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FIBONACCI DETERMINANTS. III

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In memory of Narakorn Rompurk Kanasri[**]

Abstract. There are many references to matrices or determinants that have Fibonacci numbers as elements. In this paper, we find several determinants expressing the Fibonacci and related polynomials.

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1. Introduction

Let $f_n := f_n(x)$ be the n-th Fibonacci polynomial, defined by

$$f_n = xf_{n-1} + f_{n-2} \quad (n \geq 2) \quad \text{with} \quad f_0 = 0 \quad \text{and} \quad f_1 = 1\,. \tag{1.1}$$

When $x = 1$, $F_n = f_n(1)$ is the original Fibonacci number. In the first part of Fibonacci determinants in [2], by applying the so-called Cameron's operator, several determinantal expressions of Fibonacci numbers are shown. This method is developped in [1] to obtain hypergeometric Bernoulli, Cauchy and Euler numbers with some modifications are introduced and studied. In the second part of Fibonacci determinants in [3], some determinantal expressions of f_{2n+1} and f_{2n} are given as generalizations of F_{2n+1} and F_{2n} in [2]. In fact, more general cases of Horadam-type numbers (polynomials) are considered, including $w_n = w_n(u, v)$, which is defined by

$$w_n = uw_{n-1} + vw_{n-2} \quad (n \geq 2), \quad w_0 = 0, \quad w_1 = 1\,. \tag{1.2}$$

In this paper as the third part of Fibonacci determinants, we give determinantal expressions of more general sequences f_{kn+1} and f_{kn} for any positive integer k. Namely, when $k = 2$, the results are reduced to those in [3]. Similarly, we give those of Horadam-type numbers (polynomials) as in (1.2). We also consider the corresponding Lucas cases. As applications, we consider LU decomposition and use Trudi's formula and inverse relations.

Keywords and phrases: Fibonacci polynomials, determinants, Lucas polynomials, Horadam polynomials, LU decomposition.

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[**] Ms Narakorn Rompurk Kanasri in [1] passed away in a traffic accident on January 3, 2026. Rest in peace.

2. Preliminaries

The sequence of Horadam numbers $w_n = w_n(u,v)$ is defined in (1.2), where u and v are often treated as integers with $(u,v) \neq (0,0)$. Nevertheless, similarly to the definition of Fibonacci polynomials in (1.1), we can regard them as polynomials $u := u(x)$ and $v := v(x)$. Horadam studied these numbers in depth (see, *e.g.*, [4–8]), and in [9] they are called the bivariate Fibonacci polynomials by considering both u and v as variables.

The sequence of Horadam–Lucas numbers (polynomials) $\omega_n = \omega_n(u,v)$ is defined by

$$\omega_n = u\omega_{n-1} + v\omega_{n-2} \quad (n \geq 2), \quad \omega_0 = 2, \quad \omega_1 = u\,. \tag{2.1}$$

When $v = 1$ and $u = x$, $f_n = w_n(x,1)$ and $l_n = \omega_n(x,1)$ are Fibonacci polynomials and Lucas polynomials, respectively [9]. When $u = v = 1$, $F_n = w_n(1,1)$ and $L_n = \omega_n(1,1)$ are Fibonacci numbers and Lucas numbers, respectively [10].

In fact, several fundamental formulas hold for Horadam numbers (polynomials) w_n and Horadam–Lucas numbers (polynomials) ω_n, defined as (1.2) and (2.1), respectively, below. Let $\theta = \theta(x) := \frac{u+\sqrt{u^2+4v}}{2}$ and $\phi = \phi(x) := \frac{u-\sqrt{u^2+4v}}{2}$.

Lemma 2.1 (see, *e.g.*, [9, 10])**.** *We have*

(i) $w_n = \dfrac{\theta^n - \phi^n}{\theta - \phi}$.
(ii) $\omega_n = \theta^n + \phi^n$.
(iii) $w_n \omega_n = w_{2n}$.
(iv) $w_{n+1} + v w_{n-1} = \omega_n$.
(v) $\omega_{n+1} + v\omega_{n-1} = (u^2 + 4v) w_n$.
(vi) $w_{n+1} w_{n-1} - w_n^2 = (-1)^n v^{n-1}$.
(vii) $\omega_{n+1}\omega_{n-1} - \omega_n^2 = (-v)^{n-1}(u^2 + 4v)$.
(viii) $w_{n+k} = w_{n+1} w_k + v w_n w_{k-1}$.
(ix) $\omega_{n+k} = w_{n+1}\omega_k + v w_n \omega_{k-1}$.

Concerning the last two identities, see, *e.g.*, [9], Theorem 46.3.

2.1. Horadam polynomials

Theorem 2.2. *For $n \geq 3$ and $k \geq 1$, we have the $n \times n$ determinants:*

$$w_{kn+1} = \begin{vmatrix} w_{k+1} & -v w_k & 0 & \cdots & & \cdots & 0 \\ w_k & w_{k+1} & -v w_k & & & & \vdots \\ w_{k-1} & w_k & w_{k+1} & & & & \\ \frac{w_{k-1}^2}{w_k} & w_{k-1} & & \ddots & & & \vdots \\ \vdots & & & & w_{k+1} & -w_k & 0 \\ \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & & w_{k-1} & w_k & w_{k+1} & -v w_k \\ \frac{w_{k-1}^{n-2}}{w_k^{n-3}} & \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \cdots & \frac{w_{k-1}^2}{w_k} & w_{k-1} & w_k & w_{k+1} \end{vmatrix}$$

and

$$w_{kn}=\begin{vmatrix} w_k & -vw_k & 0 & \cdots & & \cdots & 0\\ w_{k-1} & w_{k+1} & -vw_k & & & & \vdots\\ \frac{w_{k-1}^2}{w_k} & w_k & w_{k+1} & & & & \\ \frac{w_{k-1}^3}{w_k^2} & w_{k-1} & & \ddots & & & \vdots\\ \vdots & & & & w_{k+1} & -vw_k & 0\\ \frac{w_{k-1}^{n-2}}{w_k^{n-3}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & & w_{k-1} & w_k & w_{k+1} & -vw_k\\ \frac{w_{k-1}^{n-1}}{w_k^{n-2}} & \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \cdots & \frac{w_{k-1}^2}{w_k} & w_{k-1} & w_k & w_{k+1} \end{vmatrix}.$$

In order to prove Theorem 2.2, we need the following property.

