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SMOOTH CONVEX POLYIAMONDS

Toufik Mansour*

Abstract. In this paper, we present an explicit formula for the generating function that enumerates smooth convex polyiamonds with perimeter n. In particular, we show that the number of such polyiamonds grows at a rate given by $2.54442495\cdots$.

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1. Introduction and preliminaries

By a *cell* in the Cartesian plane $\mathbb{Z}^2$, we mean the interior and the boundary of a unit square with its sides parallel to the coordinate axes. A *polyomino* is a finite edge-connected collection of cells. Polyominoes were introduced by Golomb, see [1] and references therein. For some of the earliest works, we refer the reader to [2–4] and references therein. Polyominoes attracted and motivated interest mainly in solving mathematical recreational problems. More later, polyominoes arose in many different research areas of physics [5–11]), in tiling problems [12–19], in enumeration problems [20–22], etc.

Polyiamonds, animals residing on the triangular lattice, have been considered in recent years. For instance, Yang and Wilson [23] showed the minimum perimeter of a polyiamond with n cells is either $\lceil\sqrt{6n}\rceil$ or $\lceil\sqrt{6n}\rceil+1$ depending on the parity of n. Malen and Roldán [24] extended this result by studying the maximum number of holes that a polyiamond with n cells can enclose. Later, Shahleh [25] and Barequet *et al.* [26] improved the lower and upper bounds of the asymptotic growth rate with respect to the size. Recently, Mansour and Rastegar [27] provided an explicit formula for the generating function enumerates column-convex polyiamonds with perimeter n. This is mainly considered by careful recursive analyses of these objects through column-by-column decomposition and cell pruning/growing arguments which lead to explicit forms of defined generating functions. We note that one major difference between the triangular lattice and the squared/hexagonal lattice is the lack of some of the symmetrical structures in the former, which increases the level of analysis complexity. For other works on polyiamonds, we refer the reader to [28, 29].

Let $\mathcal{T}$ be the two-dimensional plane, where the angle between x-axis and y-axis is $\frac{\pi}{3}$ counter clock-wise and we mark these axes with integer points (for instance, see [27]). Then, we partition this plane with equilateral triangles; each with a horizontal edge parallel to x-axis and the sides of length one. From now on, we refer to these triangular building blocks as triangular cells or simply *cells*. Clearly, this partition is uniquely defined such that there is one cell in the first quarter with a vertex at the intersection of the axes and two edges on the axes. Moreover, let the line $x=j$ (respectively, $y=j$) denote the line parallel with the y-axis, intersection the

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Department of Mathematics, University of Haifa, 3498838 Haifa, Israel.

* Corresponding author: tmansour@univ.haifa.ac.il

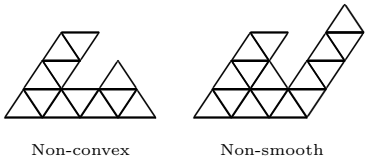

FIGURE 1. Non-convex and Non-smooth polyiamonds.

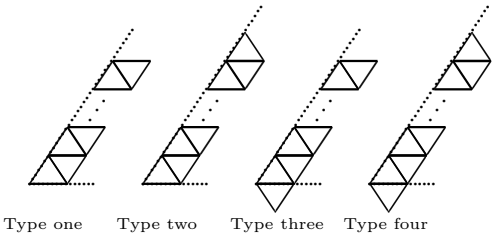

FIGURE 2. From left to right, convex columns of type one, two, three, and four.

x-axis (respectively, y-axis) at j. There are two types of cells, namely (1) down-cells whose horizontal edges are at their bases, and (2) up-cells whose horizontal sides are on the top.

A *polyiamond* ν is a finite edge-connected set of cells in $\mathcal{T}$. A *boundary edge* is a common edge for cells of ν and cells of $\nu^c := \mathcal{T} \setminus \nu$. The *perimeter* of ν is the total number of its boundary edges. A *column* (respectively, *row*) w of ν is subset of all cells in ν between lines $x = j$, called the left side of w, and $x = j + 1$, called the right side of w, (respectively, $y = j$ and $y = j + 1$) for some $j \in \mathbb{Z}$. A polyiamond is called *column-convex* (respectively, *row-convex*) if each of its columns/rows is a single contiguous block of cells. Moreover, a *convex* polyiamond is both column-convex and row-convex (see Fig. 1). Clearly, there are four types of convex columns as described in Figure 2.

