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# ON LOCAL LIMIT THEOREMS FOR $q$-MULTINOMIAL DISTRIBUTIONS

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**Abstract.** In this work, we establish local limit theorems for $q$- multinomial distributions of the first and second kind and of their discrete limits multiple Heine and multiple Euler distributions respectively. Specifically, the pointwise convergence of the $q$-multinomial distribution of the first kind, as well as for its discrete limit, the multiple Heine distribution, to a multivariate Stieltjes–Wigert type distribution, are provided. Moreover, the pointwise convergence of the $q$-multinomial distribution of the second kind, as well as for its discrete limit, the multiple Euler distribution, to a multivariate deformed Gaussian distribution, are proved. Interesting applications of the asymptotic behaviour of $q$-multinomials distributions of the two kinds are presented.

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## 1. Introduction

Recently, Vamvakari [1] introduced multivariate discrete $q$-distributions. Specifically, she derived a multivariate absorption distribution as a conditional distribution of a Heine process at a finite sequence of $q$-points in a time interval which had been defined by Kyriakoussis and Vamvakari [2]. Also, she deduced a multivariate $q$-hypergeometric distribution, as a conditional distribution of the multivariate absorption distribution.

Afterwards, Charalambides [3–5] introduced in detail $q$-multinomial, negative $q$-multinomial, multivariate $q$-Pólya and inverse $q$-Pólya distributions and also examined their limiting discrete distributions. Analytically, he considered a stochastic model of a sequence of independent Bernoulli trials with chain-composite successes (or failures), where the odds of success of a certain kind at a trial is assumed to vary geometrically, with rate $q$, with the number of previous trials and introduced the $q$-multinomial and negative $q$-multinomial distributions of the first kind as well as their discrete limit, multivariate Heine distribution. Also, he considered a stochastic model of a sequence of independent Bernoulli trials with chain-composite successes (or failures), where the probability of success of a certain kind at a trial varies geometrically, with rate $q$, with the number of previous successes and introduced the $q$-multinomial and negative $q$-multinomial distributions of the second kind as well as their discrete limit, multivariate Euler distribution.

Kyriakoussis and Vamvakari [6–8] studied the continuous limiting behaviour of the univariate discrete $q$-distributions. Analytically, they established the pointwise convergence of the $q$-binomial and the negative

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$q$-binomial distributions of the first kind, as well as of the Heine distribution, to a deformed Stieltjes–Wigert continuous one. Moreover, they proved the pointwise convergence of the $q$-binomial and the negative $q$-binomial distributions of the second kind, as well as of the Euler distribution, to a deformed Gaussian one.

Vamvakari [9] initiated the study of continuous limiting behaviour of multivariate discrete $q$-distributions inspired by the limiting behaviour of the univariate ones. Specifically, she studied the asymptotic behavior of the univariate, bivariate and multivariate absorption discrete $q$-distributions. The pointwise convergence of the univariate absorption distribution to a deformed Gaussian one and that of the bivariate and multivariate absorption distribution to a bivariate and multivariate deformed Gaussian ones, have been provided.

The aim of this work is to study further the continuous limiting behaviour of multivariate discrete $q$-distributions. Local limit theorems for $q$- multinomial distributions of the first and second kind and of their discrete limits multiple Heine and multiple Euler distributions respectively are studied. Specifically, the pointwise convergence of the $q$-multinomial distribution of the first kind, as well as for its discrete limit, the multiple Heine distribution, to a multivariate Stieltjes–Wigert type distribution, are proved. Moreover, the pointwise convergence of the $q$-multinomial distribution of the second kind, as well as for its discrete limit, the multiple Euler distribution, to a multivariate deformed Gaussian distribution, are proved. Interesting applications of the asymptotic behaviour of $q$-multinomial distributions of the two kinds are presented.

## 2. Preliminaries

Charalambides [10], had studied in details the $q$-binomial distribution of the first kind and its discrete limit, the Heine distribution with probability functions (p.f.) given respectively by

$$f_X^B(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^{n} (1+\theta q^{j-1})^{-1}, \quad x = 0, 1, \ldots, n, \tag{2.1}$$

where $\theta > 0$ and $0 < q < 1$ and

$$f_X^H(x) = e_q(-\lambda) \frac{q^{\binom{x}{2}} \lambda^x}{[x]_q!}, \quad x = 0, 1, 2, \ldots, \ 0 < q < 1, \ 0 < \lambda < \infty, \tag{2.2}$$

where

$$e_q(z) := \sum_{n=0}^{\infty} \frac{(1-q)^n z^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty}, \quad |z| < 1, \tag{2.3}$$

and

$$[n]_q! = [1]_q [2]_q \cdots [n]_q = \prod_{k=1}^{n} \frac{1-q^k}{(1-q)^n} = \frac{(q;q)_n}{(1-q)^n}, \quad 0 < q < 1, \quad [t]_q = \frac{1-q^t}{1-q}. \tag{2.4}$$

Charalambides [10], had also studied in details, the $q$-binomial distribution of the second kind with probability function and its discrete limit Euler distribution with probability functions given respectively by

$$f_X^{BS}(x) = \binom{n}{x}_q \theta^x \prod_{j=1}^{n-x} (1-\theta q^{j-1}), \quad x = 0, 1, \ldots, n, \tag{2.5}$$

where $0 < \theta < 1$ and $0 < q < 1$ or $1 < q < \infty$ with $\theta q^{n-1} < 1$ and

$$f_X^E(x) = E_q(-\lambda)\frac{\lambda^x}{[x]_q!}, \quad x = 0, 1, 2, \ldots, \; 0 < q < 1, \; 0 < \lambda(1-q) < 1, \tag{2.6}$$

where

$$E_q(z) := \sum_{n=0}^{\infty} \frac{(1-q)^n z^n q^{\binom{n}{2}}}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{z^n q^{\binom{n}{2}}}{[n]_q!} = ((1-q)z;q)_\infty, \quad |z| < 1. \tag{2.7}$$

Kyriakoussis and Vamvakari [6, 7] proved local limit theorems among others for the $q$-binomial distribution of the first kind and Heine distribution for constant $q$, by using pointwise convergence in a "$q$-analogous sense" of the classical de Moivre–Laplace limit theorem. Specifically for the needs of their study they established a $q$-Stirling formula for $n \to \infty$ of the $q$-factorial of order $n$, defined by relation (2.4). Analytically, for $0 < q < 1$ constant, it was proved that,

$$[n]_q! = \frac{q^{-1/8}(2\pi(1-q))^{1/2}}{(q\log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}(1+(q^{-n}-1)q^{j-1})} \left(1 + O(n^{-1})\right). \tag{2.8}$$

Next, the pointwise convergence of the $q$-binomial distribution of the first kind to a deformed continuous Stieltjes–Wigert distribution was established. The continuous Stieltjes–Wigert distribution has probability density function

$$v_W^{SW}(w) = \frac{q^{1/8}}{\sqrt{2\pi\log q^{-1}}\, w} e^{\frac{(\log w)^2}{2\log q}}, \quad w > 0, \tag{2.9}$$

with mean value $\mu^{SW} = q^{-1}$ and standard deviation $\sigma^{SW} = q^{-3/2}(1-q)^{1/2}$.