Lemma 2.3. *For positive integers n and k, we have*

$$w_{kn+1}=w_{k+1}w_{k(n-1)+1}+vw_k^2\sum_{j=0}^{n-2}v^j w_{k-1}^j w_{k(n-j-2)+1}\,.$$

Proof. By Lemma 2.1 (viii), we have

$$w_{kn+1}=w_{k+1}w_{k(n-1)+1}+vw_k w_{k(n-1)}\,. \tag{2.2}$$

Hence, it is enough to prove that for $n\ge 2$

$$w_{k(n-1)}=w_k\sum_{j=0}^{n-2}v^j w_{k-1}^j w_{k(n-j-2)+1}\,. \tag{2.3}$$

When $n=2$, the identity (2.3) is trivial as both sides are equal to w_k. Assume that the identity (2.3) is valid for this n. Then, by (2.2) again, we have

$$\begin{aligned} w_{kn}&=w_k f_{k(n-1)+1}+vw_{k-1}w_{k(n-1)}\\ &=w_k\left(w_{k(n-1)+1}+\sum_{j=0}^{n-2}v^{j+1}w_{k-1}^{j+1}w_{k(n-j-2)+1}\right)\\ &=w_k\sum_{j=0}^{n-1}v^j w_{k-1}^j w_{k(n-j-1)+1}\,, \end{aligned}$$

which is the identity (2.3), where n is replaced by $n+1$. By induction, the identity (2.3) is valid for all integers $n\ge 2$. □

Proof of Theorem 2.2. When $n=1$, the first result is clear because both sides are equal to w_{k+1}. Assume that the determinant expressions for w_{kn+1} are valid up to $n-1$. Expanding the right-hand side of the first equation along the first row repeatedly, we have

$$
\begin{aligned}
\mathrm{RHS} &= w_{k+1}w_{k(n-1)+1} + vw_k \begin{vmatrix} w_k & -vw_k & 0 & \cdots & \cdots & 0 \\ w_{k-1} & w_{k+1} & -vw_k & & & \vdots \\ \frac{w_k^2}{w_k} & w_k & w_{k+1} & & & \vdots \\ \vdots & & \ddots & & & 0 \\ \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \frac{w_{k-1}^{n-5}}{w_k^{n-6}} & & w_k & w_{k+1} & -vw_k \\ \frac{w_{k-1}^{n-2}}{w_k^{n-3}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & \cdots & w_{k-1} & w_k & w_{k+1} \end{vmatrix} \\
&= w_{k+1}w_{k(n-1)+1} + vw_k^2 w_{k(n-2)+1} \\
&\quad + v^2 w_k^2 \begin{vmatrix} w_{k-1} & -vw_k & 0 & \cdots & \cdots & 0 \\ \frac{w_{k-1}^2}{w_k} & w_{k+1} & -vw_k & & & \vdots \\ \frac{w_{k-1}^3}{w_k^2} & w_k & w_{k+1} & & & \vdots \\ \vdots & & \ddots & & & 0 \\ \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \frac{w_{k-1}^{n-6}}{w_k^{n-7}} & & w_k & w_{k+1} & -vw_k \\ \frac{w_{k-1}^{n-2}}{w_k^{n-3}} & \frac{w_{k-1}^{n-5}}{w_k^{n-6}} & \cdots & w_{k-1} & w_k & w_{k+1} \end{vmatrix} \\
&= \cdots \\
&= w_{k+1}w_{k(n-1)+1} + vw_k^2 w_{k(n-2)+1} + v^2 w_k^2 w_{k-1} w_{k(n-3)+1} \\
&\quad + v^3 w_k^2 w_{k-1}^2 w_{k(n-4)+1} + \cdots + v^{n-2} w_k^{n-2} \begin{vmatrix} \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & -vw_k \\ \frac{w_{k-1}^{n-2}}{w_k^{n-3}} & w_{k+1} \end{vmatrix} \\
&= w_{k+1}w_{k(n-1)+1} + vw_k^2 \sum_{j=0}^{n-2} v^j w_{k-1}^j w_{k(n-j-2)+1} \\
&= w_{kn+1} = \mathrm{LHS}.
\end{aligned}
$$

In the final calculations, we used Lemma 2.3. To prove the other identity, we see that the expansion of its determinant is exactly the same as the identity (2.3), where n is replaced by $n+1$. □

2.2. Fibonacci polynomials

By setting $v = 1$ and $u = x$ in Theorem 2.2, we have the determinantal expressions of the Fibonacci polynomials.

Corollary 2.4. *For $n \ge 3$ and $k \ge 1$, we have the $n \times n$ determinants:*

$$
f_{kn+1} = \begin{vmatrix} f_{k+1} & -f_k & 0 & \cdots & & \cdots & 0 \\ f_k & f_{k+1} & -f_k & & & & \vdots \\ f_{k-1} & f_k & f_{k+1} & & & & \\ \frac{f_{k-1}^2}{f_k} & f_{k-1} & & \ddots & & & \vdots \\ \vdots & & & & f_{k+1} & -f_k & 0 \\ \frac{f_{k-1}^{n-3}}{f_k^{n-4}} & \frac{f_{k-1}^{n-4}}{f_k^{n-5}} & & & f_k & f_{k+1} & -f_k \\ \frac{f_{k-1}^{n-2}}{f_k^{n-3}} & \frac{f_{k-1}^{n-3}}{f_k^{n-4}} & \cdots & \frac{f_{k-1}^2}{f_k} & f_{k-1} & f_k & f_{k+1} \end{vmatrix}
$$

and

$$f_{kn}=\begin{vmatrix} f_k & -f_k & 0 & \cdots & & \cdots & 0\\ f_{k-1} & f_{k+1} & -f_k & & & & \vdots\\ \frac{f_{k-1}^2}{f_k} & f_k & f_{k+1} & & & & \\ \frac{f_{k-1}^3}{f_k^2} & f_{k-1} & & \ddots & & & \vdots\\ \vdots & & & & f_{k+1} & -f_k & 0\\ \frac{f_{k-1}^{n-2}}{f_k^{n-3}} & \frac{f_{k-1}^{n-4}}{f_k^{n-5}} & & & f_k & f_{k+1} & -f_k\\ \frac{f_{k-1}^{n-1}}{f_k^{n-2}} & \frac{f_{k-1}^{n-3}}{f_k^{n-4}} & \cdots & \frac{f_{k-1}^2}{f_k} & f_{k-1} & f_k & f_{k+1} \end{vmatrix}.$$

