RAIRO-Theor. Inf. Appl. 60 (2026) 14
https://doi.org/10.1051/ita/2026015

RAIRO - Theoretical Informatics and Applications
www.rairo-ita.org

GOLDBACH–LINNIK TYPE PROBLEMS WITH ONE PRIME SQUARE AND SIX PRIME CUBES

Xue Han*

Abstract. Let $p_1, p_2, \cdots, p_7$ be prime numbers. In this paper, we first show that when $k_1 \geq 49$, any sufficiently large odd integer can be represented as the sum of a prime square, six prime cubes and k_1 powers of 2. Furthermore, we prove that for $k_2 \geq 73$, every pair of sufficiently large odd integers can be expressed as a pair of equations involving a prime square, six prime cubes and k_2 powers of 2.

Mathematics Subject Classification. 11A41, 11P05, 11P32, 11P55.

Received September 23, 2025. Accepted April 3, 2026.

1. Introduction

To advance research on the Goldbach conjecture, Linnik [1, 2] proved that every sufficiently large even integer n_1 can be written as the sum of two primes and K_1 powers of 2, specifically

$$n_1 = p_1 + p_2 + \sum_{h=1}^{K_1} 2^{v_h}. \tag{1.1}$$

Many researchers have focused on investigating the magnitude of K_1, and the current best result is $K_1 = 8$, due to Pintz and Ruzsa [3]. Such representation problems are referred to in the literature as the Goldbach–Linnik problem.

The Goldbach–Linnik problem involving four prime squares was investigated by Liu, Liu and Zhan [4], who proved that every sufficiently large even integer n_2 can be expressed as the sum of four prime squares and K_2 powers of 2, namely

$$n_2 = \sum_{\iota=1}^{4} p_\iota^2 + \sum_{h=1}^{K_2} 2^{v_h}. \tag{1.2}$$

The value of K_2 has been considered in subsequent work, and the sharpest result to date is $K_2 = 31$, due to Hathi [5].

Keywords and phrases: Waring–Goldbach problem, primes, circle method, powers of 2.

School of Cybersecurity, Shandong University of Political Science and Law, Jinan 250014, Shandong, China.

* Corresponding author: `hanxue@sdupsl.edu.cn`

In 2001, Liu and Liu [6] considered the problem of representing every sufficiently large even integer n_3 as the sum of eight prime cubes and K_3 powers of 2, namely

$$n_3 = \sum_{\iota=1}^{8} p_\iota^3 + \sum_{h=1}^{K_3} 2^{v_h}. \tag{1.3}$$

Subsequently, several scholars have improved the bound for K_3. Recently, Hu, Long and Wang [7] proved that $K_3 = 27$ is admissible.

Some other Goldbach–Linnik problems have been studied by many scholars (see [8–12], *etc.*). In particular, relying on equations (1.2) and (1.3), Liu [13] showed that every sufficiently large integer N_1 can be expressed as the sum of a prime square, six prime cubes and finitely many powers of 2, namely

$$N_1 = p_1^2 + \sum_{\iota=2}^{7} p_\iota^3 + \sum_{h=1}^{k_1} 2^{v_h}. \tag{1.4}$$

In this paper, we first give the value of k_1 by establishing the following theorem.

Theorem 1.1. *For any integer $k_1 \geq 49$, the equation (1.4) is solvable for every sufficiently large odd integer N_1.*

To establish Theorem 1.1, we combine the circle method with certain analytic techniques. We shall estimate the sum of the singular series (see Lem. 3.1 below) by applying mathematical software to handle the heavy computations. Additionally, we employ the mean value estimates for exponential sums (see Lem. 2.4 below) and Hölder's inequality in the minor arcs, which optimizes the choice of λ defined in (2.7) (see Lem. 2.3 below). These treatments collectively contribute to the determination of the value of k_1.

On the other hand, Kong [14] considered the Goldbach–Linnik problem by investigating the simultaneous representation of pairs of sufficiently large even integers. To be specific, Kong [14] demonstrated that every pair of sufficiently large even integers B_1 and B_2 with $B_2 \gg B_1 > B_2$ can be written as the sums of two primes and K_4 powers of 2, namely

$$\begin{cases} B_1 = p_1 + p_2 + \sum_{h=1}^{K_4} 2^{v_h}, \\ B_2 = p_3 + p_4 + \sum_{h=1}^{K_4} 2^{v_h}, \end{cases} \tag{1.5}$$

who showed that $K_4 = 63$ is acceptable. Later, Kong and Liu [15] improved this result by proving $K_4 = 34$. Building upon the idea introduced by Kong [14], we further explore the simultaneous representation of every pair of sufficiently large odd integers N_2 and N_3 satisfying $N_3 \gg N_2 > N_3$ in the form

$$\begin{cases} N_2 = p_1^2 + \sum_{\iota=2}^{7} p_\iota^3 + \sum_{h=1}^{k_2} 2^{v_h}, \\ N_3 = p_8^2 + \sum_{\iota=9}^{14} p_\iota^3 + \sum_{h=1}^{k_2} 2^{v_h}. \end{cases} \tag{1.6}$$

Our second result provides the value of k_2 for which the above system of equations is solvable.

Theorem 1.2. *For any integer $k_2 \geq 73$, the equations (1.6) are solvable for every pair of sufficiently large positive odd integers N_2 and N_3 satisfying*

$$N_3 \gg N_2 > N_3. \tag{1.7}$$

In order to complete the proof of Theorem 1.2, we also employ the circle method, which naturally entails dealing with some double integrals that we split into single integrals. This permits us to apply several results derived in the proof of Theorem 1.1 to the proof of Theorem 1.2.