Transferred from the random variable $X$ of the $q$-binomial distribution (2.1) to the equal-distributed deformed random variable $Y = [X]_{1/q}$ and for $n \to \infty$, the $q$-Binomial distribution of the first kind was approximated by a deformed standardized continuous Stieltjes–Wigert distribution as follows:

$$\begin{aligned} f_X^B(x) \cong{}& \frac{q^{-7/8}}{\sigma_q(2\pi)^{1/2}} \left(\frac{\log q^{-1}}{q^{-1}-1}\right)^{1/2} \left(q^{-3/2}(1-q)^{1/2}\frac{[x]_{1/q}-\mu_q}{\sigma_q} + q^{-1}\right)^{-1/2} q^{-x} \\ &\cdot \exp\left(\frac{1}{2\log q}\log^2\left(q^{-3/2}(1-q)^{1/2}\frac{[x]_{1/q}-\mu_q}{\sigma_q} + q^{-1}\right)\right), \quad x \geq 0, \end{aligned} \tag{2.10}$$

where $\theta = \theta_n$, for $n = 0, 1, 2, \ldots$, such that $\theta_n = q^{-\alpha n}$ with $0 < a < 1$ constant and $\mu_q$ and $\sigma_q^2$ the mean value and variance of the random variable $Y$, respectively. A similar asymptotic result has been provided for the Heine distribution when $\lambda \to \infty$.

Kyriakoussis and Vamvakari [8], had moreover established the pointwise convergence of the $q$-binomial distribution of the second kind and of its discrete limit, the Euler distribution to a deformed Gaussian one. Note that, for the needs of their study, they have considered $q := q(n)$ as a sequence in $n$ with $q(n) \to 1$ as $n \to \infty$. They have studied the effect of this assumption on the $q(n)$-analogue of the Stirling-type and derived the following asymptotic formula

$$[n]_q! = \frac{q^{-1/8}(2\pi(1-q))^{1/2}}{(q\log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}(1+(q^{-n}-1)q^{j-1})} \left(1 + O\left(q^n(1-q)\right)\right), \tag{2.11}$$

where $q = q(n)$ with $q(n) \to 1$ as $n \to \infty$ and $q(n)^n = \Omega(1)$.

Then, they considered the $q$-binomial distribution of the second kind with probability function given by (2.5) and they transfer to the equal-distributed deformed random variable $[X]_q$. Applying suitably the $q$-Stirling type (2.11), for $n \to \infty$, they obtained that the $q$-Binomial distribution of the second kind was approximated by a deformed standardized Gaussian distribution as follows:

$$f_X(x) \cong \frac{(\log q^{-1})^{1/2}}{\sigma_q (2\pi(1-q)^{1/2}}\, q^x \exp\left(-\frac{1}{2}\left(\left(\frac{1-q}{\log q^{-1}}\right)^{1/2} \cdot \frac{[x]_q - \mu_q}{\sigma_q}\right)^2\right),\ \ x \geq 0, \tag{2.12}$$

where $\theta = \theta_n = (1-q)^a$, $0 < a < 1$, $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, and $q(n)^n = \Omega(1)$, and $\mu_q$ and ${\sigma_q}^2$ the mean value and the variance of the random variable $[X]_q$, given respectively by

$$\mu_q = E\left([X]_q\right) = [n]_q\, \theta \quad \text{and} \quad (\sigma_q)^2 = V\left([X]_q\right) = [n]_q \theta\left(1 - \theta\left(1 - q^{n-1}\right)\right). \tag{2.13}$$

Similar asymptotic results, have been provided in [8] for the negative $q$-Binomial distribution of the second kind as well as for their discrete limit, the Euler distribution.

## 3. Main results

### 3.1. On a local limit theorem for $q$-multinomial distribution of the first kind

Let $X_j$ be the number of successes of a $j$th kind in a sequence of $n$ independent Bernoulli trials with chain composite failures, where the probability of success of the $j$th kind at the $i$th trial is given by

$$p_{j,i} = \frac{\theta_j q^{i-1}}{1 + \theta_j q^{i-1}},\ 0 < \theta_j < \infty,\ j = 1, 2, \ldots, i = 1, 2, \ldots,\ 0 < q < 1 \ \text{ or } \ 1 < q < \infty.$$

Then the joint probability function of the random vector $\mathcal{X} = (X_1, X_2, \ldots, X_k)$ is given by

$$\begin{aligned} f^B_{\mathcal{X}}(x_1, x_2, \ldots, x_k) &= P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) \\ &= \binom{n}{x_1, x_2, \ldots, x_k}_q \prod_{j=1}^{k} \frac{\theta_j^{x_j} q^{\binom{x_j}{2}}}{\prod_{i=1}^{n-s_{j-1}}(1 + \theta_j q^{i-1})} \end{aligned} \tag{3.1}$$

$x_j = 0, 1, 2, \ldots, n$, $\sum_{j=1}^{k} x_j \leq n$, $s_j = \sum_{i=1}^{j} x_i$, $\ 0 < \theta_j < 1$, $j = 1, 2, \ldots, k$, and $0 < q < 1$ or $1 < q < \infty$. This discrete $q$-distribution is known as a *$q$-multinomial distribution* (see Charalambides [3]).