Remark. When $k=1$ in Corollary 2.4, we have

$$f_{n+1}=\begin{vmatrix} x & -1 & 0 & \cdots & 0\\ 1 & x & -1 & & \vdots\\ 0 & & & & 0\\ \vdots & & & x & -1\\ 0 & \cdots & 0 & 1 & x \end{vmatrix}$$

and

$$f_n=\begin{vmatrix} 1 & -1 & 0 & \cdots & 0\\ 0 & x & -1 & & \vdots\\ 0 & & & & 0\\ \vdots & & & x & -1\\ 0 & \cdots & 0 & 1 & x \end{vmatrix}=\begin{vmatrix} x & -1 & 0 & \cdots & 0\\ 1 & x & -1 & & \vdots\\ 0 & & & & 0\\ \vdots & & & x & -1\\ 0 & \cdots & 0 & 1 & x \end{vmatrix}.$$

In fact, the second identity is essentially the same as the first one because the size of the first determinant is $n\times n$ and the second one is $(n-1)\times(n-1)$.

When $k=2$ in Corollary 2.4, we have

$$f_{2n+1}=\begin{vmatrix} x^2+1 & -x & 0 & \cdots & & \cdots & 0\\ x & x^2+1 & -x & & & & \vdots\\ 1 & x & x^2+1 & & & & \\ \frac{1}{x} & & & \ddots & & & \vdots\\ \vdots & & & & x^2+1 & -x & 0\\ \frac{1}{x^{n-4}} & \frac{1}{x^{n-5}} & & & x & x^2+1 & -x\\ \frac{1}{x^{n-3}} & \frac{1}{x^{n-4}} & \cdots & \frac{1}{x} & 1 & x & x^2+1 \end{vmatrix}$$

and

$$f_{2n}=\left|\begin{array}{ccccccc} x & -x & 0 & \cdots & & \cdots & 0\\ 1 & x^2+1 & -x & & & & \vdots\\ \frac{1}{x} & x & x^2+1 & & & & \\ \vdots & & & \ddots & & & \vdots\\ \frac{1}{x^{n-4}} & & & & x^2+1 & -x & 0\\ \frac{1}{x^{n-3}} & \frac{1}{x^{n-5}} & & & x & x^2+1 & -x\\ \frac{1}{x^{n-2}} & \frac{1}{x^{n-4}} & \frac{1}{x^{n-5}} & \cdots & 1 & x & x^2+1 \end{array}\right|.$$

There exist further different determinantal expressions of f_{kn+1} and f_{kn}. An alternative representation is given below. The proof proceeds by induction on n and the recurrence relation (2.3) with $v=1$ and $u=x$, expanding along the first row.

Proposition 2.5. *For $n\ge 1$, we have*

$$f_{kn+1}=\left|\begin{array}{cccccc} f_{k+1} & -1 & 0 & \cdots & \cdots & 0\\ f_k^2 & f_{k+1} & -1 & & & \vdots\\ f_k^2 f_{k-1} & f_k^2 & f_{k+1} & & & \vdots\\ \vdots & f_k^2 f_{k-1} & f_k^2 & & & 0\\ f_k^2 f_{k-1}^{n-3} & & & \ddots & \ddots & -1\\ f_k^2 f_{k-1}^{n-2} & f_k^2 f_{k-1}^{n-3} & \cdots & f_k^2 f_{k-1} & f_k^2 & f_{k+1} \end{array}\right|$$

and

$$f_{kn}=\left|\begin{array}{cccccc} f_k & -1 & 0 & \cdots & \cdots & 0\\ f_k f_{k-1} & f_{k+1} & -1 & & & \vdots\\ f_k f_{k-1}^2 & f_k^2 & f_{k+1} & & & \vdots\\ \vdots & & & \ddots & & 0\\ f_k f_{k-1}^{n-2} & f_k^2 f_{k-1}^{n-4} & & f_k^2 & f_{k+1} & -1\\ f_k f_{k-1}^{n-1} & f_k^2 f_{k-1}^{n-3} & f_k^2 f_{k-1}^{n-4} & \cdots & f_k^2 & f_{k+1} \end{array}\right|.$$

Remark. When $k=1$ in Proposition 2.5, again we have

$$f_{n+1}=\left|\begin{array}{ccccc} x & -1 & 0 & \cdots & 0\\ 1 & x & -1 & & \vdots\\ 0 & & & & 0\\ \vdots & & & x & -1\\ 0 & \cdots & 0 & 1 & x \end{array}\right|$$

and

$$f_n = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & & \vdots \\ 0 & & & & 0 \\ \vdots & & & x & -1 \\ 0 & \cdots & 0 & 1 & x \end{vmatrix} = \begin{vmatrix} x & -1 & 0 & \cdots & 0 \\ 1 & x & -1 & & \vdots \\ 0 & & & & 0 \\ \vdots & & & x & -1 \\ 0 & \cdots & 0 & 1 & x \end{vmatrix}.$$

When $k = 2$ in Proposition 2.5, we have

$$f_{2n+1} = \begin{vmatrix} x^2+1 & -1 & 0 & \cdots & \cdots & 0 \\ x^2 & x^2+1 & -1 & & & \vdots \\ x^2 & x^2 & x^2+1 & & & \vdots \\ x^2 & x^2 & x^2 & & & 0 \\ \vdots & & & \ddots & \ddots & -1 \\ x^2 & x^2 & \cdots & \cdots & x^2 & x^2+1 \end{vmatrix}$$

and

$$f_{2n} = \begin{vmatrix} x & -1 & 0 & \cdots & \cdots & 0 \\ x & x^2+1 & -1 & & & \vdots \\ x & x^2 & x^2+1 & & & \vdots \\ \vdots & & & \ddots & & 0 \\ x & x^2 & & x^2 & x^2+1 & -1 \\ x & x^2 & x^2 & \cdots & x^2 & x^2+1 \end{vmatrix}$$

(see, [11], p. 4, [3], Prop. 1). See also related results on Hessenberg determinants [12, 17].

