varphi(d)} \int_{-\infty}^{\infty} \Upsilon(u) P^{s-1}(u) P_J(u) e(-Nu) \mathrm{d}u.$$

We all know that

$$\begin{aligned} T_1^{s-1}T_2 =& P_1^{s-1}P_2 + (T_2-P_2)P_1^{s-1} + T_2(T_1-P_1)P_1^{s-2} + T_1T_2(T_1-P_1)P_1^{s-3} \\ &+ T_1^2T_2(T_1-P_1)P_1^{s-4} + T_1^3T_2(T_1-P_1)P_1^{s-5}\cdots \\ &+ T_1^{s-4}T_2(T_1-P_1)P_1^2 + T_1^{s-3}T_2(T_1-P_1)P_1 + T_1^{s-2}T_2(T_1-P_1). \end{aligned} \tag{5.6}$$

Combining (3.5), (5.2), (5.6) and (5.5), applying Lemmas 3.1, 3.2, Cauchy-Schwarz inequality and the trivial bounds of $T(u)$ and $P(u)$, we obtain

$$\begin{aligned}
&P^{(1)}_{l,d;J}(Y) - \Psi_{\Delta,J}(Y,d)\\
=&\int_{|u|<\Delta} \Upsilon(u)T^{s-1}(u)T_{l,d;J}(u)e(-Nu)\mathrm{d}u - \int_{|u|<\Delta} \Upsilon(u)P^{s-1}(u)P_J(u)e(-Nu)\mathrm{d}u\\
=&\int_{|u|<\Delta} \Upsilon(u)\left(T_{l,d;J}(u) - \frac{P_J(u)}{\varphi(d)}\right)P^{s-1}(u)e(-Nu)\mathrm{d}u\\
&+\int_{|u|<\Delta} \Upsilon(u)T_{l,d;J}(u)\left(T(u)-P(u)\right)P^{s-2}(u)e(-Nu)\mathrm{d}u\\
&+\int_{|u|<\Delta} \Upsilon(u)T(u)T_{l,d;J}(u)\left(T(u)-P(u)\right)P^{s-3}(u)e(-Nu)\mathrm{d}u\\
&+\int_{|u|<\Delta} \Upsilon(u)T^2(u)T_{l,d;J}(u)\left(T(u)-P(u)\right)P^{s-4}(u)e(-Nu)\mathrm{d}u\cdots\\
&+\int_{|u|<\Delta} \Upsilon(u)T^{s-4}(u)T_{l,d;J}(u)\left(T(u)-P(u)\right)P(u)^2e(-Nu)\mathrm{d}u\\
&+\int_{|u|<\Delta} \Upsilon(u)T^{s-3}(u)T_{l,d;J}(u)\left(T(u)-P(u)\right)P(u)e(-Nu)\mathrm{d}u\\
&+\int_{|u|<\Delta} \Upsilon(u)T^{s-2}(u)T_{l,d;J}(u)\left(T(u)-P(u)\right)e(-Nu)\mathrm{d}u\\
\ll&\varepsilon Y^{s-3}\max_{|u|<\Delta}\left(\left|T_{l,d;J}(u) - \frac{P_J(u)}{\varphi(d)}\right|\right)\int_{|u|<\Delta}|P(u)|^2\mathrm{d}u\\
&+\varepsilon Y^{s-2}\frac{Y}{de^{(\log Y)^{1/5}}}\int_{|u|<\Delta}|P(u)|^2\mathrm{d}u\\
&+\varepsilon Y^{s-2}\frac{Y}{de^{(\log Y)^{1/5}}}\int_{|u|<\Delta}|T(u)|^2\mathrm{d}u\\
&+\varepsilon Y^{s-2}\frac{Y}{de^{(\log Y)^{1/5}}}\int_{|u|<\Delta}|P(u)||T(u)|\mathrm{d}u\\
\ll&\varepsilon Y^{s-1-c}\log Y\max_{|t|<\Delta}\left|T_{l,d;J}(u) - \frac{P_J(u)}{\varphi(d)}\right| + \frac{\varepsilon Y^{s-c}}{de^{(\log Y)^{1/6}}}.
\end{aligned} \tag{5.7}$$

Besides, by [20], p. 71, we get

$$P_J(u) \ll \min\left(Y, \frac{Y^{1-c}}{|u|}\right), \quad P(u) \ll \min\left(\frac{Y^{1-c}}{|u|}, Y\right).$$