The discrete limit of the joint p.f. of the $q$-multinomial distribution of the 1st kind, as $n \to \infty$, is the joint p.f. of the *multiple Heine distribution*,

$$\begin{aligned} &\lim_{n\to\infty} \binom{n}{x_1, x_2, \ldots, x_k}_q \prod_{j=1}^{k} \frac{\theta_j^{x_j} q^{\binom{x_j}{2}}}{\prod_{i=1}^{n-s_{j-1}}(1 + \theta_j q^{i-1})} \\ &= \prod_{j=1}^{k} \frac{q^{\binom{x_j}{2}} {\lambda_j}^x}{[x_j]_q!} \prod_{i=1}^{\infty} (1 + \lambda_j(1-q), q^{i-1})^{-1} \end{aligned} \tag{3.2}$$

$x_j = 0, 1, 2, \ldots,\ \lambda_j > 0$, $0 < q < 1$, $\lambda_j = \theta_j/(1-q)$, $j = 1, 2, \ldots, k$ (see Charalambides [3]).

Next we will study on the asymptotic behaviour of the $q$-multinomial distribution with joint p.f. (3.1).

The marginal probability function of the random variable $X_1$, is distributed according to the $q$-binomial of the 1st kind with probability function

$$f^B_{X_1}(x_1) = \binom{n}{x_1}_q \frac{\theta_1^{x_1} q^{\binom{x_1}{2}}}{\prod_{i=1}^{n}(1+\theta_1 q^{i-1})}, \quad x_1 = 0, 1, 2, \ldots, n. \tag{3.3}$$

The mean and the variance of the deformed variable $[X_1]_{1/q}$ are given by

$$\begin{aligned} \mu_{[X_1]_{1/q}} &= E\left([X_1]_{1/q}\right) = [n]_q \frac{\theta_1}{1+\theta_1 q^{n-1}} \\ &\text{and} \\ (\sigma_{[X_1]_{1/q}})^2 &= V\left([X_1]_{1/q}\right) \\ &= \frac{1-q}{q}[n]_q^2 \frac{\theta_1^2}{(1+\theta_1 q^{n-1})^2(1+\theta_1 q^{n-2})} + [n]_q \frac{\theta_1}{(1+\theta_1 q^{n-1})(1+\theta_1 q^{n-2})} \end{aligned} \tag{3.4}$$

respectively.

The conditional random variables $X_2|X_1, X_3|(X_1, X_2), \ldots, X_k|(X_1, \ldots, X_{k-1})$ are distributed according to univariate $q$-binomial distributions of the 1st kind with probability functions

$$\begin{aligned} f^B_{X_k|(X_1,\ldots,X_{k-1})}(x_k|x_1, x_2, \ldots, x_{k-1}) = \binom{n-\sum_{j=1}^{k-1} x_j}{x_k}_q \frac{\theta_k^{x_k} q^{\binom{x_k}{2}}}{\prod_{i=1}^{n-\sum_{j=1}^{k-1} x_j}(1+\theta_k q^{i-1})}, \\ x_k = 0, 1, \ldots, n - \sum_{j=1}^{k-1} x_j, \ k \geq 2. \end{aligned} \tag{3.5}$$

Note that the joint probability (3.1) of the random vector $\mathcal{X} = (X_1, X_2, \ldots, X_k)$, in terms of (3.3) and (3.5), can be written as

$$f^B_{\mathcal{X}}(x_1, x_2, \ldots, x_k) = f^B_{X_1}(x_1) \prod_{j=2}^{k} f^B_{X_j|(X_1,\ldots,X_{j-1})}(x_j|x_1, x_2, \ldots, x_{j-1}). \tag{3.6}$$

The conditional mean and conditional variance of the deformed variables $[X_j]_{1/q}$ given $X_1 = x_1, \ldots, X_{j-1} = x_{j-1}$, $j = 2, \ldots, k$, $k \geq 2$, are given respectively by

$$\begin{aligned} \mu_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})} &= E\left([X_j]_{1/q}|(X_1, \ldots, X_{j-1})\right) = [n - s_{j-1}]_q \frac{\theta_j}{1+\theta_j q^{n-s_{j-1}-1}} \\ &\text{and} \\ \sigma^2_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})} &= V\left([[X_j]_{1/q}|(X_1, \ldots, X_{j-1})\right) \\ &= \frac{1-q}{q}[n-s_{j-1}]_q^2 \frac{\theta_j^2}{(1+\theta_j q^{n-s_{j-1}-1})^2(1+\theta_j q^{n-s_{j-1}-2})} \\ &\quad + [n-s_{j-1}]_q \frac{\theta_j}{(1+\theta_j q^{n-s_{j-1}-1})(1+\theta_j q^{n-s_{j-1}-2})}, \end{aligned} \tag{3.7}$$

where $s_{j-1} = \sum_{i=1}^{j-1} x_i$, $j = 2, \ldots, k$, $k \geq 2$. It should be noted that the conditional $q$-means, $\mu_{[X_j]_q|(X_1,\ldots,X_{j-1})}$, $3 \leq j \leq k$, $k \geq 3$, can be interpreted as *$q$-regression hyperplanes.*

Let us now consider the deformed random variables

$$[X_j]_{1/q},\, j = 1, \dots, k,\ k \geq 1$$

and the $q$-standardized random variables

$$Z_1 = \frac{[X_1]_{1/q} - \mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_q}},\ Z_j = \frac{[X_j]_{1/q} - \mu_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}}{\sigma_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}},\, j = 2, \dots, k,\ k \geq 3,$$

with $\mu_{[X_1]_{1/q}}$,$\sigma_{[X_1]_{1/q}}$ and $\mu_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}$, $\sigma_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}$ given by (3.4) and (3.7), respectively.

Then, we use formula (3.6), describing the fact that the joint probability function of the $q$-multinomial distribution of the first kind is expressed as a product of probability functions of $q$-binomial distributions of the first kind. We proceed the study of the asymptotic behaviour of $q$-multinomial distribution of the first kind, by applying similar pointwise techniques, to the joint probability function (3.1) and to the probability functions (3.3) and (3.5), as in [6] and [8], and we obtain the following local limit theorem.