Notation 1.3. *Throughout this paper, p always stands for a prime, while k denotes an integer, with or without subscripts. Set $i = 1, 2, 3$, $j = 1, 2$, $l = 2, 3$ and $e(x) = e^{2\pi ix}$. The letter ϵ stands for an arbitrarily small positive number, which may differ at each occurrence.*

2. Outline and preliminary lemmas

Let

$$P_i = N_i^{\frac{3}{20}-2\epsilon}, \quad Q_i = N_i^{\frac{17}{20}+\epsilon}, \quad L_j = \log_2 \frac{N_j}{\log N_j}, \tag{2.1}$$

and

$$\mathbb{M}_i\left(a_i, q_i\right) = \left\{\alpha_i \in [0,1] : \left|\alpha_i - \frac{a_i}{q_i}\right| \le \frac{1}{q_i Q_i}\right\},$$

we define the major arcs $\mathbb{M}_i$ and the minor arcs $\gtrdot_i$ as

$$\mathbb{M}_i = \bigcup_{1\le q_i\le P_i} \bigcup_{\substack{1\le a_i\le q_i\\(a_i,q_i)=1}} \mathbb{M}_i\left(a_i, q_i\right) \quad \text{and} \quad \gtrdot_i = [0,1]\backslash\mathbb{M}_i. \tag{2.2}$$

Observe that $\mathbb{M}_i\left(a_i, q_i\right)$ are pairwise disjoint. We further define

$$\mathbb{M}' = \mathbb{M}_2 \times \mathbb{M}_3 = \left\{(\alpha_2, \alpha_3) \in [0,1]^2 : \alpha_2 \in \mathbb{M}_2, \alpha_3 \in \mathbb{M}_3\right\} \text{ and } \gtrdot' = [0,1]^2\backslash\mathbb{M}'. \tag{2.3}$$

For a sufficiently small positive constant $\varkappa$, we write

$$U_i = \left(\frac{N_i}{16(1+\varkappa)}\right)^{\frac{1}{3}}, \quad V_i = U_i^{\frac{5}{6}}. \tag{2.4}$$

We shall estimate the sums

$$\mathbb{R}(N_1) = \sum_{\substack{N_1=p_1^2+p_2^3+\cdots+p_7^3+2^{v_1}+\cdots+2^{v_{k_1}}\\p_1^2\le N_1, U_1<p_2,p_3,p_4\le 2U_1,\\V_1<p_5,p_6,p_7\le 2V_1\\4\le v_1,\cdots,v_{k_1}\le L_1}} (\log p_1)\cdots(\log p_7), \tag{2.5}$$

which counts the weighted number of solutions to the equation (1.4) in $(p_1, \cdots, p_7, v_1, \cdots, v_{k_1})$;

$$\mathbb{R}(N_2, N_3) = \sum_{\substack{N_2=p_1^2+p_2^3\cdots+p_7^3+2^{v_1}+\cdots+2^{v_{k_2}}\\N_3=p_8^2+p_9^3+\cdots+p_{14}^3+2^{v_1}+\cdots+2^{v_{k_2}}\\p_1^2\le N_2, U_2<p_2,p_3,p_4\le 2U_2, V_2<p_5,p_6,p_7\le 2V_2\\p_8^2\le N_3, U_3<p_9,p_{10},p_{11}\le 2U_3, V_3<p_{12},p_{13},p_{14}\le 2V_3\\4\le v_1,\cdots,v_{k_2}\le L_2}} (\log p_1)\cdots(\log p_{14}), \tag{2.6}$$

which counts the weighted number of solutions to the equations (1.6) in $(p_1, \cdots, p_{14}, v_1, \cdots, v_{k_2})$. Let

$$f(\alpha_i) = \sum_{p^2\le N_i} (\log p)e(\alpha_i p^2), \quad S(\alpha_i) = \sum_{U_i<p\le 2U_i} (\log p)e\left(\alpha_i p^3\right),$$

$$T(\alpha_i)=\sum_{V_i<p\le 2V_i}(\log p)e\left(\alpha_i p^3\right), \quad G_j\left(\alpha_i\right)=\sum_{4\le v\le L_j} e\left(\alpha_i 2^v\right),$$

$$\mathbb{E}_{1\lambda}=\left\{\alpha_1\in[0,1]:|G_1(\alpha_1)|\ge \lambda L_1\right\}, \quad \mathbb{E}_{2\lambda}=\left\{(\alpha_2,\alpha_3)\in[0,1]^2:|G_2(\alpha_2+\alpha_3)|\ge \lambda L_2\right\}. \tag{2.7}$$

By (2.2), (2.3), (2.5), (2.6) and orthogonality, we have

$$\begin{aligned}\mathbb{R}(N_1)&=\left(\int\limits_{\mathbb{M}_1}+\int\limits_{\gtrdot_1\cap\mathbb{E}_{1\lambda}}+\int\limits_{\gtrdot_1\setminus\mathbb{E}_{1\lambda}}\right)f(\alpha_1)S^3(\alpha_1)T^3(\alpha_1)G_1^{k_1}(\alpha_1)e(-\alpha_1N_1)\mathrm{d}\alpha_1\\&\triangleq \mathbb{R}_1(N_1)+\mathbb{R}_2(N_1)+\mathbb{R}_3(N_1),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathbb{R}(N_2,N_3)=&\left(\iint\limits_{\mathbb{M}'}+\iint\limits_{\gtrdot'\cap\mathbb{E}_{2\lambda}}+\iint\limits_{\gtrdot'\setminus\mathbb{E}_{2\lambda}}\right)f(\alpha_2)S^3(\alpha_2)T^3(\alpha_2)f(\alpha_3)S^3(\alpha_3)\\&\times T^3(\alpha_3)G_2^{k_2}\left(\alpha_2+\alpha_3\right)e\left(-\alpha_2N_2-\alpha_3N_3\right)\mathrm{d}\alpha_2\mathrm{d}\alpha_3\\&\triangleq\mathbb{R}_1\left(N_2,N_3\right)+\mathbb{R}_2\left(N_2,N_3\right)+\mathbb{R}_3\left(N_2,N_3\right).\end{aligned} \tag{2.9}$$