Combining the above expression with Lemma 3.1, we have

$$\begin{aligned}
\Psi_{\Delta,J}(Y,d) - \Psi_J(Y,d) &\ll \frac{1}{\varphi(d)}\int_\Delta^\infty |P(u)|^{s-1}|P_J(u)||\Upsilon(u)|\mathrm{d}u\\
&\ll \varepsilon\frac{Y^{s-sc}}{\varphi(d)}\int_\Delta^\infty \frac{\mathrm{d}u}{u^s} \ll \frac{\varepsilon Y^{s-sc}}{\Delta^{s-1}\varphi(d)},
\end{aligned}$$

so

$$\Psi_{\Delta,J}(Y,d)=\Psi_J(Y,d)+O\left(\frac{\varepsilon Y^{s-sc}}{\Delta^{s-1}\varphi(d)}\right). \tag{5.8}$$

By the definition of Δ, (5.7) and (5.8), we can get

$$\begin{aligned}
&P^{(1)}_{l,d;J}(Y)\\
&=P^{(1)}_{l,d;J}(Y)-\Psi_{\Delta,J}(Y,d)+\Psi_{\Delta,J}(Y,d)-\Psi_J(Y,d)+\Psi_J(Y,d)\\
&=\Psi_J(Y,d)+O\left(\varepsilon Y^{s-1-c}\log Y\max_{|u|\le\Delta}\left|T_{l,d;J}(u)-\frac{P_J(u)}{\varphi(d)}\right|\right)+O\left(\frac{\varepsilon Y^{s-c}}{de^{(\log Y)^{1/6}}}\right).
\end{aligned} \tag{5.9}$$

For $P^{(3)}_{l,d;J}(Y)$, the trivial estimates of $T(Y)$ and $T_{l,d;J}(u)$, together with (5.4) and Lemma 3.1, yield

$$P^{(3)}_{l,d;J}(Y)\ll\frac{Y^s\log Y}{d}\int_{\mathcal{H}}^{\infty}\frac{1}{u}\left(\frac{k}{2\pi\gamma u}\right)^k\mathrm{d}u\ll\frac{Y^s\log Y}{kd}\left(\frac{k}{2\pi\gamma\mathcal{H}}\right)^k\ll\frac{1}{d}. \tag{5.10}$$

Based on the above preparation, we deal with the upper bound of the sum $\Gamma_3(Y)$ now. From (4.5), (4.6) and

$$\sum_{\substack{d|p_1-1\\ d\ge\frac{Y}{D}}}\chi_4(d)=\sum_{\substack{m|p_1-1\\ m\le(p_1-1)\frac{D}{Y}}}\chi_4\left(\frac{p_1-1}{m}\right)=\sum_{j=\pm1}\chi_4(j)\sum_{\substack{m|p_1-1\\ m\le(p_1-1)\frac{D}{Y}\\ \frac{p_1-1}{m}\equiv j(4)}}1,$$

we have

$$\Gamma_3(Y)=\sum_{\substack{m<D\\ 2|m}}\sum_{j=\pm1}\chi_4(j)P_{1+jm,4m;J_m}(Y),$$

with $J_m=\left(\max\{1+m\frac{Y}{D},\frac{Y}{2}\},Y\right]$. It follows from (5.1) that

$$\Gamma_3(Y)=\Gamma_3^{(1)}(Y)+\Gamma_3^{(2)}(Y)+\Gamma_3^{(3)}(Y), \tag{5.11}$$

where

$$\Gamma_3^{(i)}(Y)=\sum_{\substack{m<D\\ 2|m}}\sum_{j=\pm1}\chi_4(j)P^{(i)}_{1+jm,4m;J_m}(Y),\quad i=1,2,3. \tag{5.12}$$

We first consider $\Gamma_3^{(1)}(Y)$, from (5.9) with $l=1+jm, d=4m, J=J_m$ and (5.12) with $i=1$, then we obtain

$$\Gamma_3^{(1)}(Y)=\Gamma^*+O\left(\varepsilon Y^{s-1-c}\log Y\Sigma_1\right)+O\left(\frac{\varepsilon Y^{s-c}}{e^{(\log Y)^{1/6}}}\Sigma_2\right), \tag{5.13}$$

where

$$\Gamma^*=\sum_{\substack{m<D\\ 2|m}}\Psi_J(Y,4m)\sum_{j=\pm1}\chi_4(j),$$

$$\Sigma_1=\sum_{\substack{m<D\\ 2|m}}\max_{|u|<\Delta}\left|T_{1+jm,4m;J_m}(u)-\frac{P_{J_m}(u)}{\varphi(4m)}\right|, \tag{5.14}$$