**Theorem 3.1.** *Let $\theta_j = \theta_{j,n} = q^{-\alpha_j n}$ with $0 < a_j < 1,\ j = 1, 2, \dots, k$ constants and $0 < q < 1$. Then, for $n \to \infty$, the $q$-multinomial distribution of the first kind is approximated by a deformed multivariate standardized continuous Stieltjes–Wigert distribution distribution as follows:*

$$\begin{aligned}
f^B_{\mathcal{X}}(x_1, x_2, \dots, x_k) \cong{}& \left(\frac{q^{-7/8}(\log q^{-1})^{1/2}}{(2\pi)^{1/2}(q^{-1}-1)^{1/2}}\right)^k \frac{q^{-\sum_{j=1}^k x_j}}{\sigma_{[X_1]_{1/q}} \prod_{j=2}^k \sigma_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}} \\
&\cdot \left(q^{-3/2}(1-q)^{1/2}\frac{[x_1]_{1/q} - \mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}} + q^{-1}\right)^{-1/2} \\
&\cdot \prod_{j=2}^k \left(q^{-3/2}(1-q)^{1/2}\frac{[x_j]_{1/q} - \mu_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}}{\sigma_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}} + q^{-1}\right)^{-1/2} \\
&\cdot \exp\left(\frac{1}{2\log q}\left(\log^2\left(\frac{(1-q)^{1/2}}{q^{3/2}}\frac{[x_1]_{1/q} - \mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}} + q^{-1}\right)\right)\right) \\
&\cdot \exp\left(\frac{1}{2\log q}\sum_{j=2}^k \log^2\left(\frac{(1-q)^{1/2}}{q^{3/2}}\frac{[x_j]_{1/q} - \mu_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}}{\sigma_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}} + q^{-1}\right)\right), \\
& x_j \geq 0, j = 1, 2, \dots, k, k \geq 2,
\end{aligned} \tag{3.8}$$

*where $\mu_{[X_1]_{1/q}}$ and $\sigma^2_{[X_1]_{1/q}}$, given in (3.4), are the mean value and the variance of the random variable $[X_1]_{1/q}$, while $\mu_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}$ and $\sigma^2_{[X_j]_{1/q}|(X_1,\dots,X_{j-1})}$, given in (3.7), are the conditional mean values and the conditional variances of the random variables $[X_j]_{1/q}$ given $X_1 = x_1, \dots, X_{j-1} = x_{j-1},\ j = 2, \dots, k,\ k \geq 2$.*

**Remark 3.2.** *Asymptotic Behaviour of Multiple Heine Distribution.* Let $\mathcal{X} = (X_1, \dots, X_k)$ be a random vector that follows the multiple Heine distribution, defined in (3.2). Then the joint p.f. the multiple Heine distribution is given by

$$f^H_{\mathcal{X}}(x_1, x_2, \dots, x_k) = \prod_{j=1}^k \frac{q^{\binom{x_j}{2}}{\lambda_j}^x}{[x_j]_q!} \prod_{i=1}^\infty (1 + \lambda_j(1-q)q^{i-1})^{-1},$$

where $x_j = 0, 1, 2, \dots,\ \lambda_j > 0,\ 0 < q < 1,\ \lambda_j = \theta_j/(1-q),\ j = 1, 2, \dots, k,\ k \geq 2$.

Since the random variables $X_j,\ j = 1, 2, \dots, k,\ k \geq 2$, are independent, we easily derive that, for $\lambda_j \to \infty,\ j = 1, 2, \dots, k$, the multiple Heine distribution is approximated by a deformed multivariate standardized continuous Stieltjes–Wigert distribution as follows:

$$\begin{aligned}
f_{\mathcal{X}}^{H}(x_1,x_2,\ldots,x_k) \cong & \left(\frac{q^{-7/8}(\log q^{-1})^{1/2}}{(2\pi)^{1/2}(q^{-1}-1)^{1/2}}\right)^k \frac{q^{-\sum_{j=1}^k x_j}}{\prod_{j=1}^k \sigma_{[X_j]_{1/q}}} \\
& \cdot \prod_{j=1}^k \left(q^{-3/2}(1-q)^{1/2}\frac{[x_j]_{1/q}-\mu_{[X_j]_{1/q}}}{\sigma_{[X_j]_{1/q}}}+q^{-1}\right)^{-1/2} \\
& \cdot \exp\left(\frac{1}{2\log q}\left(\sum_{j=1}^k \log^2 \frac{(1-q)^{1/2}}{q^{3/2}}\frac{[x_j]_{1/q}-\mu_{[X_j]_{1/q}}}{\sigma_{[X_j]_{1/q}}}\right)\right), \\
& x_j \geq 0, j=1,\ldots,k, k\geq 2,
\end{aligned} \tag{3.9}$$

where $\mu_{[X_j]_{1/q}} = \lambda_j$ and $\sigma^2_{[X_j]_{1/q}} = \lambda_j q^{-1}(1-q)+\lambda_j$, $j=1,2,\ldots,k$, are respectively the mean values and the variances of the random variables $[X_j]_{1/q}$, $j=1,2,\ldots,k$.

**Remark 3.3.** *Asymptotic Behaviour of $q$-Trinomial Distribution of the* 1*st Kind.* Let $(X_1,X_2)$ be a discrete bivariate random variable that follows the *$q$-trinomial distribution of the first kind*, defined in (3.1) with $k=2$. Then the joint probability function of the $q$-trinomial distribution of the first kind is given by

$$\begin{aligned}
f_{X_1,X_2}^{B}(x_1,x_2) &= P\left(X_1=x_1, X_2=x_2\right) \\
&= \binom{n}{x_1,x_2}_q \frac{\theta_1^{x_1}\theta_2^{x_2} q^{\binom{x_1}{2}+\binom{x_2}{2}}}{\prod_{i=1}^n (1+\theta_1 q^{i-1})\prod_{i=1}^{n-x_1}(1+\theta_2 q^{i-1})}
\end{aligned} \tag{3.10}$$

$x_j = 0,1,2,\ldots,n$, $j=1,2$, $x_1+x_2\leq n$, $0<\theta_1,\theta_2<1$ and $0<q<1$ or $1<q<\infty$.

The mean and the variance of the deformed variable $[X_1]_{1/q}$ are given by

$$\begin{aligned}
\mu_{[X_1]_{1/q}} &= E\left([X_1]_{1/q}\right) = [n]_q \frac{\theta_1}{1+\theta_1 q^{n-1}} \\
&\text{and} \\
(\sigma_{[X_1]_{1/q}})^2 & + \left([X_1]_{1/q}\right) = \frac{1-q}{q}[n]_q^2 \frac{\theta_1^2}{(1+\theta_1 q^{n-1})^2(1+\theta_1 q^{n-2})} \\
& + [n]_q \frac{\theta_1}{(1+\theta_1 q^{n-1})(1+\theta_1 q^{n-2})}
\end{aligned} \tag{3.11}$$

respectively.