Let π be any nonempty polyiamond with m columns such that the lower horizontal edge/vertex of the bottom cell of the first column lies on the x-axis. We say that a bottom (respectively, upper) cell of the i-th column of π is at position k if it lies above and touches (respectively, lies below and touches) the line $y = k$ and we denote this by $b_i = k$ (respectively, $u_i = k$), for $i = 1, 2, \ldots, m$. Now, we are ready to define our main object. A *smooth convex polyiamond* is a convex polyiamond such that $b_i - b_{i-1}, u_i - u_{i-1} \in \{0, 1, -1\}$, for all $i = 2, 3, \ldots, m$. The main result of this paper can be formulated as follows[1].

Theorem 1.1. *Define* $B_1^{U(1)}(p) = \frac{g(p,t_2)}{p(1+p)(1+p^2t_2)}$, *where*

$$g(p,t) = \frac{p^4t(p^{10}t^4 + p^7(p-1)(p+2)t^3 + 2p^5(p^2-p-1)t^2 + p^3(p^3-3p-1)t + p^4 + (1+p)(1-p^2))}{(1-p^2t)(1-p^2-p^3(1+2p)t - p^5(p+2)t^2 - p^7t^3)},$$

$$t_2 = \frac{2}{3p^2}\sqrt{\frac{(p+1)(p^4-2p+3)}{p}}\cos\left(\frac{\pi}{3} + \frac{1}{3}\arccos\left(\frac{(2p^3+p^2+3p+2)(p^3-5p+7)-5}{2\sqrt{\frac{(p+1)(p^4-2p+3)^3}{p}}}\right)\right) - \frac{(p+1)^2}{3p^2},$$

Then, the generating function $B(p)$ *for the number of smooth convex polyiamonds according to their perimeter is given by*

$$B(p) = \frac{a(p)}{b(p)}B_1^{U(1)}(p) - \frac{c(p)}{d(p)}\sqrt{((p^2+1)^2-p^2)((p^2-1)^2-p^2)} + \frac{e(p)}{f(p)},$$

[1]For deriving the results of this paper, we used mathematical programming as described at http://math.haifa.ac.il/toufik/enumerative/MapleSmoothConvexPoly.pdf.

where

$$\begin{aligned}a(p) &= p^{17}+3p^{16}+3p^{15}-p^{14}-7p^{13}-2p^{12}+10p^{11}+15p^{10}+14p^9+12p^8+7p^7\\&\quad+16p^6+34p^5+25p^4-p^3-8p^2-p+1,\end{aligned}$$

$$\begin{aligned}b(p) &= (1-p)(2p^5+3p^4+3p^3+3p^2+p-1)(p^{12}+p^{11}-3p^{10}-2p^9+p^8+4p^7\\&\quad+3p^6-2p^5-p^4+5p^3+5p^2-1),\end{aligned}$$

$$\begin{aligned}c(p) &= p^5(1+p)^2(p^9+p^8+p^6-3p^5-2p^4-2p^3-2p^2+p+1)(p^{14}-2p^{13}+4p^{12}\\&\quad-7p^{11}+4p^{10}-2p^9-5p^8+8p^7-2p^6+2p^5+5p^4-p^3+p^2-1),\end{aligned}$$

$$\begin{aligned}d(p) &= 2(p^9+p^8-2p^7-2p^6-2p^5-p^4-2p^3-2p^2+p+1)(2p^4+2p^2-1)^2(p^{12}\\&\quad-p^{11}-2p^{10}+2p^9-2p^8+p^7+3p^5+2p^4-p^3+2p^2-1),\end{aligned}$$