$$\Sigma_2 = \sum_{m<D} \frac{1}{\varphi(4m)}.$$

It is clear that

$$\Gamma^* = 0 \quad \text{and} \quad \Sigma_2 \ll \log Y. \tag{5.15}$$

Combining (5.14) with (3.8) in Lemma 3.8, we derive

$$\Sigma_1 \ll \frac{Y}{(\log Y)^{A-5}}. \tag{5.16}$$

By (5.13), (5.15) and (5.16), we can get

$$\Gamma_3^{(1)}(Y) \ll \frac{\varepsilon Y^{s-c}}{\log Y}. \tag{5.17}$$

We now estimate $\Gamma_3^{(2)}(Y)$. Applying (5.3) with $l = 1 + jm, d = 4m, J = J_m$ and (5.12) with $i = 2$, we obtain

$$\Gamma_3^{(2)}(Y) = \int_{\Delta \le |u| \le \mathcal{H}} \Upsilon(u) T^{s-1}(u) K(u) e(-Nu) \mathrm{d}u, \tag{5.18}$$

$K(u)$ is the same as the definition of $K(u)$ in Lemma 3.3.

Lemma 5.1. *Suppose*

$$\Delta \le |u| \le \mathcal{H}, \quad |a(m)| \ll m^\eta, \quad \mathcal{L} \gg Y^{\frac{12}{25}}, \quad \mathcal{L}\mathcal{M} \asymp Y.$$

Let

$$T_I = \sum_{m \sim \mathcal{M}} a(m) \sum_{l \sim \mathcal{L}} e(um^c l^c),$$

then

$$T_I \ll Y^{\frac{24s+5}{25s+8}+\eta}.$$

Proof. For $\mathcal{M} \ll Y^{\frac{128s+47}{450s+144}}$, we apply (3.10) with the exponent pair $\left(\frac{4}{11}, \frac{6}{11}\right)$ (see [21]) to obtain

$$\begin{aligned} T_I &\ll Y^\eta \sum_{m \sim \mathcal{M}} \left| \sum_{l \sim \mathcal{L}} e(um^c l^c) \right| \\ &\ll Y^\eta \sum_{m \sim \mathcal{M}} \left((|u| Y^c \mathcal{L}^{-1})^{\frac{4}{11}} \mathcal{L}^{\frac{6}{11}} + \frac{1}{|u| Y^c \mathcal{L}^{-1}} \right) \\ &\ll Y^{\frac{24s+5}{25s+8}+\eta}. \end{aligned} \tag{5.19}$$

For $Y^{\frac{128s+47}{450s+144}} \ll \mathcal{M} \ll Y^{\frac{13}{25}}$, by Lemma 3.5, we can get

$$T_I \ll Y^{\frac{24s+5}{25s+8}+\eta}. \tag{5.20}$$

We prove this lemma from (5.19) and (5.20). □

Lemma 5.2. *Suppose that*

$$\Delta \leq |u| \leq \mathcal{H}, \quad |b(l)| \ll l^{\eta}, \quad |a(m)| \ll m^{\eta}, \quad Y^{\frac{1}{25}} \ll \mathcal{L} \ll Y^{\frac{1}{3}}, \quad \mathcal{L}\mathcal{M} \asymp Y.$$

Let

$$T_{II} = \sum_{m \sim \mathcal{M}} a(m) \sum_{l \sim \mathcal{L}} b(l) e(u m^c l^c),$$

then

$$T_{II} \ll Y^{\frac{24s+5}{25s+8}+\eta}.$$

Proof. An application of Cauchy-Schwarz inequality and (3.9) with $\mathcal{Q} = Y^{\frac{1}{5}}$ gives

$$|T_{II}|^2 \ll Y^{\eta} \left(\frac{Y^2}{\mathcal{Q}} + \frac{Y}{\mathcal{Q}} \sum_{1 \leq q \leq \mathcal{Q}} \sum_{l \sim \mathcal{L}} \left| \sum_{m \sim \mathcal{M}} e(f(l,m,q)) \right| \right),$$

where $f(l,m,q) = u m^c((l+q)^c - l^c)$. Applying (3.10) with the exponent pair $\left(\frac{1}{6}, \frac{2}{3}\right)$, we can get

$$\begin{aligned} T_{II} &\ll Y^{\eta} \left(\frac{Y^2}{\mathcal{Q}} + \frac{Y}{\mathcal{Q}} \sum_{1 \leq q \leq \mathcal{Q}} \sum_{l \sim \mathcal{L}} \left(\left(|u| q Y^{c-1}\right)^{\frac{1}{6}} \mathcal{M}^{\frac{2}{3}} + \frac{1}{|u| q Y^{c-1}} \right) \right)^{\frac{1}{2}} \\ &\ll Y^{\eta} \left(\frac{Y^2}{\mathcal{Q}} + \frac{Y}{\mathcal{Q}} \left(\mathcal{H}^{\frac{1}{6}} \mathcal{Q}^{\frac{7}{6}} Y^{\frac{c-1}{6}} \mathcal{M}^{\frac{2}{3}} \mathcal{L} + \Delta^{-1} Y^{1-c} \mathcal{L} \log \mathcal{Q} \right) \right)^{\frac{1}{2}} \\ &\ll Y^{\frac{24s+5}{25s+8}+\eta}. \end{aligned}$$