The conditional random variable $X_2|X_1$, is distributed according to the univariate $q$-binomial of the 1st kind with probability function

$$f_{X_2|X_1}^{B}(x_2|x_1) = \binom{n-x_1}{x_2}_q \frac{\theta_2^{x_2} q^{\binom{x_2}{2}}}{\prod_{i=1}^{n-x_1}(1+\theta_2 q^{i-1})}, \quad x_2 = 0,1,2,\ldots,n-x_1.$$

The conditional mean and conditional variance of the deformed variable $[X_2]_{1/q}$ given $X_1=x_1$, are given by

$$\begin{aligned}
\mu_{[X_2]_{1/q}|X_1} &= E\left([X_2]_{1/q}|X_1\right) = [n-x_1]_q \frac{\theta_2}{1+\theta_2 q^{n-x_1-1}}, \\
&\text{and} \\
(\sigma_{[X_2]_{1/q}|X_1})^2 &= V\left([X_2]_{1/q}|X_1\right) = \frac{1-q}{q}[n-x_1]_q^2 \frac{\theta_2^2}{(1+\theta_2 q^{n-x_1-1})^2(1+\theta_2 q^{n-x_1-2})} \\
& + [n-x_1]_q \frac{\theta_2}{(1+\theta_2 q^{n-x_1-1})(1+\theta_2 q^{n-x_1-2})},
\end{aligned} \tag{3.12}$$

respectively.

Suppose that $\theta_1 = \theta_{1,n} = q^{-\alpha_1 n}$ and $\theta_2 = \theta_{2,n} = q^{-\alpha_2 n}$ with $0 < a_1,\, a_2 < 1$ constants and $0 < q < 1$. Then, for $n \to \infty$, the $q$-trinomial distribution of the first kind from theorem 3.1, is approximated by a deformed standardized bivariate continuous Stieltjes–Wigert distribution as follows:

$$\begin{aligned} f^B_{X_1,X_2}(x_1,x_2) \cong{} & \frac{q^{-7/4}\log q^{-1}}{2\pi(q^{-1}-1)\sigma_{[X_1]_{1/q}}\sigma_{[X_2]_{1/q}|X_1}} q^{-(x_1+x_2)} \\ & \cdot \left( q^{-3/2}(1-q)^{1/2}\frac{[x_1]_{1/q}-\mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}} + q^{-1}\right)^{-1/2} \\ & \cdot \left( q^{-3/2}(1-q)^{1/2}\frac{[x_2]_{q}-\mu_{[X_2]_{1/q}|X_1}}{\sigma_{[X_2]_{1/q}|X_1}} + q^{-1}\right)^{-1/2} \\ & \cdot \exp\left(\frac{1}{2\log q} log^2\left( q^{-3/2}(1-q)^{1/2}\frac{[x_1]_{1/q}-\mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}} + q^{-1}\right)\right) \\ & \cdot \exp\left(\frac{1}{2\log q} \log^2\left( q^{-3/2}(1-q)^{1/2}\frac{[x_2]_{q}-\mu_{[X_2]_{1/q}|X_1}}{\sigma_{[X_2]_{1/q}|X_1}} + q^{-1}\right)\right), \\ & x_1,\, x_2 \geq 0, \end{aligned} \tag{3.13}$$

where $\mu_{[X_1]_{1/q}}$ and $\sigma^2_{[X_1]_{1/q}}$, given in (3.11), are the mean value and the variance of the random variable $[X_1]_{1/q}$ while $\mu_{[X_2]_{1/q}|X_1}$ and $\sigma^2_{[X_2]_{1/q}|X_1}$, given in (3.12), are the conditional mean value and the conditional variance of the random variable $[X_2]_{1/q}$ given $X_1 = x_1$.

An interesting application of Theorem 3.1, regarding the asymptotic behaviour of the $q$-multinomial distribution of the 1st kind is presented in the following example.

**Example 3.4.** *Asymptotic Behaviour of Proofreading a Manuscript with Heine-Distributed Random Number of Errors.* Consider a manuscript of $k$ chapters and suppose that the number $M_j$ of typographical errors in chapter $c_j$ is a random variable obeying a Heine distribution with parameters $\lambda_j$ and $q$, for $j = 1, 2, \ldots, k$. Furthermore, suppose that the random variables $M_1, M_2, \ldots, M_k$ are independent. Let $Y_j$ be the number of successes of $j$th kind, for $j = 1, 2, \ldots, k$, in a sequence of $n$ scans (readings) of the manuscript. Then the joint probability function of the random vector $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$ obeys a $q$-multinomial distribution of the 1st kind with parameters $\theta_j = \lambda_j(1-q)$, $0 < q < 1$ (see Charalambides [5]).

Suppose that $\lambda_j = \lambda_{j,n}$, then according to the local limit Theorem 3.1, the joint probability function of the random vector $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$, for $\theta_j = \theta_{j,n} = q^{-\alpha_j n}$ with $0 < a_j < 1$, $j = 1, 2, \ldots, k$, $0 < q < 1$ and for $n \to \infty$, is approximated by a deformed multivariate standardized continuous Stieltjes–Wigert distribution, given in (3.8).

Note that the joint probability function of the random vector $\mathcal{M} = (M_1, \ldots, M_k)$ obeys a multiple Heine distribution with parameters $\lambda_j$, $j = 1, 2, \ldots, k$. According to Remark 3.2, the joint probability function of the random vector $\mathcal{M} = (M_1, \ldots, M_k)$ for $\lambda_j \to \infty$, $j = 1, 2, \ldots, k$, is approximated by a deformed multivariate standardized continuous Stieltjes–Wigert distribution, given in (3.9).

### 3.2. On a local limit theorem for $q$-multinomial distribution of the second kind

Let $X_j$ be the number of failures of a $j$th kind in a sequence of $n$ independent Bernoulli trials with chain composite failures, where the conditional probability of success of the $j$th kind at any trial, given that $i-1$ successes of the $j$th kind occur in the previous trials, is given by,

$$p_{j,i} = 1 - \theta_j q^{i-1},\ 0 < \theta_j < 1,\ \ j = 1, 2, \ldots, k, i = 1, 2, \ldots,\ 0 < q < 1. \text{ or } \ 1 < q < \infty.$$

The distribution of the random vector$\mathcal{X} = (X_1, X_2, \ldots, X_k)$ is called *$q$-multinomial distribution of the second kind*, with parameters $n$, $(\theta_1, \theta_2, \ldots, \theta_k)$, and $q$ (see Charalambides [3]).