We write

$$\mathbb{S}(n)=\sum_{q=1}^{\infty}A(n,q), \tag{2.10}$$

where

$$A(n,q)=\frac{1}{\varphi^7(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}C_2(a,q)C_3^6(a,q)e\left(-\frac{an}{q}\right), \tag{2.11}$$

and for $l=2,3$,

$$C_l(a,q)=\sum_{\substack{m=1\\(m,q)=1}}^{q}e\left(\frac{am^l}{q}\right).$$

Lemma 2.1. *For $N_1/2\le n_1\le N_1$ and $N_l/2\le n_l\le N_l$ $(l=2,3)$, we have*

$$\int_{\mathbb{M}_1}f(\alpha_1)S^3(\alpha_1)T^3(\alpha_1)e(-\alpha_1n_1)\mathrm{d}\alpha_1=\frac{1}{1458}\mathbb{S}(n_1)\mathbb{J}(n_1)+O\left(N_1^{\frac{4}{3}}L_1^{-1}\right),$$

$$\int_{\mathbb{M}_l}f(\alpha_l)S^3(\alpha_l)T^3(\alpha_l)e(-\alpha_ln_l)\mathrm{d}\alpha_l=\frac{1}{1458}\mathbb{S}(n_l)\mathbb{J}(n_l)+O\left(N_l^{\frac{4}{3}}L_2^{-1}\right),$$

where $\mathbb{S}(n_i) \gg 1$ *for* $i = 1, 2, 3$ *and* $n_i \equiv 1 \pmod 2$, *while* $\mathbb{J}(n_i)$ *is given by*

$$\mathbb{J}(n_i) = \sum_{\substack{m_1+\cdots+m_7=n_i \\ m_1 \le N_i \\ U_i^3 < m_2, m_3, m_4 \le (2U_i)^3 \\ V_i^3 < m_5, m_6, m_7 \le (2V_i)^3}} m_1^{-\frac{1}{2}} (m_2 m_3 m_4 m_5 m_6 m_7)^{-\frac{2}{3}} \tag{2.12}$$

with $N_i^{\frac{4}{3}} \ll \mathbb{J}(n_i) \ll N_i^{\frac{4}{3}}$.

Proof. This lemma follows from a standard analysis using the circle method for enlarged major arcs (see [16, 17], *etc.*). Hence, we omit the proof. □

Lemma 2.2. *For* $(1 - \varkappa)N_i \le n_i \le N_i$, *we have*

$$\mathbb{J}(n_i) \ge 2.06491 N_i^{\frac{4}{3}}.$$

Proof. The domain over which the sum $\mathbb{J}(n_i)$ is taken can be expressed as

$$\mathbb{D} = \left\{ (m_1, \cdots, m_7) : \begin{array}{l} m_1 \le N_i, \quad U_i^3 < m_2, m_3, m_4 \le 8U_i^3, \\ V_i^3 < m_5, m_6, m_7 \le 8V_i^3 \end{array} \right\},$$

with $m_1 = n_i - m_2 - \cdots - m_7$. Let

$$\mathbb{D}^* = \left\{ (m_1, \cdots, m_7) : \begin{array}{l} m_1 \le 0.813N_i, \ U_i^3 < m_2, m_3 \le 5U_i^3, \\ U_i^3 < m_4 \le 6U_i^3, \ V_i^3 < m_5, m_6, m_7 \le 8V_i^3 \end{array} \right\}.$$

For $(m_1, \cdots, m_7) \in \mathbb{D}^*$, it follows from $(1 - \varkappa)N_i \le n_i \le N_i$ that

$$1 \le m_1 = n_i - m_2 - \cdots - m_7 \le 0.813N_i.$$

Hence $\mathbb{D}^*$ is contained in $\mathbb{D}$, so we obtain

$$\begin{aligned} \mathbb{J}(n_i) \ge & \sum_{(m_1, \cdots, m_7) \in \mathbb{D}^*} m_1^{-\frac{1}{2}} (m_2 m_3 m_4 m_5 m_6 m_7)^{-\frac{2}{3}} \\ \ge & (0.813N_i)^{-\frac{1}{2}} \sum_{U_i^3 < m_2, m_3 \le 5U_i^3} (m_2 m_3)^{-\frac{2}{3}} \sum_{U_i^3 < m_4 \le 6U_i^3} m_4^{-\frac{2}{3}} \\ & \times \sum_{V_i^3 < m_5, m_6, m_7 \le 8V_i^3} (m_5 m_6 m_7)^{-\frac{2}{3}} \\ \ge & (0.813N_i)^{-\frac{1}{2}} 3^6 (5^{\frac{1}{3}} - 1)^2 (6^{\frac{1}{3}} - 1) U_i^3 V_i^3 \\ \ge & 2.06491 N_i^{\frac{4}{3}}, \end{aligned}$$

where the sum $\sum_{a < m \le b} m^{-c}$ is well approximated by the corresponding integral $\int_a^b x^{-c} \mathrm{d}x$. □

Lemma 2.3. *Let* $\mathrm{meas}(\mathbb{E}_{j\lambda})$ *be the Lebesgue measure of* $\mathbb{E}_{j\lambda}$. *We have*

$$\mathrm{meas}\left(\mathbb{E}_{j\lambda}\right) \ll N_j^{-\frac{2}{3} - 10^{-20}}$$

with $\lambda = 0.83372$.

Proof. This lemma follows from Lemma 5 and (3.10) in [18]. ☐

Lemma 2.4. *We have*

$$\int_0^1 |S^2(\alpha_i)T^6(\alpha_i)|\mathrm{d}\alpha_i \ll N_i^{\frac{4}{3}+\epsilon}, \tag{2.13}$$

$$\int_{\mathfrak{>}_i} |f^2(\alpha_i)S^6(\alpha_i)|\mathrm{d}\alpha_i \ll N_i^{\frac{23}{12}+\epsilon}, \tag{2.14}$$

$$\int_0^1 |S^4(\alpha_i)T^4(\alpha_i)|\mathrm{d}\alpha_i \leq 0.134694091 N_i^{\frac{13}{9}}, \tag{2.15}$$

$$\int_0^1 |f^4(\alpha_i)G_1^4(\alpha_i)|\mathrm{d}\alpha_i \leq \left(\frac{32^4 \times 101 \times 1.620767}{3} + \frac{8\log^2 2}{\pi^2}\right)\frac{(1+\epsilon)^9\pi^2}{16} N_i L_1^4. \tag{2.16}$$

Proof. By (5.6) and (5.9) in [19], we can get the estimate (2.13). The proof of the estimate (2.14) is analogous to that of (4.12) in [20]. The estimates (2.15) and (2.16) can be found in Lemma 3.6 in [21] and Lemma 2.3 in [22], respectively. ☐