This completes the proof. □

Lemma 5.3. *Let $\Delta \leq |u| \leq \mathcal{H}$. We obtain*

$$T(u) \ll Y^{\frac{24s+5}{25s+8}+\eta}.$$

Proof. It is sufficient to establish the bound

$$T^*(u) = \sum_{\frac{Y}{2} < n \leq Y} \Lambda(n) e(u n^c) \ll Y^{\frac{24s+5}{25s+8}+\eta}.$$

Define

$$A = Y^{\frac{1}{25}}, \quad B = Y^{\frac{1}{3}}, \quad Z = \left[Y^{\frac{12}{25}} \right] + \frac{1}{2}.$$

By [22], Lemma 3, the sum $T^*(u)$ can be expressed as a combination of $O\left(\log^{10} Y\right)$ sums, each belonging to one of the following types.

Type I:

$$\sum_{m \sim \mathcal{M}} a(m) \sum_{l \sim \mathcal{L}} e(um^c l^c),$$

with

$$\mathcal{L} \gg Z, \quad \mathcal{L}\mathcal{M} \asymp Y, \quad |a(m)| \ll m^\eta.$$

Type II:

$$\sum_{m \sim \mathcal{M}} a(m) \sum_{l \sim \mathcal{L}} b(l) e(um^c l^c),$$

with

$$A \ll \mathcal{L} \ll B, \quad \mathcal{L}\mathcal{M} \asymp Y, \quad |a(m)| \ll m^\eta, \quad |b(l)| \ll l^\eta.$$

Applying Lemmas 5.1 and 5.2 to these sums yields

$$T^*(u) \ll Y^{\frac{24s+5}{25s+8}+\eta},$$

thus completing the proof of the lemma. □

Lemma 5.4. *We obtain*

$$\int_{\Delta \le |u| \le \mathcal{H}} |T(u)|^8 |\Upsilon(u)| \mathrm{d}u \ll Y^{\frac{1553s+371}{200s+64}-c+\eta}.$$

Proof. For a function $\psi(x)$ continuous on $[-\mathcal{H}, \mathcal{H}]$, an argument analogous to Lemma 19 in Dimitrov [19] applied to the integral $\int_{\Delta \le |u| \le \mathcal{H}} T(u)\psi(u)\mathrm{d}u$ yields that

$$\begin{aligned} \left| \int_{\Delta \le |u| \le \mathcal{H}} T(u)\psi(u)\mathrm{d}u \right|^2 \le & Y^{2-c+\eta} \max_{\Delta \le |u| \le \mathcal{H}} |\psi(u)| \int_{\Delta \le |u| \le \mathcal{H}} |\psi(u)|\mathrm{d}u \\ & + Y^{1+\frac{c}{2}+\eta} \left(\int_{\Delta \le |u| \le \mathcal{H}} |\psi(u)|\mathrm{d}u \right)^2. \end{aligned} \tag{5.21}$$

Choosing

$$\psi_1(u) = \overline{T(u)} |T(u)|^3 |\Upsilon(u)|,$$

from (5.21), (3.4), Lemmas 3.1 and 5.3, we can get

$$\begin{aligned}\int_{\Delta\le|u|\le\mathcal{H}}|T(u)|^5|\Upsilon(u)|\mathrm{d}t=&\int_{\Delta\le|u|\le\mathcal{H}}T(u)\psi_1(u)\mathrm{d}u\\ \ll&Y^{1-\frac{c}{2}+\eta}\left(\max_{\Delta\le|u|\le\mathcal{H}}|\psi(u)|\right)^{\frac12}\left(\int_{\Delta\le|u|\le\mathcal{H}}|T(u)|^4|\Upsilon(u)|\mathrm{d}u\right)^{\frac12}\\ &+Y^{\frac12+\frac{c}{4}+\eta}\int_{\Delta\le|u|\le\mathcal{H}}|T(u)|^4|\Upsilon(u)|\mathrm{d}u\\ \ll&Y^{\frac{123s+34}{25s+8}-c+\eta}+Y^{\frac92-\frac{3c}{4}+\eta}\\ \ll&Y^{\frac{123s+34}{25s+8}-c+\eta}.\end{aligned}$$

Next, we repeat the above process three times, then we can get

$$\int_{\Delta\le|u|\le\mathcal{H}}|T(u)|^8|\Upsilon(u)|\mathrm{d}u\ll Y^{\frac{1153s+371}{200s+60}-c+\eta}.$$