The joint probability function of the random vector $\mathcal{X} = (X_1, X_2, \dots, X_k)$ is given by

$$\begin{aligned} f_{\mathcal{X}}^{BS}(x_1, x_2, \dots, x_k) &= P(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k) \\ &= \binom{n}{x_1, x_2, \dots, x_k}_q \prod_{j=1}^{k} \theta_j^{x_j} \prod_{i=1}^{n-y_j} (1 - \theta_j q^{i-1}) \end{aligned} \tag{3.14}$$

$x_j = 0, 1, 2, \dots, n,$ with $\sum_{j=1}^{k} x_j \leq n,$ where $y_j = \sum_{i=1}^{j} x_i,$ $0 < \theta_j < 1,$ $j = 1, 2, \dots, k,$ and $0 < q < 1$ or $1 < q < \infty$ and $0 < \theta_j < q^{-(n-1)}$ (see Charalambides [3]).

The discrete limit of the joint p.f. of the $q$-multinomial distribution of the 2nd kind, as $n \to \infty$, is the joint p.f. of the *multiple Euler distribution*,

$$\lim_{n\to\infty} \binom{n}{x_1, x_2, \dots, x_k}_q \prod_{j=1}^{k} \theta_j^{x_j} \prod_{i=1}^{n-y_j} (1 - \theta_j q^{i-1}) = \prod_{j=1}^{k} E_q(-\lambda_j) \frac{\lambda_j{}^x}{[x_j]_q!}, \tag{3.15}$$

$x_j = 0, 1, 2, \dots,$ $0 < \lambda_j < 1/(1-q)$, $0 < q < 1$ with $\lambda_j = \theta_j/(1-q),$ $j = 1, 2, \dots, k, k \geq 2$ (see Charalambides [3]).

Next we will study the asymptotic behaviour of the $q$-multinomial distribution of the second kind with joint p.f. (3.14).

The marginal probability function of the random variable $X_1$, is distributed according to the $q$-binomial of the 2nd kind with probability function

$$f_{X_1}^{BS}(x_1) = \binom{n}{x_1}_q \theta_1^{x_1} \prod_{i=1}^{n-x_1} (1 - \theta_1 q^{i-1}), \quad x_1 = 0, 1, 2, \dots, n. \tag{3.16}$$

The mean and the variance of the deformed variable $[X_1]_q$ are given respectively by

$$\begin{gathered} \mu_{[X_1]_q} = E\left([X_1]_q\right) = [n]_q \theta_1 \\ \text{and} \\ (\sigma_{[X_1]_q})^2 = V\left([X_1]_q\right) = [n]_q \theta_1 \left(1 - \theta_1 (1 - q^{n-1})\right). \end{gathered} \tag{3.17}$$

The conditional random variables $X_2|X_1, X_3|(X_1, X_2), \dots, X_k|(X_1, \dots, X_{k-1})$ are distributed according to univariate $q$-binomial distributions of the 2nd kind with probability functions

$$\begin{aligned} f_{X_k|(X_1,\dots,X_{k-1})}^{BS}(x_k|x_1, \dots, x_{k-1}) &= \binom{n - \sum_{j=1}^{k-1} x_j}{x_k}_q \theta_k^{x_k} \prod_{i=1}^{n - \sum_{j=1}^{k} x_j} (1 - \theta_k q^{i-1}) \\ & x_k = 0, 1, \dots, n - \sum_{j=1}^{k-1} x_j, \ k \geq 2. \end{aligned} \tag{3.18}$$

Note that the joint probability (3.14) of the random vector $\mathcal{X} = (X_1, X_2, \dots, X_k)$ in terms of (3.16) and (3.18), can be written as

$$f_{\mathcal{X}}^{BS}(x_1, x_2, \dots, x_k) = f_{X_1}^{BS}(x_1) \prod_{j=2}^{k} f_{X_j|(X_1,\dots,X_{j-1})}^{BS}(x_j|x_1, x_2, \dots, x_{j-1}). \tag{3.19}$$

The conditional mean and conditional variance of the deformed variables $[X_j]_{1/q}$ given $X_1 = x_1, \dots, X_{j-1} = x_{j-1}$, $j = 2, \dots, k,\ \ k \geq 2$, are given respectively by

$$\begin{aligned} \mu_{[X_j]_q|(X_1,\dots,X_{j-1})} &= E\left([X_j]_q|(X_1,\dots,X_{j-1})\right) = [n - \sum_{j=1}^{k-1} x_j]_q\theta_j \\ &\text{and} \\ \sigma^2_{[X_j]_q|(X_1,\dots,X_{j-1})} &= V\left([[X_j]_q|(X_1,\dots,X_{j-1})\right) \\ &= [n - \sum_{j=1}^{k-1} x_j]_q \theta_j \left(1 - \theta_j(1 - q^{n-1-\sum_{j=1}^{k-1} x_j})\right). \end{aligned} \tag{3.20}$$

It should be noted that the conditional $q$-means, $\mu_{[X_j]_q|(X_1,\dots,X_{j-1})}$, $3 \leq j \leq k,\ k \geq 3$, can be interpreted as $q$-regression hyperplanes.

Let us now consider the deformed random variables

$$[X_j]_q,\ j = 1, \dots, k,\ \ k \geq 1$$

and the $q$-standardized random variables

$$Z_1 = \frac{[X_1]_q - \mu_{[X_1]_q}}{\sigma_{[X_1]_q}},\ \ Z_j = \frac{[X_j]_q - \mu_{[X_j]_q|(X_1,\dots,X_{j-1})}}{\sigma_{[X_j]_q|(X_1,\dots,X_{j-1})}},\ j = 2, \dots, k,\ \ k \geq 3,$$

with $\mu_{[X_1]_q}$,$\sigma_{[X_1]_q}$ and $\mu_{[X_j]_q|(X_1,\dots,X_{j-1})}$, $\sigma_{[X_j]_q|(X_1,\dots,X_{j-1})}$ given by (3.17) and (3.20), respectively.