3. Proof of Theorem 1.1

Lemma 3.1. *Let* $\Theta(N_1, k_1) = \{n_1 \geq 2 : n_1 = N_1 - 2^{v_1} - 2^{v_2} - \cdots - 2^{v_{k_1}}\}$ *with* $k_1 \geq 30$. *Then for* $N_1 \equiv 1 \pmod 2$, *we have*

$$\sum_{\substack{n_1 \in \Theta(N_1,k_1) \\ n_1 \equiv 1 \pmod 2}} \mathbb{S}(n_1) \geq 1.97501 L_1^{k_1}. \tag{3.1}$$

Proof. By (2.11), we obtain that $A(n_i, p)$ is multiplicative and when $\nu \geq 2$,

$$A(n_i, p^\nu) = 0.$$

Then

$$\mathbb{S}(n_i) = \prod_{p \geq 2} (1 + A(n_i, p)). \tag{3.2}$$

For $p = 2$, we have

$$1 + A(n_i, 2) = 0,$$

if $n_i \equiv 0 \pmod 2$, and

$$1 + A(n_i, 2) = 2, \tag{3.3}$$

if $n_i \equiv 1 \pmod 2$).

For $p \in [3, 200]$, with the aid of mathematical software, we compute $\min_{1\le n_i \le p}(1 + A(n_i, p))$ and present the following table.

p	$1+A(n_i,p)$	p	$1+A(n_i,p)$	p	$1+A(n_i,p)$	p	$1+A(n_i,p)$
$p=3$	≥ 0.984375	$p=43$	≥ 0.999084	$p=101$	≥ 0.999999	$p=163$	≥ 0.999988
$p=5$	≥ 0.999755	$p=47$	≥ 0.999999	$p=103$	≥ 0.999965	$p=167$	≥ 0.999999
$p=7$	≥ 0.473958	$p=53$	≥ 0.999999	$p=107$	≥ 0.999999	$p=173$	≥ 0.999999
$p=11$	≥ 0.999999	$p=59$	≥ 0.999999	$p=109$	≥ 0.999968	$p=179$	≥ 0.999999
$p=13$	≥ 0.914591	$p=61$	≥ 0.999823	$p=113$	≥ 0.999999	$p=181$	≥ 0.999994
$p=17$	≥ 0.999999	$p=67$	≥ 0.999807	$p=127$	≥ 0.999963	$p=191$	≥ 0.999999
$p=19$	≥ 0.994741	$p=71$	≥ 0.999999	$p=131$	≥ 0.999999	$p=193$	≥ 0.99999
$p=23$	≥ 0.999999	$p=73$	≥ 0.9999	$p=137$	≥ 0.999999	$p=197$	≥ 0.999999
$p=29$	≥ 0.999999	$p=79$	≥ 0.999805	$p=139$	≥ 0.999967	$p=199$	≥ 0.999993
$p=31$	≥ 0.998658	$p=83$	≥ 0.999999	$p=149$	≥ 0.999999		
$p=37$	≥ 0.997606	$p=89$	≥ 0.999999	$p=151$	≥ 0.999987		
$p=41$	≥ 0.999999	$p=97$	≥ 0.999948	$p=157$	≥ 0.999985		

From the above table, we can find the fact that when $p \in [3, 200]$, the values of $1 + A(n_i, p)$ are very close to 1, except for when $p = 3, 7, 13$. In addition, we can obtain

$$(1 + A(n_i, 5))(1 + A(n_i, 11)) \prod_{p\in[17,200]} (1 + A(n_i, p)) \geq 0.98894. \tag{3.4}$$

For $p > 200$, we employ the estimate $|C_l(a, q)| \leq (l - 1)\sqrt{p} + 1$. If $p \equiv 1 \pmod 3$, then

$$\begin{aligned} 1 + A(n_i, p) &\geq 1 - \frac{\sum_{a=1}^{p-1} |C_2(p, a)C_3^6(p, a)|}{(p-1)^7} \\ &\geq 1 - \frac{(\sqrt{p} + 1)(2\sqrt{p} + 1)^6}{(p-1)^6}. \end{aligned} \tag{3.5}$$

If $p \equiv 2 \pmod 3$ and $(a, p) = 1$, then

$$1 + A(n_i, p) \geq 1 - \frac{\sum_{a=1}^{p-1} |C_2(p, a)|}{(p-1)^7} \geq 1 - \frac{\sqrt{p} + 1}{(p-1)^6}, \tag{3.6}$$

where $C_3(a, q) = -1$ is used. By (3.5) and (3.6), we further employ mathematical software to get

$$\begin{aligned} \prod_{p\in(200,10^6)} (1 + A(n_i, p)) \geq & \prod_{\substack{p\in(200,10^6) \\ p\equiv 1 \pmod 3}} \left(1 - \frac{(\sqrt{p} + 1)(2\sqrt{p} + 1)^6}{(p-1)^6}\right) \\ & \times \prod_{\substack{p\in(200,10^6) \\ p\equiv 2 \pmod 3}} \left(1 - \frac{\sqrt{p} + 1}{(p-1)^6}\right) \\ \geq & 0.99858, \end{aligned} \tag{3.7}$$

$$\prod_{p\geq 10^6}(1+A(n_i,p)) \geq \prod_{p\geq 10^6}\left(1-\frac{1}{(p-1)^2}\right)^{17} \geq 0.99998. \tag{3.8}$$

From (3.4), (3.7) and (3.8), we have

$$(1+A(n_i,5))(1+A(n_i,11))\prod_{p\geq 17}(1+A(n_i,p)) \geq 0.98751 \triangleq C. \tag{3.9}$$

Let $\mathbf{q} = 3\times 7\times 13 = 273$. By (3.2), (3.3) and (3.9), we get

$$\begin{aligned}\sum_{\substack{n_1\in\Theta(N_1,k_1)\\ n_1\equiv 1 \pmod 2}} \mathbb{S}(n_1) &\geq 2\mathbf{C}\sum_{\substack{n_1\in\Theta(N_1,k_1)\\ n_1\equiv 1 \pmod 2}}\prod_{p=3,7,13}(1+A(n_1,p))\\ &\geq 2\mathbf{C}\sum_{1\leq b\leq \mathbf{q}}\prod_{p=3,7,13}(1+A(b,p))\sum_{\substack{n_1\in\Theta(N_1,k_1)\\ n_1\equiv 1 \pmod 2\\ n_1\equiv b \pmod{\mathbf{q}}}}1.\end{aligned} \tag{3.10}$$