The lemma is proved. □

Considering Lemmas 5.3 and 5.4, we apply the Cauchy–Schwarz inequality to (5.18) and (3.6), then we can get

$$\begin{aligned}\Gamma_3^{(2)}(Y)\ll&\max_{\Delta\le u\le\mathcal{H}}|T(u)|^{s-5}\left(\int_{\Delta\le|u|\le\mathcal{H}}|K(u)|^2|\Upsilon(u)|\mathrm{d}u\right)^{\frac12}\left(\int_{\Delta\le|u|\le\mathcal{H}}|T(u)|^8|\Upsilon(u)|\mathrm{d}u\right)^{\frac12}\\ \ll&\frac{\varepsilon Y^{s-c}}{\log Y}.\end{aligned}\tag{5.22}$$

Then, we deal with $\Gamma_3^{(3)}(Y)$. By (5.10) with $l=1+jm, d=4m, J=J_m$ and (5.12) with $i=3$, we derive

$$\Gamma_3^{(3)}(Y)\ll\sum_{m<d}\frac{1}{m}\ll\log Y.\tag{5.23}$$

Putting (5.11), (5.17), (5.22) and (5.23) together, we can get

$$\Gamma_3(Y)\ll\frac{\varepsilon Y^{s-c}}{\log Y}.\tag{5.24}$$

Next, we now establish a lower bound for the sum $\Gamma_1(Y)$. From (4.3), (4.6) and (5.1), we can derive

$$\Gamma_1(Y)=\Gamma_1^{(1)}(Y)+\Gamma_1^{(2)}(Y)+\Gamma_1^{(3)}(Y),\tag{5.25}$$

where

$$\Gamma_1^{(i)}=\sum_{d\le D}\chi_4(d)P_{1,d}^{(i)}(Y),\quad i=1,2,3.$$

For $\Gamma_1^{(1)}(Y)$, analogous to (5.13), we obtain

$$\Gamma_1^{(1)}(Y) = \Psi(Y) \sum_{d \le D} \frac{\chi_4(d)}{\varphi(d)} + O\left(\frac{\varepsilon Y^{s-c}}{\log Y}\right),$$

with

$$\Psi(Y) = \int_{-\infty}^{\infty} \Upsilon(t) P^s(u) e(-Nu) \mathrm{d}u.$$

From Lemma 3.4, we have

$$\Psi(Y) \gg \varepsilon Y^{s-c}. \tag{5.26}$$

As shown in [23],

$$\sum_{d \le D} \frac{\chi_4(d)}{\varphi(d)} = \frac{\pi}{4} \prod_p \left(1 + \frac{\chi_4(d)}{p(p-1)}\right) + O\left(Y^{-\frac{1}{20}}\right),$$

so, we can get

$$\Gamma_1^{(1)}(Y) = \frac{\pi}{4} \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) \Psi(Y) + O\left(\frac{\varepsilon Y^{6-c}}{\log Y}\right) + O\left(\Psi(Y) Y^{-\frac{1}{20}}\right).$$

By (5.26), we obtain the lower bound

$$\Gamma_1^{(1)}(Y) \gg \varepsilon Y^{s-c}. \tag{5.27}$$

For $\Gamma_3^{(2)}(Y)$, an estimate analogous to (5.22) gives

$$\Gamma_1^{(2)}(Y) \ll \frac{\varepsilon Y^{s-c}}{\log Y}. \tag{5.28}$$

As for $\Gamma_3^{(3)}(Y)$, reasoning similar to that in (5.23) leads to

$$\Gamma_1^{(3)}(Y) \ll \sum_{d \le D} \frac{1}{d} \ll \log Y. \tag{5.29}$$

Substituting the bounds (5.27), (5.28) and (5.29) into (5.25), we conclude that

$$\Gamma_1(Y) \gg \varepsilon Y^{s-c}. \tag{5.30}$$

6. The Estimate of $\Gamma_2(Y)$

This section is devoted to establishing an upper bound for the sum $\Gamma_2(Y)$. We begin by presenting a lemma that pertains to the number of solutions to the Diophantine inequality (1.1) in $s-1$ variables.