Then, we use formula (3.19), describing the fact that the joint probability function of the $q$-multinomial distribution of the second kind is expressed as a product of probability functions of $q$-binomial distributions of the second kind. We proceed the study of the asymptotic behaviour of $q$-multinomial distribution of the second kind, by applying similar pointwise techniques to the joint probability function (3.14) and to the probability functions (3.16) and (3.18), as in [6], [8] and [9], and we obtain the following local limit theorem.

**Theorem 3.5.** *Let $\theta_j = \theta_{j,n} = (1-q)^{a_j}$, $0 < a_j < 1$, $j = 1, 2$, $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, and $q(n)^n = \Omega(1)$. Then, for $n \to \infty$, the $q$-multinomial distribution of the second kind is approximated by a deformed multivariate standardized Gaussian distribution as follows*

$$\begin{aligned} f^{BS}_{\mathcal{X}}(x_1, x_2, \dots, x_k) &\cong \left(\frac{\log q^{-1}}{2\pi(q^{-1}-1)}\right)^{k/2} \frac{q^{\sum_{i=1}^k x_i}}{\sigma_{[X_1]_q} \prod_{j=2}^k \sigma_{[X_j]_q|\left(X_1,\dots,X_{j-1}\right)}} \\ &\cdot \exp\left(-\frac{1-q}{2\log q^{-1}}\left(\left(\frac{[x_!]_q - \mu_{[X_1]_q}}{\sigma_{[X_1]_q}}\right)^2 + \sum_{j=2}^k \left(\frac{[x_j]_q - \mu_{[X_j]_q|\left(X_1,\dots,X_{j-1}\right)}}{\sigma_{[X_j]_q|\left(X_1,\dots,X_{j-1}\right)}}\right)^2\right)\right), \\ &x_j \geq 0, j = 1, 2, \dots, k, k \geq 2, \end{aligned} \tag{3.21}$$

*where $\mu_{[X_1]_q}$ and $\sigma^2_{[X_1]_q}$, given in (3.17), are the mean value and the variance of the random variable $[X_1]_q$, while $\mu_{[X_j]_q|(X_1,\dots,X_{j-1})}$ and $\sigma^2_{[X_j]_q|(X_1,\dots,X_{j-1})}$, given in (3.20), are the conditional mean values and the conditional variances of the random variables $[X_j]_q$ given $X_1 = x_1, \dots, X_{j-1} = x_{j-1}$, $j = 2, \dots, k,\ k \geq 2$.*

**Remark 3.6.** *Asymptotic Behaviour of Multiple Euler Distribution.* Let $\mathcal{X} = (X_1, \dots, X_k)$ be a random vector that follows the multiple Euler distribution, defined in (3.15). Then the joint p.f. of the multiple Euler distribution is given by

$$f_{\mathcal{X}}^{E}(x_1,x_2,\ldots,x_k)=P\left(X_1=x_1,X_2=x_2,\ldots,X_k=x_k\right)=\prod_{j=1}^{k}E_q(-\lambda_j)\frac{{\lambda_j}^x}{[x_j]_q!}, \tag{3.22}$$

where $x_j=0,1,2,\ldots,\ 0<\lambda_j<1/(1-q)$, $0<q<1,\ j=1,2,\ldots,k,\ k\geq 2$.

Let $q:=q(\min(\lambda_j))$ with $1-1/\lambda_j<q<1,\ j=1,2,\ldots,k$. Then from the independencies of the random variables $X_j,\ j=1,2,\ldots,k,\ k\geq 2,$ , we easily derive that, for $\lambda_j\to\infty$ with $\lambda_j(1-q)\to 1$, $j=1,2,\ldots,k$, the multiple Euler distribution is approximated by a deformed multivariate standardized Gaussian distribution as follows:

$$\begin{aligned} f_{\mathcal{X}}^{E}(x_1,x_2,\ldots,x_k)\cong\left(\frac{\log q^{-1}}{2\pi(q^{-1}-1)}\right)^{k/2}&\frac{q^{\sum_{i=1}^{k}x_i}}{\prod_{j=1}^{k}\sigma_{[X_j]_q}}\\ &\cdot\exp\left(-\frac{1-q}{2\log q^{-1}}\left(\sum_{j=1}^{k}\left(\frac{[x_j]_q-\mu_{[X_j]_q}}{\sigma_{[X_j]_q}}\right)^2\right)\right),\\ &x_j\geq 0, j=1,2,\ldots,k,k\geq 2 \end{aligned} \tag{3.23}$$

where $\mu_{[X_j]_q}$ and $(\sigma_{[X_j]_q})^2$ are the mean values and the variances of the random variables $[X_j]_q$, $j=1,2,\ldots,k$, given respectively by

$$\begin{aligned} \mu_{[X_j]_q}&=E\left([X_j]_q\right)=\lambda_j\\ &\text{and}\\ (\sigma_{[X_j]_q})^2&=V\left([X_j]_q\right)=\lambda_j(1-\lambda_j(1-q)). \end{aligned} \tag{3.24}$$

**Remark 3.7.** *Asymptotic Behaviour of $q$-Trinomial Distribution of the Second Kind.* Let $(X_1,X_2)$ be a discrete bivariate random variable that follows the *$q$-trinomial distribution of the second kind*, defined in (3.14) with $k=2$. Then the joint probability function of the $q$-trinomial distribution of the second kind is given by

$$\begin{aligned} f_{X_1,X_2}^{BS}(x_1,x_2)&=P\left(X_1=x_1,X_2=x_2\right)\\ &=\binom{n}{x_1,x_2}_q\theta_1^{x_1}\theta_2^{x_2}\prod_{i=1}^{n-x_1}(1-\theta_1q^{i-1})\prod_{i=1}^{n-x_1-x_2}(1-\theta_2q^{i-1}) \end{aligned} \tag{3.25}$$

$x_j=0,1,2,\ldots,n$, $j=1,2$, $x_1+x_2\leq n$, $0<\theta_1,\theta_2<1$ and $0<q<1$ or $1<q<\infty$ and $0<\theta_j<q^{-(n-1)}$.