Let

$$\mathbf{S} = \sum_{\substack{n_1\in\Theta(N_1,k_1)\\ n_1\equiv 1 \pmod 2\\ n_1\equiv b \pmod{\mathbf{q}}}}1.$$

We adopt an argument similar to that in Lemma 4.4 of [23] and get

$$\mathbf{S} = \left(\frac{L_1}{\varpi(\mathbf{q})}+O(1)\right)^{k_1}\sum_{\substack{1\leq v_1,\cdots,v_{k_1}\leq\varpi(\mathbf{q})\\ 2^{v_1}+\cdots+2^{v_{k_1}}\equiv N_1-b \pmod{\mathbf{q}}}}1,$$

where $\varpi(\mathbf{q})$ stands for the least positive integer ϖ such that $2^{\varpi}\equiv 1 \pmod{\mathbf{q}}$. Observing that

$$\mathbf{S} = \frac{1}{\mathbf{q}}\left(\frac{L_1}{\varpi(\mathbf{q})}+O(1)\right)^{k_1}\sum_{t=0}^{\mathbf{q}-1}e\left(\frac{t(N_1-b)}{\mathbf{q}}\right)\theta^{k_1}(t),$$

we get

$$\begin{aligned}\mathbf{S} &\geq \frac{1}{\mathbf{q}}\left(\frac{L_1}{\varpi(\mathbf{q})}+O(1)\right)^{k_1}\left(\varpi(\mathbf{q})^{k_1}-(\mathbf{q}-1)\left(\max_{0<t\leq\mathbf{q}-1}|\theta(t)|\right)^{k_1}\right)\\ &\geq \frac{L_1^{k_1}}{\mathbf{q}}\left(1-(\mathbf{q}-1)\left(\frac{\max_{0<t\leq\mathbf{q}-1}|\theta(t)|}{\varpi(\mathbf{q})}\right)^{k_1}\right)+O\left(L_1^{k_1-1}\right),\end{aligned}$$

where the function $\theta(t)$ is given by

$$\theta(t) = \sum_{1\le s\le \varpi(\mathbf{q})} e\left(\frac{t2^s}{\mathbf{q}}\right).$$

By the definition of $\varpi(\mathbf{q})$, we have

$$\varpi(\mathbf{q}) = 12 \quad \text{and} \quad \max_{0<t\le \mathbf{q}-1} |\theta(t)| \approx 6.00000.$$

Therefore, we can get

$$\mathbf{S} \ge 3.663\times 10^{-3} L_1^{k_1}. \tag{3.11}$$

From (3.10) and

$$\sum_{1\le b\le p} (1+A(b,p)) = p + \sum_{1\le b\le p} A(b,p) = p,$$

we have

$$\sum_{\substack{n_1\in\Theta(N_1,k_1)\\ n_1\equiv 1 \pmod 2}} \mathbb{S}(n_1) \ge 2\times 3.663\times 10^{-3}\mathbf{Cq}L_1^{k_1} \ge 1.97501 L_1^{k_1}.$$

□

Remark 3.2. The numerical computations for the proof of this lemma were carried out in Python.

Proposition 3.3. *Let* $\mathbb{R}_1(N_1)$ *be defined as in* (2.8). *We have*

$$\mathbb{R}_1(N_1) \ge 2.79713\times 10^{-3} N_1^{\frac{4}{3}} L_1^{k_1}.$$

Proof. We deduce from Lemmas 2.1, 2.2 and 3.1 that

$$\begin{aligned}\mathbb{R}_1(N_1) \ge& \frac{1}{1458} \sum_{\substack{n_1\in\Theta(N_1,k_1)\\ n_1\equiv 1 \pmod 2}} \mathbb{S}(n_1)\mathbb{J}(n_1)\\ \ge& \frac{1.97501\times 2.06491}{1458} N_1^{\frac{4}{3}} L_1^{k_1}\\ \ge& 2.79713\times 10^{-3} N_1^{\frac{4}{3}} L_1^{k_1}.\end{aligned}$$

□

Proposition 3.4. *Let* $\mathbb{R}_2(N_1)$ *be defined as in* (2.8). *We have*

$$\mathbb{R}_2(N_1) \ll N_1^{\frac{4}{3}} L_1^{k_1-1}.$$

Proof. From the trivial bound $G_1(\alpha_1) \ll L_1$, Hölder's inequality, (2.13), (2.14) and Lemma 2.3, we have

$$\begin{aligned}\mathbb{R}_2\left(N_1\right) \ll & L_1^{k_1}\left(\int_0^1 |f^4(\alpha_1)| \mathrm{d}\alpha_1\right)^{\frac{1}{12}}\left(\int_{\gtrdot_1} |f^2(\alpha_1) S^6(\alpha_1)| \mathrm{d}\alpha_1\right)^{\frac{1}{3}} \\ & \times\left(\int_0^1 |S^2(\alpha_1) T^6(\alpha_1)| \mathrm{d}\alpha_1\right)^{\frac{1}{2}}\left(\int_{\mathbb{E}_{1\lambda}} 1 \mathrm{d}\alpha_1\right)^{\frac{1}{12}} \\ \ll & N_1^{\frac{4}{3}+\frac{1}{18}+\epsilon}(\operatorname{meas}(\mathbb{E}_{1\lambda}))^{\frac{1}{12}} L_1^{k_1} \\ \ll & N_1^{\frac{4}{3}} L_1^{k_1-1},\end{aligned}$$

where we use the estimate

$$\int_0^1\left|f^4(\alpha_i)\right| \mathrm{d}\alpha_i \ll N_i^{1+\epsilon}, \tag{3.12}$$

which is given in [24]. □

Proposition 3.5. *Let* $\mathbb{R}_3(N_1)$ *be defined as in* (2.8) *and* $\lambda = 0.83372$. *We have*

$$\mathbb{R}_3(N_1) \leq 17.13706 \lambda^{k_1-1} N_1^{\frac{4}{3}} L_1^{k_1}.$$

Proof. By the definition of $\mathbb{E}_{1\lambda}$, Hölder's inequality, (2.4), (2.15) and (2.16), we get

$$\begin{aligned}\mathbb{R}_3\left(N_1\right) \leq & (\lambda L_1)^{k_1-1}\left(\int_0^1 |f^4(\alpha_1) G_1^4(\alpha_1)| \mathrm{d}\alpha_1\right)^{\frac{1}{4}}\left(\int_0^1 |S^4(\alpha_1) T^4(\alpha_1)| \mathrm{d}\alpha_1\right)^{\frac{3}{4}} \\ \leq & 17.13706 \lambda^{k_1-1} N_1^{\frac{4}{3}} L_1^{k_1}.\end{aligned}$$