Lemma 6.1. *Consider $1 < c < 3$, $c \neq 2$, and a large enough positive number N_0. Define $B_0(N_0)$ as the number of solutions to the Diophantine inequality*

$$|p_1^c + \cdots + p_{s-1}^c - N_0| \leq \varepsilon$$

in primes $p_1, \cdots, p_{s-1}$ is bounded above by

$$B_0(N_0) \ll \frac{\varepsilon N_0^{\frac{s-1}{c}-1}}{\log^{s-1} N_0}.$$

Proof. Define

$$B(Y_0) = \sum_{\substack{\frac{Y_0}{2} < p_1, \cdots, p_{s-1} \leq Y_0 \\ |p_1^c + \cdots + p_{s-1}^c - N_0| \leq \varepsilon}} \log p_1 \cdots \log p_{s-1}, \tag{6.1}$$

with

$$Y_0 = \left(\frac{N_0}{2}\right)^{\frac{1}{c}}. \tag{6.2}$$

By Lemma 3.1 for $k_0 = [\log Y_0]$, $h_0 = \frac{5\varepsilon}{4}$ and $\gamma_0 = \frac{\varepsilon}{4}$, we obtain a k_0 times continuously differentiable function $\vartheta_0(y)$, its Fourier transform and the bound which satisfies the conditions we choose. The bounds allow

$$|\Upsilon_0(x)| \leq \min\left(\frac{1}{\pi|x|}, \frac{5\varepsilon}{2}, \frac{1}{\pi|x|}\left(\frac{2k_0}{\pi|x|\varepsilon}\right)^{k_0}\right). \tag{6.3}$$

By combining (6.1), the definition of $\vartheta_0(y)$ and the inverse Fourier transformation formula we obtain

$$\begin{aligned} B(Y_0) &\leq \sum_{\frac{Y_0}{2} < p_1, \cdots, p_{s-1} \leq Y_0} \vartheta_0\left(p_1^c + \cdots + p_{s-1}^c - N_0\right) \log p_1 \cdots \log p_{s-1} \\ &= \int_{-\infty}^{\infty} \Upsilon_0(u) T_0^{s-1}(u) e\left(-N_0 u\right) \mathrm{d}u = B_1\left(Y_0\right) + B_2\left(Y_0\right) + B_3\left(Y_0\right), \end{aligned} \tag{6.4}$$

where

$$T_0(u) = \sum_{\frac{Y_0}{2} < p \leq Y_0} e(up^c) \log p, \tag{6.5}$$

$$\Delta_0 = \frac{(\log Y_0)^{A_0}}{Y_0^c}, \quad A_0 > 10, \tag{6.6}$$

$$B_1(Y_0) = \int_{|u| < \Delta_0} \Upsilon_0(u) T_0^{s-1}(u) e(-N_0 u) \mathrm{d}u,$$

$$B_2(Y_0) = \int_{\Delta_0 \leq |u| \leq \mathcal{H}} \Upsilon_0(u) T_0^{s-1}(u) e(-N_0 u) \mathrm{d}u, \tag{6.7}$$

$$B_3(Y_0) = \int_{|t|\geq \mathcal{H}} \Upsilon_0(u) T_0^{s-1}(u) e(-N_0 u) \mathrm{d}u. \tag{6.8}$$

We begin by estimating $B_1(Y_0)$. Define

$$\psi_{\Delta_0}(Y_0) = \int_{|u|<\Delta_0} \Upsilon_0(u) I_0^{s-1}(u) e(-N_0 u) \mathrm{d}u,$$

$$\psi(Y_0) = \int_{-\infty}^{\infty} \Upsilon_0(u) P_0^{s-1}(u) e(-N_0 u) \mathrm{d}u,$$

$$P_0(u) = \int_{\frac{Y_0}{2}}^{Y_0} e(uy^c) \mathrm{d}y.$$

From (6.3) and [20], p. 71, it follows that

$$\begin{aligned}
\psi(Y_0) &= \int_{|u|<Y_0^{-c}} \Upsilon_0(t) P_0^{s-1}(u) e(-N_0 u) \mathrm{d}u + \int_{|u|\geq Y_0^{-c}} \Upsilon_0(u) P_0^{s-1}(u) e(-N_0 u) \mathrm{d}u \\
&\ll \int_{|u|<Y_0^{-c}} \varepsilon Y_0^{s-1} \mathrm{d}u + \int_{|u|\geq Y_0^{-c}} \frac{\varepsilon Y_0^{s-sc}}{u^{s-1}} \mathrm{d}u, \\
&\ll \varepsilon Y_0^{s-1-c}.
\end{aligned} \tag{6.9}$$

Using (6.3), Lemmas 3.2 and 3.3, together with the trivial bounds of $T_0(u)$ and $P_0(u)$, we obtain

$$\begin{aligned}
B_1(Y_0) - \psi_{\Delta_0}(Y_0) &\ll \int_{|u|<\Delta_0} \left| T_0^{s-1}(u) - P_0^{s-1}(u) \right| |\Upsilon_0(u)| \, \mathrm{d}u \\
&\ll \varepsilon \int_{|u|<\Delta_0} |T_0(u) - P_0(u)| \left(|T_0(u)|^{s-2} + |P_0(u)|^{s-2} \right) \mathrm{d}u \\
&\ll \varepsilon \frac{Y_0}{e^{(\log Y_0)^{1/5}}} \left(\int_{|u|<\Delta_0} |T_0(u)|^{s-2} \, \mathrm{d}u + \int_{|u|<\Delta_0} |P_0(u)|^{s-2} \, \mathrm{d}u \right) \\
&\ll \frac{\varepsilon Y_0^{s-1-c}}{e^{(\log Y_0)^{1/6}}}.
\end{aligned} \tag{6.10}$$