The marginal probability function of the random variable $X_1$, is distributed according to the $q$-binomial of the 2nd kind with probability function

$$f_{X_1}^{BS}(x_1)=\binom{n}{x_1}_q\theta_1^{x_1}\prod_{i=1}^{n-x_1}(1-\theta_1q^{i-1}),\quad x_1=0,1,2,\ldots,n.$$

The mean and the variance of the deformed variable $[X_1]_q$ are given by

$$\begin{aligned} \mu_{[X_1]_q}&=E\left([X_1]_q\right)=[n]_q\theta_1\\ &\text{and}\\ (\sigma_{[X_1]_q})^2&=V\left([X_1]_q\right)=[n]_q\theta_1\left(1-\theta_1\left(1-q^{n-1}\right)\right). \end{aligned} \tag{3.26}$$

respectively.

The conditional random variable $X_2|X_1$, is distributed according to the univariate $q$-binomial of the 2nd kind with probability function

$$f^{BS}_{X_2|X_1}(x_2|x_1) = \binom{n-x_1}{x_2}_q \theta_2^{x_2} \prod_{i=1}^{n-x_1-x_2} (1+\theta_2 q^{i-1}), \quad x_2 = 0,1,2,\dots,n-x_1.$$

The conditional mean and conditional variance of the deformed variable $[X_2]_q$ given $X_1 = x_1$, are given by

$$\begin{aligned} \mu_{[X_2]_q|X_1} &= E\left([X_2]_q|X_1\right) = [n-x_1]_q\theta_2, \\ &\text{and} \\ (\sigma_{[X_2]_q|X_1})^2 &= V\left([X_2]_q|X_1\right) = [n-x_1]_q\theta_1\left(1-\theta_1\left(1-q^{n-x_1-1}\right)\right) \end{aligned} \tag{3.27}$$

respectively.

Suppose that $\theta_j = \theta_{j,n} = (1-q)^{a_j}$, $0 < a_j < 1$, $j = 1,2$, $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, and $q(n)^n = \Omega(1)$. Then, for $n \to \infty$, the $q$-trinomial of the second kind from the local limit theorem 2, is approximated by a deformed standardized bivariate Gaussian distribution as follows:

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) \cong{}& \frac{\log q^{-1}}{2\pi(1-q)\sigma_{[X_1]_q}\sigma_{[X_2]_q|X_1}}\, q^{x_1+x_2} \\ &\cdot \exp\left(\frac{1-q}{2\log q}\left(\left(\frac{[x_1]_q - \mu_{[X_1]_q}}{\sigma_{[X_1]_q}}\right)^2 + \left(\frac{[x_2]_q - \mu_{[X_2]_q|X_1}}{\sigma_{[X_2]_q|X_1}}\right)^2\right)\right), \\ & x_1, x_2 \geq 0, \end{aligned} \tag{3.28}$$

where the mean $\mu_{[X_1]_q}$ and the variance $(\sigma_{[X_1]_q})^2$ are given in (3.26), while the conditional mean $\mu_{[X_2]_q|X_1}$ and the conditional variance $(\sigma_{[X_2]_q|X_1})^2$ are given in (3.27).

**Example 3.8.** *Asymptotic Behaviour of Proofreading a Manuscript with Euler-Distributed Random Number of Errors.* An alternative option of the Example 3.4 of *proofreading a manuscript,* is by assuming that the random number $M_j$ of the typographical errors in chapter $c_j$, obeys an Euler distribution with parameters $\lambda_j$ and $q$, for $j = 1,2,\dots,k$. Suppose that the random variables $M_1, M_2, \dots, M_k$ are independent. Let $Y_j$ be the number of successes of $j$th kind, for $j = 1,2,\dots,k$, in a sequence of $n$ scans (readings) of the manuscript. Then the joint probability function of the random vector $\mathcal{Y} = (Y_1, Y_2, \dots, Y_k)$, obeys a $q$-multinomial distribution of the 2nd kind with parameters $\theta_j = \lambda_j(1-q)$, $0 < q < 1$ and $0 < \theta_j < 1$. (see Charalambides [5]).

Suppose that $\lambda_j = \lambda_{j,n}$, then according to the local limit Theorem 3.5, the joint probability function of the random vector $\mathcal{Y} = (Y_1, Y_2, \dots, Y_k)$, for $\theta_j = \theta_{j,n} = (1-q)^{\alpha_j}$ with $0 < a_j < 1$, $j = 1,2,\dots,k$ and $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, and $q(n)^n = \Omega(1)$, is approximated, for $n \to \infty$, by a deformed multivariate standardized Gaussian distribution, given in (3.21).

Note that the joint probability function of the random vector $\mathcal{M} = (M_1, \dots, M_k)$ obeys a multiple Euler distribution with parameters $\lambda_j$, $j = 1,2,\dots,k$. According to Remark 3.7, the joint probability function of the random vector $\mathcal{M} = (M_1, \dots, M_k)$, for $q := q(\min(\lambda_j))$ with $1 - 1/\lambda_j < q < 1$, $j = 1,2,\dots,k$ and for $\lambda_j \to \infty$ with $\lambda_j(1-q) \to 1$, $j = 1,2,\dots,k$, $\lambda_j \to \infty$, $j = 1,2,\dots,k$, is approximated by a deformed multivariate standardized Gaussian distribution, given in (3.23.)

## 4. Concluding remarks – further study

The aim of this work was to study the continuous limiting behaviour of multivariate discrete $q$-distributions inspired by the limiting behaviour of the univariate ones. Local limit theorems for $q$-multinomial distributions of the first and second kind and of their discrete limits multiple Heine and multiple Euler distributions respectively were studied. Specifically, the pointwise convergence of the $q$-multinomial distribution of the first kind, as well as for its discrete limit, the multiple Heine distribution, to a multivariate Stieltjes–Wigert type distribution, were proved. Moreover, the pointwise convergence of the $q$-multinomial distribution of the second kind, as well as for its discrete limit, the multiple Euler distribution, to a multivariate deformed Gaussian distribution, were derived. Interesting applications of the asymptotic behaviour of $q$-multinomials distributions of the two kinds were presented.

An intriguing question that arises from this study is whether it is possible to develop a unified framework for analyzing the asymptotic behavior of both univariate and multivariate discrete $q$-distributions without the need to prove local limit theorems separately for each $q$-distribution. Establishing such a general approach could provide deeper insights and more efficient methods for studying discrete $q$-probabilistic models

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## Data availability statement

No data were used in this study.

## Authors' contributions

Sole author.

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