□

Inserting Propositions 3.3, 3.4 and 3.5 into (2.8), we have

$$\begin{aligned}\mathbb{R}(N_1) & \geq \mathbb{R}_1(N_1)-\mathbb{R}_3(N_1)+O\left(N_1^{\frac{4}{3}} L_1^{k_1-1}\right) \\ & >\left(2.79713 \times 10^{-3}-17.13706 \lambda^{k_1-1}\right) N_1^{\frac{4}{3}} L_1^{k_1}.\end{aligned}$$

Recalling that $\lambda = 0.83372$, we can deduce that $\mathbb{R}\left(N_1\right) > 0$ provided that $k_1 \geq 49$, which completes the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.2

Lemma 4.1. *Let* $\Theta(N_l, k_2) = \{n_l \geq 2 : n_l = N_l - 2^{v_1} - 2^{v_2} - \cdots - 2^{v_{k_2}}\}$ *with* $k_2 \geq 30$. *Then for* $N_2 \equiv N_3 \equiv 1 \pmod 2$, *we have*

$$\sum_{\substack{n_2 \in \Theta(N_2, k_2) \\ n_3 \in \Theta(N_3, k_2) \\ n_2 \equiv n_3 \equiv 1 \pmod 2}} \mathbb{S}\left(n_2\right) \mathbb{S}\left(n_3\right) \geq 3.9007 L_2^{k_2}.$$

Proof. The argument employed to establish this lemma runs parallel to that of Lemma 3.1. Setting $\mathbf{q} = 273$, by (3.2), (3.3) and (3.9), we get

$$\begin{aligned}
&\sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_3\in\Theta(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2}} \mathbb{S}\left(n_2\right)\mathbb{S}\left(n_3\right)\\
&\geq \left(2\mathbf{C}\right)^2 \sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_3\in\Theta(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2}} \prod_{p=3,7,13}\left(1+A\left(n_2,p\right)\right)\prod_{p=3,7,13}\left(1+A\left(n_3,p\right)\right)\\
&\geq \left(2\mathbf{C}\right)^2 \sum_{1\leq b\leq \mathbf{q}} \sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_3\in\Theta(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2\\ n_2\equiv n_3\equiv b \pmod{\mathbf{q}}}} \prod_{p=3,7,13}\left(1+A\left(n_2,p\right)\right)\prod_{p=3,7,13}\left(1+A\left(n_3,p\right)\right)\\
&\geq \left(2\mathbf{C}\right)^2 \sum_{1\leq b\leq \mathbf{q}} \prod_{p=3,7,13}\left(1+A\left(b,p\right)\right)\prod_{p=3,7,13}\left(1+A\left(b,p\right)\right) \sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_3\in\Theta(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2\\ n_2\equiv n_3\equiv b \pmod{\mathbf{q}}}} 1\\
&\geq \left(2\mathbf{C}\right)^2 \sum_{1\leq b\leq \mathbf{q}} \prod_{p=3,7,13}\left(1+A(b,p)\right)^2 \sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_2\equiv 1 \pmod 2\\ n_2\equiv b \pmod{\mathbf{q}}}} 1.
\end{aligned} \tag{4.1}$$

We infer from (3.11) that

$$\sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_2\equiv 1 \pmod 2\\ n_2\equiv b \pmod{\mathbf{q}}}} 1 \geq 3.663\times 10^{-3}L_2^{k_2}.$$

From (4.1) and

$$\sum_{1\leq b\leq p}\left(1+A(b,p)\right)^2 = p+\sum_{1\leq b\leq p}A^2(b,p)\geq p,$$

we have

$$\sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_3\in\Theta(N_3,k_2)\\ n_2\equiv n_3\equiv 1 \pmod 2}} \mathbb{S}\left(n_2\right)\mathbb{S}\left(n_3\right) \geq 4\times 3.663\times 10^{-3}\mathbf{C}^2\mathbf{q}L_2^{k_2}\geq 3.9007L_2^{k_2}.$$

□

Proposition 4.2. *Let* $\mathbb{R}_1(N_2,N_3)$ *be defined as in* (2.9). *We have*

$$\mathbb{R}_1(N_2,N_3)\geq 7.82401\times 10^{-6}N_2^{\frac{4}{3}}N_3^{\frac{4}{3}}L_2^{k_2}.$$

Proof. We deduce from Lemmas 2.1, 2.2 and 4.1 that

$$\begin{aligned}\mathbb{R}_1(N_2,N_3) \geq& \frac{1}{1458^2}\sum_{\substack{n_2\in\Theta(N_2,k_2)\\ n_3\in\Theta(N_3,k_2)}}\mathbb{S}(n_2)\mathbb{S}(n_3)\mathbb{J}(n_2)\mathbb{J}(n_3)\\ \geq& \frac{3.9007\times(2.06491)^2}{1458^2}N_2^{\frac{4}{3}}N_3^{\frac{4}{3}}L_2^{k_2}\\ \geq& 7.82401\times 10^{-6}N_2^{\frac{4}{3}}N_3^{\frac{4}{3}}L_2^{k_2}.\end{aligned}$$