3. Horadam–lucas polynomials

Theorem 3.1. *For $n \geq 1$ and $k \geq 2$, we have the $n \times n$ determinants:*

$$\omega_{kn-1} = \begin{vmatrix} \omega_{k-1} & -vw_k & 0 & \cdots & & \cdots & 0 \\ \omega_{k-2} & w_{k+1} & -vw_k & & & & \vdots \\ \frac{\omega_{k-2}w_{k-1}}{w_k} & w_k & w_{k+1} & & & & \\ \frac{\omega_{k-2}w_{k-1}^2}{w_k^2} & w_{k-1} & w_k & & & & \vdots \\ \vdots & & & \ddots & & & 0 \\ \frac{\omega_{k-2}w_{k-1}^{n-3}}{w_k^{n-3}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & & & w_k & w_{k+1} & -vw_k \\ \frac{\omega_{k-2}w_{k-1}^{n-2}}{w_k^{n-2}} & \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & \cdots & w_{k-1} & w_k & w_{k+1} \end{vmatrix}$$

and

$$\omega_{kn}=\begin{vmatrix}\omega_k & -vw_k & 0 & \cdots & & \cdots & 0\\ \omega_{k-1} & w_{k+1} & -vw_k & & & & \vdots\\ \frac{\omega_{k-1}w_{k-1}}{w_k} & w_k & w_{k+1} & & & & \\ \frac{\omega_{k-1}w_{k-1}^2}{w_k^2} & w_{k-1} & w_k & & & & \vdots\\ \vdots & & & \ddots & & & 0\\ \frac{\omega_{k-1}w_{k-1}^{n-3}}{w_k^{n-3}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & & & w_k & w_{k+1} & -vw_k\\ \frac{\omega_{k-1}w_{k-1}^{n-2}}{w_k^{n-2}} & \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & \cdots & w_{k-1} & w_k & w_{k+1}\end{vmatrix}.$$

Proof of Theorem 3.1. Notice that

$$\omega_{n+m}=w_{m+1}\omega_n+vw_m\omega_{n-1} \tag{3.1}$$

(see, *e.g.*, [9], Thm. 46.7). The result is trivial for $n=1$. It is also valid for $n=2$ because by using (3.1) with $m=k$ and $n=k-1$,

$$\omega_{2k-1}=\begin{vmatrix}\omega_{k-1} & -vw_k\\ \omega_{k-2} & w_{k+1}\end{vmatrix}.$$

Assume that the determinant expressions are valid up to $n-1$. Remember the determinant expression of w_{kn+1} in Theorem 2.2. Expanding the right-hand side of the first equation along the first row repeatedly gives us

$$\begin{aligned}\mathrm{RHS}&=\omega_{k-1}w_{k(n-1)+1}+vw_k\begin{vmatrix}\omega_{k-2} & -vw_k & 0 & \cdots & \cdots & 0\\ \frac{\omega_{k-2}w_{k-1}}{w_k} & w_{k+1} & -vw_k & & & \vdots\\ \frac{\omega_{k-2}w_{k-1}^2}{w_k^2} & w_k & w_{k+1} & & & \vdots\\ \vdots & & \ddots & & & 0\\ \frac{\omega_{k-2}w_{k-1}^{n-3}}{w_k^{n-3}} & \frac{w_{k-1}^{n-5}}{w_k^{n-6}} & & w_k & w_{k+1} & -vw_k\\ \frac{\omega_{k-2}w_{k-1}^{n-2}}{w_k^{n-2}} & \frac{w_{k-1}^{n-4}}{w_k^{n-5}} & \cdots & w_{k-1} & w_k & w_{k+1}\end{vmatrix}\\
&=\omega_{k-1}w_{k(n-1)+1}+\omega_{k-2}\Bigg(vw_kw_{k(n-2)+1}\\
&\quad+v^2w_k^2\begin{vmatrix}\frac{w_{k-1}}{w_k} & -vw_k & 0 & \cdots & \cdots & 0\\ \frac{w_{k-1}^2}{w_k^2} & w_{k+1} & -vw_k & & & \vdots\\ \frac{w_{k-1}^3}{w_k^3} & w_k & w_{k+1} & & & \vdots\\ \vdots & & \ddots & & & 0\\ \frac{w_{k-1}^{n-3}}{w_k^{n-3}} & \frac{w_{k-1}^{n-6}}{w_k^{n-7}} & & w_k & w_{k+1} & -vw_k\\ \frac{w_{k-1}^{n-2}}{w_k^{n-2}} & \frac{w_{k-1}^{n-5}}{w_k^{n-6}} & \cdots & w_{k-1} & w_k & w_{k+1}\end{vmatrix}\Bigg)\\
&=\omega_{k-1}w_{k(n-1)+1}+\omega_{k-2}\big(vw_kw_{k(n-2)+1}+v^2w_kw_{k-1}w_{k(n-3)+1}\end{aligned}$$

$$
\begin{aligned}
&+ v^3 w_k w_{k-1}^2 w_{k(n-4)+1} + \cdots + v^{n-2} w_k^{n-2} \begin{vmatrix} \frac{w_{k-1}^{n-3}}{w_k^{n-3}} & -v w_k \\ \frac{w_{k-1}^{n-2}}{w_k^{n-2}} & w_{k+1} \end{vmatrix} \Bigg) \\
&= \omega_{k-1} w_{k(n-1)+1} + v \omega_{k-2} w_k \sum_{j=0}^{n-2} v^j w_{k-1}^j w_{k(n-j-2)+1} \\
&= \omega_{k-1} w_{k(n-1)+1} + v \omega_{k-2} w_{k(n-1)} = \omega_{kn-1} = \text{LHS}\,.
\end{aligned}
$$

In the final steps, we used (2.3) and the identity (3.1) with $n = k-1$ and $m = k(n-1)$.