By (6.6) and [20], p. 71 we can get

$$\begin{aligned}
|\psi(Y_0) - \psi_{\Delta_0}(Y_0)| &\ll \int_{\Delta_0}^{\infty} |P_0(u)|^{s-1} |\Upsilon_0(u)| \mathrm{d}u \ll \frac{\varepsilon}{Y_0^{(s-1)(c-1)}} \int_{\Delta_0}^{\infty} \frac{\mathrm{d}u}{u^{s-1}} \\
&\ll \frac{\varepsilon}{Y_0^{(s-1)(c-1)} \Delta_0^{s-2}} \ll \frac{\varepsilon Y_0^{s-1-c}}{\log Y_0}.
\end{aligned} \tag{6.11}$$

Combining (6.9), (6.10) and (6.11) with the equality

$$B_1(Y_0) = B_1(Y_0) - \psi_{\Delta_0}(Y_0) + \psi_{\Delta_0}(Y_0) - \psi(Y_0) + \psi(Y_0),$$

we conclude that

$$B_1(Y_0) \ll \varepsilon Y_0^{s-1-c}. \tag{6.12}$$

We now estimate $B_2(Y_0)$. Define

$$B_2^*(Y_0) = \int_{\Delta_0 \le t \le \mathcal{H}} \Upsilon_0(u) T_0^{s-1}(u) e(-N_0 u) \mathrm{d}u. \tag{6.13}$$

Using (6.2), (6.3) and partial integration, we have

$$\begin{aligned} B_2^*(Y_0) &= -\frac{1}{2\pi i} \int_{\Delta_0 \le u \le \mathcal{H}} \frac{\Upsilon_0(t) T_0^{s-1}(u)}{N_0} \mathrm{d}e(-N_0 u) \\ &= -\frac{\Upsilon_0(t) T_0^{s-1}(u) e(-N_0 u)}{2\pi i N_0} \Big|_{\Delta_0}^{\mathcal{H}} + \frac{1}{2\pi i N_0} \int_{\Delta_0 \le u \le \mathcal{H}} e(-N_0 u) \mathrm{d}\left(\Upsilon_0(u) T_0^{s-1}(u)\right) \\ &\ll \varepsilon Y_0^{s-1-c} + Y_0^{-c} |\Omega|, \end{aligned} \tag{6.14}$$

where

$$\Omega = \int_{\Delta_0 \le u \le \mathcal{H}} e(-N_0 u) \mathrm{d}\left(\Upsilon_0(u) T_0^{s-1}(u)\right).$$

Now, considering the curve

$$\Gamma_0: \ z = g(u) = \Upsilon_0(u) T_0^{s-1}(u), \quad g'(u) \ne 0, \quad \Delta_0 \le u \le \mathcal{H},$$

since $g(u)$ is holomorphic and $g'(u) \ne 0$ on $u \in [\Delta_0, \mathcal{H}]$, the inverse $g^{-1}(t)$ exists for $t \in \Gamma_0$. By (6.3), we have

$$\Omega = \int_{\Gamma_0} e\left(-N_0 g^{-1}(t)\right) \mathrm{d}t \ll \int_{\Gamma_0} |\mathrm{d}t| \ll (|g(\Delta_0)| + |g(\mathcal{H})|) \ll \varepsilon Y_0^{s-1}. \tag{6.15}$$

Combining (6.7), (6.13), (6.14) and (6.15), we obtain

$$B_2(Y_0) \ll \varepsilon Y_0^{s-1-c}. \tag{6.16}$$

We now estimate $B_3(Y_0)$. By the definition of ε, (6.3), (6.5) and (6.8), it follows that

$$B_3(Y_0) \ll Y_0^{s-1} \int_{\mathcal{H}}^{\infty} \frac{1}{u} \left(\frac{2k_0}{\pi u \varepsilon}\right)^{k_0} \mathrm{d}u \ll Y_0^{s-1} \left(\frac{k_0}{\mathcal{H}\varepsilon}\right)^{k_0} \ll 1. \tag{6.17}$$