□

Proposition 4.3. *Let* $\mathbb{R}_2(N_2,N_3)$ *be defined as in* (2.9). *We have*

$$\mathbb{R}_2(N_2,N_3) \ll N_2^{\frac{4}{3}}N_3^{\frac{4}{3}}L_2^{k_2-1}.$$

Proof. By (2.2) and (2.3), we obtain

$$\gtrdot' \subset \{(\alpha_2,\alpha_3):\alpha_2\in\gtrdot_2,\alpha_3\in[0,1]\}\cup\{(\alpha_2,\alpha_3):\alpha_2\in[0,1],\alpha_3\in\gtrdot_3\}.$$

From the trivial bound $G_2(\alpha_2+\alpha_3)\ll L_2$, we have

$$\begin{aligned}\mathbb{R}_2\left(N_2,N_3\right) \ll& L_2^{k_2}\left(\iint\limits_{\substack{(\alpha_2,\alpha_3)\in\gtrdot_2\times[0,1]\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}} + \iint\limits_{\substack{(\alpha_2,\alpha_3)\in[0,1]\times\gtrdot_3\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}}\right)\left|f(\alpha_2)S^3(\alpha_2)T^3(\alpha_2)\right.\\ &\left.\times f(\alpha_3)S^3(\alpha_3)T^3(\alpha_3)\right|\mathrm{d}\alpha_2\mathrm{d}\alpha_3 \\ =& L_2^{k_2}\left(\int_0^1\left|f(\alpha_2)S^3(\alpha_2)T^3(\alpha_2)J_1(\alpha_2)\right|\mathrm{d}\alpha_2\right.\\ &\left.+\int_0^1\left|f(\alpha_3)S^3(\alpha_3)T^3(\alpha_3)J_2(\alpha_3)\right|\mathrm{d}\alpha_3\right),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}J_1(\alpha_2) =& \int\limits_{\substack{\alpha_3\in\gtrdot_3\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}}\left|f(\alpha_3)S^3(\alpha_3)T^3(\alpha_3)\right|\mathrm{d}\alpha_3,\\ J_2(\alpha_3) =& \int\limits_{\substack{\alpha_2\in\gtrdot_2\\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}}\left|f(\alpha_2)S^3(\alpha_2)T^3(\alpha_2)\right|\mathrm{d}\alpha_2.\end{aligned}$$

Combining Hölder's inequality, the estimates (2.13), (2.14), (3.12) together with the periodicity of $G_2(\alpha_2+\alpha_3)$, we get

$$\begin{aligned} J_1(\alpha_2) \ll & \left(\int_0^1 |f^4(\alpha_3)|\mathrm{d}\alpha_3\right)^{\frac{1}{12}} \left(\int_{\gtrdot_3} |f^2(\alpha_3)S^6(\alpha_3)|\mathrm{d}\alpha_3\right)^{\frac{1}{3}} \\ & \times \left(\int_0^1 |S^2(\alpha_3)T^6(\alpha_3)|\mathrm{d}\alpha_3\right)^{\frac{1}{2}} \left(\int\limits_{\substack{\alpha_3\in\gtrdot_3 \\ |G_2(\alpha_2+\alpha_3)|\geq\lambda L_2}} 1\mathrm{d}\alpha_3\right)^{\frac{1}{12}} \\ \ll & N_3^{\frac{4}{3}+\frac{1}{18}+\epsilon} \left(\int\limits_{\substack{\digamma\in[\alpha_3,1+\alpha_3] \\ |G_2(\digamma)|\geq\lambda L_2}} 1\ \mathrm{d}\digamma\right)^{\frac{1}{12}}, \end{aligned} \tag{4.3}$$

where $\digamma=\alpha_2+\alpha_3$. By Hölder's inequality, (1.7), (2.4), (2.15), (3.12), (4.3) and Lemma 2.3, we have

$$\begin{aligned} & \int_0^1 \left|f(\alpha_2)S^3(\alpha_2)T^3(\alpha_2)J_1(\alpha_2)\right| \mathrm{d}\alpha_2 \\ & \ll N_3^{\frac{4}{3}+\frac{1}{18}+\epsilon} \left(\operatorname{meas}\left(\mathbb{E}_{2\lambda}\right)\right)^{\frac{1}{12}} \left(\int_0^1 \left|f^4(\alpha_2)\right| \mathrm{d}\alpha_2\right)^{\frac{1}{4}} \left(\int_0^1 \left|S^4(\alpha_2)T^4(\alpha_2)\right| \mathrm{d}\alpha_2\right)^{\frac{3}{4}} \\ & \ll N_2^{\frac{4}{3}-10^{-22}} N_3^{\frac{4}{3}+\epsilon}. \end{aligned} \tag{4.4}$$

Following the same strategy, we can obtain

$$\int_0^1 \left|f(\alpha_3)S^3(\alpha_3)T^3(\alpha_3)J_2(\alpha_3)\right| \mathrm{d}\alpha_3 \ll N_2^{\frac{4}{3}+\epsilon} N_3^{\frac{4}{3}-10^{-22}}. \tag{4.5}$$

By (2.4), and inserting (4.4) and (4.5) into (4.2), we get

$$\mathbb{R}_2(N_2,N_3) \ll N_2^{\frac{4}{3}} N_3^{\frac{4}{3}} L_2^{k_2-1}.$$

□

Proposition 4.4. *Let* $\mathbb{R}_3(N_2,N_3)$ *be defined as in* (2.9) *and* $\lambda=0.83372$. *We have*

$$\mathbb{R}_3(N_2,N_3) \leq 1.22976\lambda^{k_2-7} N_2^{\frac{4}{3}} N_3^{\frac{4}{3}} L_2^{k_2}.$$