The determinantal expression of ω_{kn} is different from that of ω_{kn-1} in that the subscript of ω_k in the first column is just increased by one, so expanding the determinant in the same way and using induction, we get

$$
\begin{aligned}
\text{RHS} &= \omega_k w_{k(n-1)+1} + v \omega_{k-1} w_k \sum_{j=0}^{n-2} v^j w_{k-1}^j w_{k(n-j-2)+1} \\
&= \omega_k w_{k(n-1)+1} + v \omega_{k-1} w_{k(n-1)} = \omega_{kn} = \text{LHS}\,.
\end{aligned}
$$

□

3.1. Lucas polynomials

When $v = 1$ and $u = x$ in Theorem 3.1, we have the determinantal expressions of Lucas polynomials.

Corollary 3.2. *For $n \ge 1$ and $k \ge 2$, we have the $n \times n$ determinants:*

$$
l_{kn-1} = \begin{vmatrix}
l_{k-1} & -f_k & 0 & \cdots & & \cdots & 0 \\
l_{k-2} & f_{k+1} & -f_k & & & & \vdots \\
\frac{l_{k-2} f_{k-1}}{f_k} & f_k & f_{k+1} & & & & \\
\frac{l_{k-2} f_{k-1}^2}{f_k^2} & f_{k-1} & f_k & & & & \vdots \\
\vdots & & & \ddots & & & 0 \\
\frac{l_{k-2} f_{k-1}^{n-3}}{f_k^{n-3}} & \frac{f_{k-1}^{n-4}}{f_k^{n-5}} & & & f_k & f_{k+1} & -f_k \\
\frac{l_{k-2} f_{k-1}^{n-2}}{f_k^{n-2}} & \frac{f_{k-1}^{n-3}}{f_k^{n-4}} & \frac{f_{k-1}^{n-4}}{f_k^{n-5}} & \cdots & f_{k-1} & f_k & f_{k+1}
\end{vmatrix}
$$

and

$$
l_{kn} = \begin{vmatrix}
l_k & -f_k & 0 & \cdots & & \cdots & 0 \\
l_{k-1} & f_{k+1} & -f_k & & & & \vdots \\
\frac{l_{k-1} f_{k-1}}{f_k} & f_k & f_{k+1} & & & & \\
\frac{l_{k-1} f_{k-1}^2}{f_k^2} & f_{k-1} & f_k & & & & \vdots \\
\vdots & & & \ddots & & & 0 \\
\frac{l_{k-1} f_{k-1}^{n-3}}{f_k^{n-3}} & \frac{f_{k-1}^{n-4}}{f_k^{n-5}} & & & f_k & f_{k+1} & -f_k \\
\frac{l_{k-1} f_{k-1}^{n-2}}{f_k^{n-2}} & \frac{f_{k-1}^{n-3}}{f_k^{n-4}} & \frac{f_{k-1}^{n-4}}{f_k^{n-5}} & \cdots & f_{k-1} & f_k & f_{k+1}
\end{vmatrix}\,.
$$

Remark. When $k=2$ in Corollary 3.2, for $n\ge 1$, we have the $n\times n$ determinants:

$$l_{2n-1}=\begin{vmatrix} x & -x & 0 & \cdots & & \cdots & 0\\ 2 & x^2+1 & -x & & & & \vdots\\ \frac{2}{x} & x & x^2+1 & & & & \\ \vdots & & & \ddots & & & \vdots\\ \frac{2}{x^{n-4}} & & & & x^2+1 & -x & 0\\ \frac{2}{x^{n-3}} & \frac{1}{x^{n-5}} & & & x & x^2+1 & -x\\ \frac{2}{x^{n-2}} & \frac{1}{x^{n-4}} & \frac{1}{x^{n-5}} & \cdots & 1 & x & x^2+1 \end{vmatrix}$$

and

$$l_{2n}=\begin{vmatrix} x^2+2 & -x & 0 & \cdots & & \cdots & 0\\ x & x^2+1 & -x & & & & \vdots\\ 1 & x & x^2+1 & & & & \\ \vdots & & & \ddots & & & \vdots\\ \frac{1}{x^{n-5}} & & & & x^2+1 & -x & 0\\ \frac{1}{x^{n-4}} & \frac{1}{x^{n-5}} & & & x & x^2+1 & -x\\ \frac{1}{x^{n-3}} & \frac{1}{x^{n-4}} & \frac{1}{x^{n-5}} & \cdots & 1 & x & x^2+1 \end{vmatrix}.$$

When $k=2$ and $x=1$ in Corollary 3.2, we have determinant expressions of Lucas numbers. For $n\ge 1$, we have the $n\times n$ determinants:

$$L_{2n-1}=\begin{vmatrix} 1 & -1 & 0 & \cdots & & \cdots & 0\\ 2 & 2 & -1 & & & & \vdots\\ 2 & 1 & 2 & & & & \\ \vdots & & & \ddots & & & \vdots\\ 2 & & & & 2 & -1 & 0\\ 2 & 1 & & & 1 & 2 & -1\\ 2 & 1 & 1 & \cdots & 1 & 1 & 2 \end{vmatrix}$$

and

$$L_{2n}=\begin{vmatrix} 3 & -1 & 0 & \cdots & & \cdots & 0\\ 1 & 2 & -1 & & & & \vdots\\ 1 & 1 & 2 & & & & \\ \vdots & & & \ddots & & & \vdots\\ 1 & & & & 2 & -1 & 0\\ 1 & 1 & & & 1 & 2 & -1\\ 1 & 1 & 1 & \cdots & 1 & 1 & 2 \end{vmatrix}.$$

4. Hessenberg matrix

We recall that a lower Hessenberg matrix has the form

$$H_n = \begin{bmatrix} h_{11} & h_{12} & 0 & \cdots & \cdots & \dots & 0 \\ h_{21} & h_{22} & h_{23} & 0 & \cdots & \dots & \vdots \\ h_{31} & h_{32} & h_{33} & h_{34} & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & & & 0 \\ h_{n-1,1} & h_{n-1,2} & \cdots & \cdots & \cdots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n1} & h_{n2} & \cdots & \cdots & \cdots & h_{n,n-1} & h_{nn} \end{bmatrix}$$

and note the fact that a tridiagonal matrix is a special Hessenberg matrix. By considering this fact here, we will present an interesting generalization of the first determinant equality given in Theorem 2.2.