Substituting the bounds (6.12), (6.16) and (6.17) into (6.4) yields

$$B(Y_0) \ll \varepsilon Y_0^{s-1-c},$$

together with (6.1) and (6.2), this implies

$$B_0(N_0) \ll \frac{\varepsilon N_0^{\frac{s-1}{c}-1}}{\log^{s-1} N_0},$$

which completes the proof of the lemma. □

A prime p in the interval $(\frac{Y}{2}, Y]$ belongs to $\mathcal{F}(Y)$ if $p-1$ has a divisor in $\left(D, \frac{Y}{D}\right)$.Then by (4.4) and the inequality $xy \le x^2 + y^2$ we can get

$$\Gamma_2^2(Y) \ll \log^{2s} Y \sum_{\substack{\frac{Y}{2}<p_1,\cdots,p_{2s}\le Y \\ |p_1^c+\cdots+p_s^c-N|<\varepsilon \\ \left|p_{s+1}^c+\cdots+p_{2s}^c-N\right|<\varepsilon}} \left|\sum_{\substack{d|p_1-1 \\ D<d<\frac{Y}{D}}} \chi_4(d)\right| \left|\sum_{\substack{t|p_{s+1}-1 \\ D<t<\frac{Y}{D}}} \chi_4(t)\right|$$

$$\ll \log^{2s} Y \sum_{\substack{\frac{Y}{2}<p_1,\cdots,p_{2s}\le Y \\ |p_1^c+\cdots+p_s^c-N|<\varepsilon \\ \left|p_{s+1}^c+\cdots+p_{2s}^c-N\right|<\varepsilon \\ p_{s+1}\in\mathcal{F}(Y)}} \left|\sum_{\substack{d|p_1-1 \\ D<d<\frac{Y}{D}}} \chi_4(d)\right|^2.$$

The contribution from terms with $p_1 = p_{s+1}$ is $O\left(Y^{2s-3+\varepsilon}\right)$, leading to

$$\Gamma_2^2(Y) \ll \Xi \log^{2s} Y + Y^{2s-3+\varepsilon}, \tag{6.18}$$

where

$$\Xi = \sum_{\frac{Y}{2}<p_1\le Y} \left|\sum_{\substack{d|p_1-1 \\ D<d<\frac{Y}{D}}} \chi_4(d)\right|^2 \sum_{\substack{\frac{Y}{2}<p_{s+1}\le Y \\ p_{s+1}\in\mathcal{F}(Y) \\ p_{s+1}\ne p_1}} \sum_{\substack{\frac{Y}{2}<p_2,\cdots,p_s,p_{s+2},\cdots,p_{2s}\le Y \\ |p_1^c+\cdots+p_s^c-N|<\varepsilon \\ |p_{s+1}^c+\cdots+p_{2s}^c-N|<\varepsilon}} 1.$$

Applying Lemma 6.1, we obtain

$$\Xi \ll \frac{Y^{2s-2-2c}}{\log^{10} Y} \Xi_1 \Xi_2, \tag{6.19}$$

where

$$\Xi_1 = \sum_{\frac{Y}{2}<p\le Y} \left|\sum_{\substack{d|p-1 \\ D<d<\frac{Y}{D}}} \chi_4(d)\right|^2 \quad \text{and} \quad \Xi_2 = \sum_{\substack{\frac{Y}{2}<p\le Y \\ p\in\mathcal{F}(Y)}} 1.$$

From (3.11) and (3.12), it follows that

$$\Xi_1 \ll \frac{Y(\log\log Y)^7}{\log Y} \quad \text{and} \quad \Xi_2 \ll \frac{Y(\log\log Y)^3}{(\log Y)^{1+2\vartheta_0}}. \tag{6.20}$$

By the definition of ε, (6.18), (6.19) and (6.20), we can get

$$\Gamma_2(Y) \ll \frac{Y^{s-c}(\log\log Y)^5}{(\log Y)^{\vartheta_0}} = \frac{\varepsilon Y^{s-c}}{\log\log Y}. \tag{6.21}$$

7. Proof of Theorem 1.1

Combining (4.1), (4.2), (5.24), (5.30) and (6.21), we derive

$$\Gamma(Y) \gg \varepsilon Y^{s-c},$$

which yields

$$\Gamma(Y) \to \infty \quad \text{as} \quad Y \to \infty.$$

We have now established Theorem 1.1.

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Conflicts of interest

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Data availability statement

This article has no associated data generated.

Author contribution statement

All authors have reviewed, discussed, and agreed to their individual contributions ahead of this time. Conceptualization, Y.K. Liu. and J. Huang.; Methodology, Y.K. Liu.; Validation, Y.K. Liu. and J. Huang.; Supervision, J. Huang.; Writing – Original Draft Preparation, Y.K. Liu.; Writing – Review & Editing, J. Huang.

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