$$\begin{aligned}e(p) &= 18p^{67}+97p^{66}+232p^{65}+294p^{64}-192p^{63}-1721p^{62}-3926p^{61}-5366p^{60}\\&\quad-2118p^{59}+8820p^{58}+25666p^{57}+41008p^{56}+36278p^{55}-2736p^{54}-77296p^{53}\\&\quad-165612p^{52}-210538p^{51}-157218p^{50}+29919p^{49}+329910p^{48}+632126p^{47}\\&\quad+801752p^{46}+693888p^{45}+238630p^{44}-483734p^{43}-1319357p^{42}-2019670p^{41}\\&\quad-2365863p^{40}-2276561p^{39}-1756419p^{38}-977166p^{37}-156234p^{36}+525524p^{35}\\&\quad+906875p^{34}+975741p^{33}+781698p^{32}+419918p^{31}+50463p^{30}-238635p^{29}\\&\quad-380179p^{28}-352243p^{27}-227038p^{26}-62480p^{25}+78896p^{24}+140191p^{23}\\&\quad+131329p^{22}+77450p^{21}+11195p^{20}-31636p^{19}-45749p^{18}-32958p^{17}\\&\quad-10179p^{16}+1025p^{15}+10078p^{14}+13612p^{13}+1987p^{12}-3661p^{11}+529p^{10}\\&\quad-836p^9-2601p^8+363p^7+1381p^6+21p^5-332p^4-20p^3+40p^2+2p-2,\end{aligned}$$

$$\begin{aligned}f(p) &= 2\left(p^7+p^6+2p^5+2p^4+p^3+p^2-1\right)\left(2p^5+3p^4+3p^3+3p^2+p-1\right)\\&\quad\cdot\left(p^6+p^5+2p^4+p^3+p^2-1\right)\left(p^9+p^8-2p^7-2p^6-2p^5-p^4-2p^3-2p^2+p+1\right)\\&\quad\cdot\left(p^{12}+p^{11}-3p^{10}-2p^9+p^8+4p^7+3p^6-2p^5-p^4+5p^3+5p^2-1\right)\\&\quad\cdot(p-1)^2(p+1)\left(2p^4+2p^2-1\right)^2\\&\quad\cdot\left(p^{12}-p^{11}-2p^{10}+2p^9-2p^8+p^7+3p^5+2p^4-p^3+2p^2-1\right).\end{aligned}$$

Note that in [30], the authors counted the number of smooth column convex polyominoes (in square lattice) according to the perimeter. So our main result presents the enumeration of smooth convex polyiamonds according to the perimeter.

The growth rate of a sequence a_n with respect to n is simply $\lim_{n\mapsto\infty}(a_n)^{1/n}$ provided that this limit exists. As a corollary of Theorem 1.1, we see that the growth rate of the number of smooth convex polyiamonds with perimeter n is given by $\frac{1}{\alpha}\approx 2.54442495\cdots$. More precisely,

$$\lim_{n\mapsto\infty}([p^n]t_2)^{1/n}=\lim_{n\mapsto\infty}([p^n]B_1^{U(1)}(p))^{1/n}=\lim_{n\mapsto\infty}([p^n]B(p))^{1/n}=\frac{1}{\alpha},$$

where α is the smallest root of the polynomial equation $p^{11}-2p^{10}-p^9+6p^8-9p^7+8p^6-8p^5+2p^4+5p^3-10p^2-7p+4=0$, namely, $\alpha\approx 0.39301611082311\cdots$.

2. Proof of Theorem 1.1

In this section, we enumerate the smooth convex polyiamonds with respect to their perimeter. In order to do that, we introduce three subfamilies of smooth convex polyiamonds, and for each, we enumerate with respect

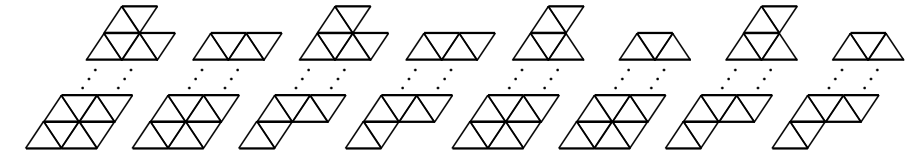

FIGURE 3. All possibilities of two columns of a bounded smooth convex polyiamond such that the first column is of type 1.

to their perimeter. Following [30], the strategy on counting a such subfamily of smooth convex polyiamonds is based on "adding a slice" method, this method has been used in [31]. The methodology is as follows:

- For fixed values of d and k, we define the generating function – according to perimeter – for the number of polyiamonds with k columns, where the first column is of type d.
- We derive recurrence relations for these generating functions by reading the columns of the polyiamonds from left to right.
- We convert the recurrence relations into functional equations and solve them by using the kernel method (see [32]).