Define the lower Hessenberg matrix $M_n(k) = M_n(k,u,v)$ of order n as

$$M_n(k) = \begin{bmatrix} w_{k+1} & -vw_k & 0 & \cdots & \cdots & \cdots & 0 \\ w_k & w_{k+1} & -vw_k & 0 & \cdots & & \vdots \\ w_{k-1} & w_k & w_{k+1} & -vw_k & 0 & \ddots & \vdots \\ \frac{w_{k-1}^2}{w_k} & w_{k-1} & w_k & \ddots & \ddots & \ddots & \vdots \\ \frac{w_{k-1}^3}{w_k^2} & \frac{w_{k-1}^2}{w_k} & w_{k-1} & \ddots & w_{k+1} & -vw_k & 0 \\ \vdots & \vdots & \vdots & \ddots & w_k & w_{k+1} & -vw_k \\ \frac{w_{k-1}^{n-2}}{w_k^{n-3}} & \frac{w_{k-1}^{n-3}}{w_k^{n-4}} & \cdots & \frac{w_{k-1}^2}{w_k} & w_{k-1} & w_k & w_{k+1} \end{bmatrix},$$

which arrangement of elements is the same as the determinant giving w_{kn+1} in Theorem 2.2. Namely, $\det M_n(k) = w_{kn+1}$.

We give some special cases of the Hessenberg matrix $M_n(k,u,v)$. If we take $k = v = 1$ and $u = x$, then we get

$$M_n(1,x,1) = \begin{bmatrix} x & -1 & 0 & \cdots & \cdots & 0 \\ 1 & x & -1 & 0 & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & x & -1 & 0 \\ \vdots & & \ddots & 1 & x & -1 \\ 0 & \cdots & \cdots & 0 & 1 & x \end{bmatrix}$$

and so we already know that

$$\det M_n(1,x,1) = f_{n+1}.$$

We now generalize the above result and present one of our main results, namely

$$\det M_n(k) = w_{kn+1}.$$

We define the Horadam–Lucas analogue of the matrix $M_n(k)$ denoted by $T_n(k)$ and we will show that

$$\det T_n(k) = \omega_{kn+1},$$

where ω_n is the nth Horadam–Lucas polynomial.

In this paper we shall derive the matrices L and U arising from the LU-decompositions of the matrices $M_n(k)$ and $T_n(k)$ as well as the inverse matrices L^{-1} and U^{-1} for the matrices $M_n(k)$ and $T_n(k)$. By the way, one can formulate the inverses of the matrices $M_n(k)$ and $T_n(k)$ in closed form by using the matrices L^{-1} and U^{-1} and their products. But here we can explicitly formulate the inverse matrices $M_n^{-1}(k)$ and $T_n^{-1}(k)$. We can also evaluate the determinants of the matrices $M_n(k)$ and $T_n(k)$ by using their LU-decompositions.

Then we have its LU-decomposition with the matrices U and L

$$(U)_{1\le i,j\le n} = \begin{cases} \dfrac{w_{ik+1}}{w_{(i-1)k+1}} & \text{if } i=j, \\ -vw_k & \text{if } j=i+1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(L)_{1\le i,j\le n} = \begin{cases} 1 & \text{if } i=j, \\ \left(\dfrac{w_{k-1}}{w_k}\right)^{i-j-1} \dfrac{w_{jk}}{w_{jk+1}} & \text{if } i>j, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $n=5$, then we have that

$$U = \begin{bmatrix} w_{k+1} & -vw_k & 0 & 0 & 0 \\ 0 & \frac{w_{2k+1}}{w_{k+1}} & -vw_k & 0 & 0 \\ 0 & 0 & \frac{w_{3k+1}}{w_{2k+1}} & -vw_k & 0 \\ 0 & 0 & 0 & \frac{w_{4k+1}}{w_{3k+1}} & -vw_k \\ 0 & 0 & 0 & 0 & \frac{w_{5k+1}}{w_{4k+1}} \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{w_k}{w_{k+1}} & 1 & 0 & 0 & 0 \\ \frac{w_{k-1}}{w_{k+1}} & \frac{w_{2k}}{w_{2k+1}} & 1 & 0 & 0 \\ \frac{w_{k-1}^2}{w_k w_{k+1}} & \frac{w_{k-1}w_{2k}}{w_k w_{2k+1}} & \frac{w_{3k}}{f_{3k+1}} & 1 & 0 \\ \frac{w_{k-1}^3}{w_k^2 w_{k+1}} & \frac{w_{k-1}^2 w_{2k}}{w_k^2 w_{2k+1}} & \frac{w_{k-1}w_{3k}}{w_k f_{3k+1}} & \frac{w_{4k}}{w_{4k+1}} & 1 \end{bmatrix}.$$

We have the inverse matrices U^{-1} and L^{-1} as

$$(L^{-1})_{1\le i,j\le n} = \begin{cases} 1 & \text{if } i=j, \\ (-1)^{i+j}\dfrac{w_{jk}}{w_k^{i-j-1}w_{(i-1)k+1}} & \text{if } i>j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(U^{-1})_{1\le i,j\le n} = \begin{cases} \dfrac{(vw_k)^{j-i}w_{(i-1)k+1}}{w_{jk+1}} & \text{if } j\ge i, \\ 0 & \text{otherwise.} \end{cases}$$