Proof. By the definition of $\mathbb{E}_{2\lambda}$, Hölder's inequality and (2.15), we get

$$\begin{aligned}\mathbb{R}_3\left(N_2,N_3\right)\leq&(\lambda L_2)^{k_2-7}\left(\iint\limits_{(\alpha_2,\alpha_3)\in[0,1]^2}\left|f^4(\alpha_2)f^4(\alpha_3)G_2^{28}(\alpha_2+\alpha_3)\right|\mathrm{d}\alpha_2\mathrm{d}\alpha_3\right)^{\frac{1}{4}}\\&\times\left(\iint\limits_{(\alpha_2,\alpha_3)\in[0,1]^2}\left|S^4(\alpha_2)T^4(\alpha_2)S^4(\alpha_3)T^4(\alpha_3)\right|\mathrm{d}\alpha_2\mathrm{d}\alpha_3\right)^{\frac{3}{4}}\\\leq&1.22976\lambda^{k_2-7}N_2^{\frac{4}{3}}N_3^{\frac{4}{3}}L_2^{k_2},\end{aligned}$$

where we use the estimate

$$\iint\limits_{(\alpha_2,\alpha_3)\in[0,1]^2}\left|f^4(\alpha_2)f^4(\alpha_3)G_2^{28}(\alpha_2+\alpha_3)\right|\mathrm{d}\alpha_2\mathrm{d}\alpha_3\leq 3.83\times10^5N_2N_3L_2^{28}+O(N_2N_3L_2^{26}),$$

which is given in Lemma 2.5 of [25]. □

Inserting Propositions 4.2, 4.3 and 4.4 into (2.9), we have

$$\begin{aligned}\mathbb{R}(N_2,N_3)&\geq\mathbb{R}_1(N_2,N_3)-\mathbb{R}_3(N_2,N_3)+O\left(N_2^{\frac{4}{3}}N_3^{\frac{4}{3}}L_2^{k_2-1}\right)\\&>\left(7.82401\times10^{-6}-1.22976\lambda^{k_2-7}\right)N_2^{\frac{4}{3}}N_3^{\frac{4}{3}}L_2^{k_2}.\end{aligned}$$

Recalling that $\lambda=0.83372$, we can deduce that $\mathbb{R}\left(N_2,N_3\right)>0$ provided that $k_2\geq 73$, which establishes the proof of Theorem 1.2.

ACKNOWLEDGMENTS

I would like to thank the referees for their careful reading of the manuscript and their valuable comments, which have greatly improved the presentation of this paper.

FUNDING

This work was supported by the National Natural Science Foundation of China (Grant No. 12171286).

CONFLICTS OF INTEREST

The author declares that they have no conflicts of interest to report regarding the present study.

DATA AVAILABILITY STATEMENT

No data were used to support this study.

AUTHOR CONTRIBUTION STATEMENT

All authors have reviewed, discussed, and agreed to their individual contributions ahead of this time. Conceptualization, X. Han; Methodology, X. Han; Validation, X. Han; Formal Analysis, X. Han; Writing – Original Draft Preparation, X. Han; Writing – Review & Editing, X. Han.

REFERENCES

[1] Yu.V. Linnik, Prime numbers and powers of two. *Tr. Mat. Inst. Steklova* **38** (1951) 151–169 (in Russian).

[2] Yu.V. Linnik, Addition of prime numbers with powers of one and the same number. *Mat. Sb. (N.S.)* **32** (1953) 3–60 (in Russian).

[3] J. Pintz and I.Z. Ruzsa, On Linnik's approximation to Goldbach's problem II. *Acta Math. Hungar.* **161** (2020) 569–582.

[4] J.Y. Liu, M.C. Liu and T. Zhan, Squares of primes and powers of 2. *Monatsh. Math.* **128** (1999) 283–313.

[5] S. Hathi, Representation of even integers as a sum of squares of primes and powers of two. *Acta Arith.* **206** (2022) 353–372.

[6] J.Y. Liu and M.C. Liu, Representation of even integers by cubes of primes and powers of 2. *Acta Math. Hungar.* **91** (2001) 217–243.

[7] L.Q. Hu, X. Long and H.M. Wang, Goldbach–Linnik Problem on Unlike Powers of Primes and Powers of Two. *Front. Math.* **20** (2025) 1305–1323.

[8] H.F. Liu, Two results on powers of two in Waring-Goldbach type problems. *Int. J. Number Theory* **12** (2016) 1813–1825.

[9] Z.X. Liu and G.S. Lü, Two results on powers of 2 in Waring–Goldbach problem. *J. Number Theory* **131** (2011) 716–736.

[10] X.D. Lü, On unequal powers of primes and powers of 2. *Ramanujan J.* **50** (2019) 111–121.

[11] R. Zhang, Goldbach–Linnik type problems with mixed powers of primes. *Ramanujan J.* **60** (2023) 1069–1080.

[12] L. Zhu, Goldbach–Linnik type problems with unequal powers of primes. *J. Korean Math. Soc.* **59** (2022) 407–420.

[13] Z.X. Liu, Goldbach–Linnik type problems with unequal powers of primes. *J. Number Theorey* **176** (2017) 439–448.

[14] Y.F. Kong, On pairs of linear equations in four prime variables and powers of two. *Bull. Aust. Math. Soc.* **87** (2013) 55–67.

[15] Y.F. Kong and Z.X. Liu, On pairs of Goldbach–Linnik equations. *Bull. Aust. Math. Soc.* **95** (2017) 199–208.

[16] J.Y. Liu, Enlarged major arcs in additive problems II. *Proc. Steklov Inst. Math.* **276** (2012) 176–192.

[17] J.Y. Liu and T. Zhan, Sums of five almost equal prime squares II. *Sci. China Ser. A* **41** (1998) 710–722.

[18] A. Languasco and A. Zaccagnini, On a Diophantine problem with two primes and s powers of two. *Acta Arith.* **145** (2010) 193–208.

[19] J. Brüdern, A sieve approach to the Waring-Goldbach problem, I: sums of four cubes. *Ann. Sci. ÉcoleNorm. Sup.* **28** (1995) 461–476.

[20] X. Han and H.F. Liu, Slim exceptional sets of Waring–Goldbach problems involving squares and cubes of primes. *Sb. Mat.* **214** (2023) 744–756.

[21] X. Han and H.F. Liu, A pair of equations in eight prime cubes and powers of 2. *Bull. Aust. Math. Soc.* **108** (2023) 244–253.

[22] Z.X. Liu and G.S. Lü, On unlike powers of primes and powers of 2. *Acta Math. Hungar.* **132** (2011) 125–139.

[23] L.L. Zhao, Four squares of primes and powers of 2. *Acta Arith.* **162** (2014) 255–271.

[24] L.K. Hua, *Additive Theory of Prime Numbers*. American Mathematical Society, Providence, RI (1965).

[25] L.Q. Hu, A pair of equations in one prime, two prime squares and powers of 2. *Acta Arith.* **191** (2019) 191–200.