2.1. Bounded smooth convex polyiamonds

Recall the types of convex columns in Figure 2. Let π be any nonempty smooth convex polyiamond with m columns such that the lower horizontal edge/vertex of the bottom cell of the first column lies on the x-axis. We say that π is *bounded* if the cells of the jth column lie between the lines

- $y = u_{j-1}$ and $y = b_{j-1}$ if the $(j-1)$-st column is of type $1, 3$;
- $y = u_{j-1} - 1$ and $y = b_{j-1}$ if the $(j-1)$-st column is of type $2, 4$,

for all $j = 2, 3, \ldots, m$. Let $B_k^{B(i)}(p)$ be the generating function for the number of bounded smooth convex polyiamonds whose first columns are of type i and have k down-cells according to their perimeter. It is clear that the generating function $B^B(p)$ for the number of bounded smooth convex polyiamonds according to their perimeter is

$$B^B(p) = 1 + \sum_{k\geq 0}\sum_{i=1}^{4} B_k^{B(i)}(p).$$

In order to study the generating function $B_k^{B(i)}(p)$, we use *cell growing/pruning* method. First, let ν be any bounded smooth convex polyiamond whose first column is of type 1 and has k edges on its right side. If ν has another column, then it can be described as in Figure 3.

By adding up the contribution of each case in Figure 3, we obtain

$$\begin{aligned} B_k^{B(1)}(p) = {} & p^{2k+2} + 2p^4B_{k-1}^{B(1)}(p) + p^2B_k^{B(1)}(p) + p^6B_{k-2}^{B(1)}(p) \\ & + 2p^4B_{k-2}^{B(2)}(p) + p^2B_{k-1}^{B(2)}(p) + p^6B_{k-3}^{B(2)}(p). \end{aligned} \tag{2.1}$$

Next, we describe the relation between $B_k^{B(i)}(p)$, $i = 2, 3, 4$, and $B_k^{(1)}(p)$:

- By removing the up-cell from the upper of the first column of a bounded smooth convex polyiamond whose first column is of type 2 and has k edges on its right side, we have a bounded smooth convex polyiamond

whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{B(2)}(p) = pB_k^{B(1)}(p). \tag{2.2}$$

- By adding an up-cell to the bottom of the first column of a bounded smooth convex polyiamond whose first column is of type 3 and has k edges on its right side, we have a bounded smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{B(3)}(p) = \frac{1}{p}B_k^{B(1)}(p). \tag{2.3}$$

- By adding an up-cell to the bottom and removing an up-cell to the upper of the first column of a bounded smooth convex polyiamond whose first column is of type 4 and has k edges on its right side, we have a bounded smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{B(4)}(p) = B_k^{B(1)}(p). \tag{2.4}$$

- Note that the initial conditions are given by $B_0^{B(2)}(p) = p^3$ and $B_0^{B(i)}(p) = 0$ for all $i = 1, 3, 4$.

Define $B^{B(i)}(p;t) = \sum_{k\geq 0} B_k^{B(i)}(p)t^k$ for all $i = 1, 2, 3, 4$. Hence, by multiplying (2.1)–(2.4) by t^k and summing over $k \geq 1$, we have

$$\begin{aligned}
B^{B(1)}(p;t) &= \frac{p^4}{1-p^2t} + p^2(1+p^2t)^2B^{B(1)}(p;t) + p^2t(1+p^2t)^2B^{B(2)}(p;t),\\
B^{B(2)}(p;t) &= p^3 + pB^{B(1)}(p;t),\\
B^{B(3)}(p;t) &= \frac{1}{p}B^{B(1)}(p;t),\\
B^{B(4)}(p;t) &= B^{B(1)}(p;t).
\end{aligned}$$

Solving this system for $B^{B(i)}(p;t)$, $i = 1, 2, 3, 4$, we obtain the following result.