When $n=5$, we have

$$U^{-1} = \begin{bmatrix} \frac{1}{w_{k+1}} & \frac{vw_k}{w_{2k+1}} & \frac{v^2w_k^2}{w_{3k+1}} & \frac{v^3w_k^3}{w_{4k+1}} & \frac{v^4w_k^4}{w_{5k+1}} \\ 0 & \frac{w_{k+1}}{w_{2k+1}} & \frac{vw_kw_{k+1}}{w_{3k+1}} & \frac{v^2w_k^2w_{k+1}}{w_{4k+1}} & \frac{v^3w_k^3w_{k+1}}{w_{5k+1}} \\ 0 & 0 & \frac{w_{2k+1}}{w_{3k+1}} & \frac{vw_kw_{2k+1}}{w_{4k+1}} & \frac{v^2w_k^2w_{2k+1}}{w_{5k+1}} \\ 0 & 0 & 0 & \frac{w_{3k+1}}{w_{4k+1}} & \frac{vw_kw_{3k+1}}{w_{5k+1}} \\ 0 & 0 & 0 & 0 & \frac{w_{4k+1}}{w_{5k+1}} \end{bmatrix}$$

and

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\dfrac{w_k}{w_{k+1}} & 1 & 0 & 0 & 0 \\ \dfrac{1}{w_{2k+1}} & -\dfrac{w_{2k}}{w_{2k+1}} & 1 & 0 & 0 \\ -\dfrac{1}{w_kw_{3k+1}} & \dfrac{w_{2k}}{w_kw_{3k+1}} & -\dfrac{w_{3k}}{w_{3k+1}} & 1 & 0 \\ \dfrac{1}{w_k^2w_{4k+1}} & -\dfrac{w_{2k}}{w_k^2w_{4k+1}} & \dfrac{w_{3k}}{w_kw_{4k+1}} & -\dfrac{w_{4k}}{w_{4k+1}} & 1 \end{bmatrix}.$$

4.1. Explicit formula for the inverse matrix

We mentioned before that the inverse matrix $M_n^{-1}(k)$ could be derived by using $M_n^{-1}(k) = U^{-1}L^{-1}$ in closed form. But as we mentioned before, we will present an explicit formula for the inverse matrix $M_n^{-1}(k)$ in the following result.

Theorem 4.1. *For $n>2$, the inverse matrix $M_n^{-1}(k)$ of order n is given by*

$$\left(M_n^{-1}(k)\right)_{1\le i,j\le n} = \frac{w_k^{j-i}}{w_{kn+1}} \begin{cases} (-1)^{i+j}v^{(n-1)(i-j-1)}w_{jk}w_{k(n-i+1)} & \text{if } i>j, \\ v^{j-i}w_{k(i-1)+1}w_{k(n-j)+1} & \text{if } j\ge i. \end{cases}$$

For example, when $n = 5$, it has the form

$$M_5^{-1}(k) = \frac{1}{w_{5k+1}} \begin{bmatrix} w_{4k+1} & v w_k w_{3k+1} & v^2 w_k^2 w_{2k+1} & v^3 w_k^3 w_{k+1} & v^4 w_k^4 \\ -w_{4k} & w_{k+1} w_{3k+1} & v w_k w_{k+1} w_{2k+1} & v^2 w_k^2 w_{k+1}^2 & v^3 w_k^3 w_{k+1} \\ \frac{v^4 w_{3k}}{w_k} & -\frac{w_{2k} w_{3k}}{w_k} & w_{2k+1}^2 & v w_k w_{k+1} w_{2k+1} & v^2 w_k^2 w_{2k+1} \\ -\frac{v^8 w_{2k}}{w_k^2} & v^4 w_{2k}^2 w_k & -\frac{w_{3k} w_{2k}}{w_k} & w_{3k+1} w_{k+1} & v w_k w_{3k+1} \\ \frac{v^{12}}{w_k^2} & -\frac{v^8 w_{2k}}{w_k^2} & \frac{v^4 w_{3k}}{w_k} & -w_{4k} & w_{4k+1} \end{bmatrix}.$$

5. Applications by Trudi's formula

The forms of determinant expressions in Theorem 2.2 remind us Trudi's formula [13], Vol. 3, p. 214, [14], and the case $a_0 = 1$ of this formula is known as Brioschi's formula [13], Vol. 3, pp. 208–209, [15].

Lemma 5.1. *For a positive integer n, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}$$

$$= \sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1,t_2,\ldots,t_n\ge 0}} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ *are the multinomial coefficients.*

By applying Lemma 5.1 to the identity of w_{kn+1} in Theorem 2.2, we have the following.

Proposition 5.2. *For $n \ge 1$, we have*

$$w_{kn+1} = \sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1,t_2,\ldots,t_n\ge 0}} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} v^{n-t_1-\cdots-t_n} w_{k+1}^{t_1} w_k^{n-t_1-T_n} w_{k-1}^{T_n},$$

where $T_n = \sum_{i=2}^n (i-2)t_i$.

When $u = v = 1$ in Proposition 5.2, we have the following Fibonacci expression.

Corollary 5.3. *For $n \ge 1$, we have*

$$F_{kn+1} = \sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1,t_2,\ldots,t_n\ge 0}} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} F_{k+1}^{t_1} F_k^{n-t_1-T_n} F_{k-1}^{T_n}.$$

Remark. If $k = 2$ in Corollary 5.3, we have

$$F_{2n+1} = \sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1,t_2,\ldots,t_n\ge 0}} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} 2^{t_1}$$

([3], Cor. 3).

Proof of Proposition 5.2. By Lemma 5.1, we have

$$
\begin{aligned}
&w_{kn+1}\\
&=\sum_{\substack{t_1+2t_2+\cdots+nt_n=n\\ t_1,t_2,\dots,t_n\ge 0}}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(vw_k)^{n-t_1-\cdots-t_n}w_{k+1}^{t_1}w_k^{t_2}\\
&\quad\times w_{k-1}^{t_3}\left(\frac{w_{k-1}^2}{w_k}\right)^{t_4}\left(\frac{w_{k-1}^3}{w_k^2}\right)^{t_5}\cdots\left(\frac{w_{k-1}^{n-2}}{w_k^{n-3}}\right)^{t_n}\\
&=\sum_{\substack{t_1+2t_2+\cdots+nt_n=n\\ t_1,t_2,\dots,t_n\ge 0}}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}v^{n-t_1-\cdots-t_n}w_{k+1}^{t_1}w_k^{n-t_1-\sum_{i=2}^n(i-2)t_i}w_{k-1}^{\sum_{i=2}^n(i-2)t_i}\,.
\end{aligned}
$$

□

6. Applications of the inversion formula

One basic inversion formula is explained as follows (see, *e.g.*, [16]).

Lemma 6.1. *For two sequences* $\{\alpha_n\}_{n\ge 0}$ *and* $\{\beta_n\}_{n\ge 0}$ *with* $\alpha_0=\beta_0=1$, *we have for* $n\ge 1$

$$
\begin{aligned}
&\alpha_n=\begin{vmatrix}
\beta_1 & 1 & 0 & \cdots & 0\\
\beta_2 & \beta_1 & 1 & & \vdots\\
\vdots & & \ddots & & 0\\
\beta_{n-1} & & & \beta_1 & 1\\
\beta_n & \beta_{n-1} & \cdots & \beta_2 & \beta_1
\end{vmatrix}\\
\Longleftrightarrow\quad &\sum_{k=0}^n(-1)^{n-k}\alpha_k\beta_{n-k}=0\\
\Longleftrightarrow\quad &\beta_n=\begin{vmatrix}
\alpha_1 & 1 & 0 & \cdots & 0\\
\alpha_2 & \alpha_1 & 1 & & \vdots\\
\vdots & & \ddots & & 0\\
\alpha_{n-1} & & & \alpha_1 & 1\\
\alpha_n & \alpha_{n-1} & \cdots & \alpha_2 & \alpha_1
\end{vmatrix}\,.
\end{aligned}
$$

To apply the determinant expression of f_{kn+1} in Theorem 2.2, we need a different inversion formula ([3], Lem. 7). When $c=\alpha_1=\beta_1$ and $d=1$, it is reduced to Lemma 6.1.

Lemma 6.2. *Let* c *and* d *be some constants or polynomials. Then, for* $n\ge 2$, *we have*

$$
\alpha_n=\begin{vmatrix}
c & d & 0 & \cdots & 0\\
\beta_2 & c & d & & \vdots\\
\vdots & & \ddots & & 0\\
\beta_{n-1} & & & c & d\\
\beta_n & \beta_{n-1} & \cdots & \beta_2 & c
\end{vmatrix}
$$

$$\Longleftrightarrow \quad \alpha_n - c\alpha_{n-1} + \sum_{k=0}^{n-2}(-1)^{n-k} d^{n-k-1}\alpha_k\beta_{n-k} = 0 \quad (\alpha_0 = 1)$$

$$\Longleftrightarrow \quad \beta_n = \frac{1}{d^{n-1}}\begin{vmatrix} c & 1 & 0 & \cdots & 0 \\ \alpha_2 & c & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ \alpha_{n-1} & & & c & 1 \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_2 & c \end{vmatrix}.$$

Finally, we can get the determinant whose entries are odd Horadam polynomials. It is interesting to see that even as the size increases, the absolute value of the determinant remains the same.

Proposition 6.3. *For $n \ge 2$, we have*

$$\begin{vmatrix} w_{k+1} & 1 & 0 & \cdots & 0 \\ w_{2k+1} & w_{k+1} & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ w_{k(n-1)+1} & & & w_{k+1} & 1 \\ w_{kn+1} & w_{k(n-1)+1} & \cdots & w_{2k+1} & w_{k+1} \end{vmatrix} = (-v)^{n-1} w_k^2 w_{k-1}^{n-2}.$$

When $u = v = 1$ in Proposition 6.3, we can get the determinants whose entries are odd Fibonacci numbers.

Corollary 6.4. *For $n \ge 2$, we have*

$$\begin{vmatrix} F_{k+1} & 1 & 0 & \cdots & 0 \\ F_{2k+1} & F_{k+1} & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ F_{k(n-1)+1} & & & F_{k+1} & 1 \\ F_{kn+1} & F_{k(n-1)+1} & \cdots & F_{2k+1} & F_{k+1} \end{vmatrix} = (-1)^{n-1} F_k^2 F_{k-1}^{n-2}.$$

Proof of Proposition 6.3. Set $\alpha_n = w_{kn+1}$, $\beta_n = w_{k-1}^{n-2}/w_k^{n-3}$ $(n \ge 2)$, $c = w_{k+1}$ and $d = -vw_k$ in Lemma 6.2. Then, by the determinantal expression of w_{kn+1} in Theorem 2.2, we have

$$\frac{w_{k-1}^{n-2}}{w_k^{n-3}} = \frac{1}{(-vw_k)^{n-1}}\begin{vmatrix} w_{k+1} & 1 & 0 & \cdots & 0 \\ w_{2k+1} & w_{k+1} & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ w_{k(n-1)+1} & & & w_{k+1} & 1 \\ w_{kn+1} & w_{k(n-1)+1} & \cdots & w_{2k+1} & w_{k+1} \end{vmatrix}.$$

□

7. Final comments

For $k > 2$, the indices kn and $kn + 1$ do not cover the entire set of positive integers. However, for Horadam numbers (polynomials) w_n, by Lemma 2.1 (viii), we have the identity

$$w_{kn+m} = w_{kn+1}w_m + vw_{kn}w_{m-1}\,,$$

which shows that the values w_{kn} and w_{kn+1} are fundamental, since all Horadam numbers (polynomials) can be obtained as linear combinations of them. It is similar for Horadam–Lucas numbers (polynomials) ω_n because of Lemma 2.1 (ix).

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