Theorem 2.1. *We have*

$$\begin{aligned}
B^{B(1)}(p;t) &= pB^{B(3)}(p;t) = B^{B(4)}(p;t)\\
&= \frac{p^4(1+pt+p^3t^2-p^5t^3-p^7t^4)}{(1-p^2t)(1-p^2-p^3(1+2p)t-p^5(2+p)t^2-p^7t^3)},\\
B^{B(2)}(p;t) &= \frac{p^3(1-p^2(1+p^2)t+p^6t^2+p^8t^3)}{(1-p^2t)(1-p^2-p^3(1+2p)t-p^5(2+p)t^2-p^7t^3)}.
\end{aligned}$$

Moreover, the generating function $B^B(p)$ for the number of bounded smooth convex polyiamonds according to their perimeter is given by

$$\begin{aligned}
B^B(p) &= 1 + \frac{p^3(2+p+p^3-p^5-p^7)}{(1-p)(1-p^2-p^3-2p^4-2p^5-p^6-p^7)}\\
&= 1 + 2p^3 + 3p^4 + 5p^5 + 9p^6 + 16p^7 + 27p^8 + 46p^9 + 78p^{10} + \cdots.
\end{aligned}$$

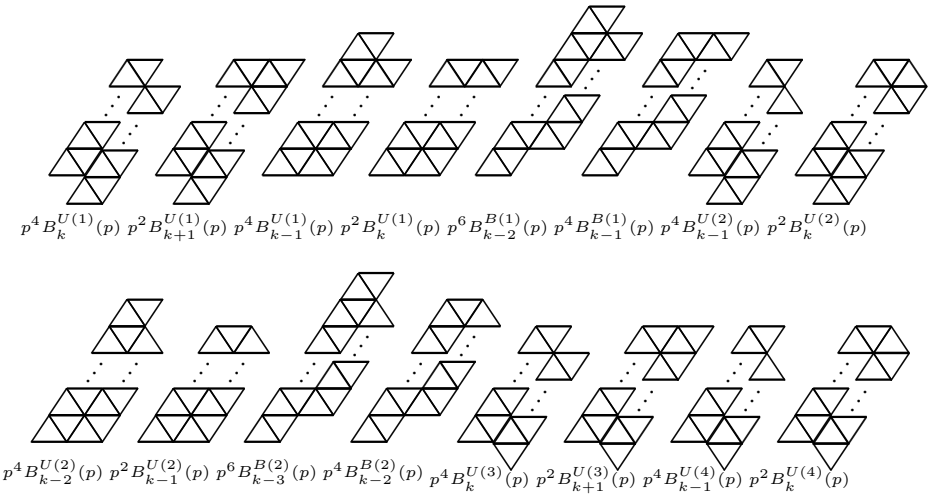

FIGURE 4. All possibilities of two columns of an upper smooth polyiamond such that the first column is of type 1.

2.2. Upper smooth convex polyiamonds

Recall the types of convex columns in Figure 2. Let π be any nonempty smooth convex polyiamond with m columns such that the lower horizontal edge/vertex of the bottom cell of the first column lies on the x-axis. We say that π is *upper* if the cells of the jth column lie below the line

- $y = u_{j-1}$ if the $(j-1)$-st column is of type $1, 3$;
- $y = u_{j-1} - 1$ if the $(j-1)$-st column is of type $2, 4$,

for all $j = 2, 3, \ldots, m$. Let $B_k^{U(i)}(p)$ be the generating function for the number of upper smooth convex polyiamonds whose first columns are of type i and have k down-cells according to their perimeter. It is clear that the generating function $B^U(p)$ for the number of upper smooth convex polyiamonds according to their perimeter is

$$B^U(p) = 1 + \sum_{k\geq 0}\sum_{i=1}^{4} B_k^{U(i)}(p).$$

Let ν be any upper smooth convex polyiamond whose first column is of type 1 and has k edges on its right side. If ν has another column, then it can be described as in Figure 4.

By adding up the contribution of each case in Figure 4, we obtain

$$\begin{aligned} B_k^{U(1)}(p) = {} & p^{2k+2} + p^4B_{k-1}^{U(1)}(p) + (p^2 + p^4\delta_{k\geq 2})B_k^{U(1)}(p) + p^2B_{k+1}^{U(1)}(p) \\ & + p^6B_{k-2}^{B(1)}(p) + p^4B_{k-1}^{B(1)}(p) + (p^2 + p^4\delta_{k\geq 2})B_{k-1}^{U(2)}(p) + p^2B_k^{U(2)}(p) \\ & + p^4\delta_{k\geq 3}B_{k-2}^{U(2)}(p) + p^6B_{k-3}^{B(2)}(p) + p^4B_{k-2}^{B(2)}(p) + p^4B_k^{U(3)}(p) \\ & + p^2B_{k+1}^{U(3)}(p) + p^4B_{k-1}^{U(4)}(p) + p^2B_k^{U(4)}(p). \end{aligned} \tag{2.5}$$

Next, we describe the relation between $B_k^{U(i)}(p)$, $i = 2, 3, 4$, and $B_k^{U(1)}(p)$:

- By removing an up-cell from the upper of the first column of an upper smooth convex polyiamond whose first column is of type 2 and has k edges on its right side, we have an upper smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{U(2)}(p) = pB_k^{U(1)}(p). \tag{2.6}$$

- By adding an up-cell to the bottom of the first column of an upper smooth convex polyiamond whose first column is of type 3 and has k edges on its right side, we have an upper smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{U(3)}(p) = \frac{1}{p}B_k^{U(1)}(p). \tag{2.7}$$

- By adding an up-cell to the bottom and removing an up-cell to the upper of the first column of an upper smooth convex polyiamond whose first column is of type 4 and has k edges on its right side, we have an upper smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{U(4)}(p) = B_k^{U(1)}(p). \tag{2.8}$$

- Note that the initial conditions are given by $B_0^{U(2)}(p) = p^3$ and $B_0^{U(i)}(p) = 0$ for all $i = 1, 3, 4$.

Define $B^{U(i)}(p;t) = \sum_{k\geq 0} B_k^{U(i)}(p)t^k$ for all $i = 1, 2, 3, 4$. Hence, by multiplying (2.5)–(2.8) by t^k and summing over $k \geq 1$, we have

$$\begin{aligned}
B^{U(1)}(p;t) &= \frac{p^4 t}{1-p^2 t} + p^2(1+p^2+p^2t+t^{-1})B^{U(1)}(p;t) - p^2 B_1^{U(1)}(p) \\
&\quad + p^4 t(1+p^2t)B^{B(1)}(p;t) + p^2 t(1+p^2+p^2t)B^{U(2)}(p;t) \\
&\quad + p^2(B^{U(2)}(p;t) - p^3) + p^4t^2(1+p^2t)B^{B(2)}(p;t) + p^4 B^{U(3)}(p;t) \\
&\quad + \frac{p^2}{t}B^{U(3)}(p;t) - p^2 B_1^{U(3)}(p) + p^4 t B^{U(4)}(p;t) + p^2 B^{U(4)}(p;t) \\
&\quad - p^4 t B_1^{U(1)}(p) - p^4 t B_1^{U(3)}(p) - p^7 t(1+t), \\
B^{U(2)}(p;t) &= p^3 + pB^{U(1)}(p;t), \\
B^{U(3)}(p;t) &= \frac{1}{p}B^{U(1)}(p;t), \\
B^{U(4)}(p;t) &= B^{U(1)}(p;t).
\end{aligned}$$

Thus, by Theorem 2.1, we have

$$\begin{aligned}
&\frac{t - p(1+p+pt)(1+pt)(1+p^2t)}{t}B^{U(1)}(p;t) \\
&\qquad + p(1+p)(1+p^2t)B_1^{U(1)}(p) = g(p,t),
\end{aligned} \tag{2.9}$$

where $g(p,t) = \frac{p^4t(p^{10}t^4+p^7(p-1)(p+2)t^3+2p^5(p^2-p-1)t^2+p^3(p^3-3p-1)t+p^4+(1+p)(1-p^2))}{(1-p^2t)(1-p^2-p^3(1+2p)t-p^5(p+2)t^2-p^7t^3)}$. This type of functional equation can be solved systematically using the kernel method (see [32]). Here, the kernel is given by

$K(p,t) = 1 - \frac{p(1+p+pt)(1+pt)(1+p^2t)}{t}$. Note that the equation $K(p,t) = 0$ has three solutions

$$t_k = \frac{2}{3p^2}\sqrt{\frac{(p+1)(p^4-2p+3)}{p}}\cos\left(\frac{\pi(2k+1)}{3} - \frac{1}{3}\arccos(\theta(p))\right) - \frac{(p+1)^2}{3p^2},$$

where $\theta(p) = \frac{(2p^3+p^2+3p+2)(p^3-5p+7)-5}{2\sqrt{\frac{(p+1)(p^4-2p+3)^3}{p}}}$, for $k = 0, 1, 2$. By taking $t = t_2$ into (2.9), we obtain

$$B_1^{U(1)}(p) = \frac{g(p,t_2)}{p(1+p)(1+p^2t_2)}. \tag{2.10}$$

Hence, by (2.5)–(2.8), we can state the following result.

Theorem 2.2. *We have* $B^{U(2)}(p;t) = p^3 + pB^{U(1)}(p;t)$, $B^{U(3)}(p;t) = \frac{1}{p}B^{U(1)}(p;t)$, $B^{U(4)}(p;t) = B^{U(1)}(p;t)$, *and*

$$B^{U(1)}(p;t) = \frac{t(1+p^2t_2)g(p,t) - t(1+p^2t)g(p,t_2)}{(1+p^2t_2)(t-p(1+p+pt)(1+pt)(1+p^2t))},$$

with

$$t_2 = \frac{2}{3p^2}\sqrt{\frac{(p+1)(p^4-2p+3)}{p}}\cos\left(\frac{\pi}{3} + \frac{1}{3}\arccos(\theta(p))\right) - \frac{(p+1)^2}{3p^2}.$$

Moreover, the generating function $B^U(p)$ *for the number of upper smooth convex polyiamonds according to their perimeter is given by*

$$\begin{aligned} B^U(p) &= 1 + p^3 + \frac{(1+p)^2((1+p^2t_2)g(p,1) - (1+p^2)g(p,t_2))}{p(1+p^2t_2)(1-p(1+2p)(1+p)(1+p^2))} \\ &= 1 + 2p^3 + 3p^4 + 6p^5 + 13p^6 + 30p^7 + 70p^8 + 164p^9 + 384p^{10} + \cdots. \end{aligned}$$

2.3. Lower smooth convex polyiamonds

Let π be any nonempty smooth convex polyiamond with m columns such that the lower horizontal edge/vertex of the bottom cell of the first column lies on the x-axis. We say that π is *lower* if the cells of the jth column lie above the line

- $y = b_{j-1}$ if the $(j-1)$-st column is of type $1, 2$;
- $y = b_{j-1} + 1$ if the $(j-1)$-st column is of type $3, 4$,

for all $j = 2, 3, \ldots, m$. Let $B_k^{L(i)}(p)$ be the generating function for the number of lower smooth convex polyiamonds whose first column of type i has k edges on its right side according to their perimeter. It is clear that the generating function $B^L(p)$ for the number of lower smooth convex polyiamonds according to their perimeter is

$$B^L(p) = 1 + \sum_{k\geq 0}\sum_{i=1}^{4} B_k^{L(i)}(p).$$

Let ν be any upper smooth convex polyiamond whose first column is of type 1 and has k edges on its right side. If ν has another column, then it can be described as in Figure 5.

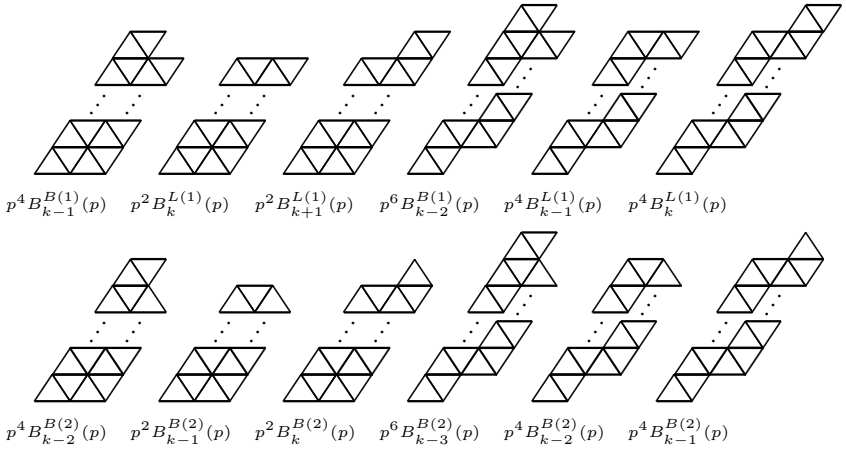

FIGURE 5. All possibilities of two columns of a lower smooth polyiamonds such that the first column is of type 1.

By adding up the contribution of each case in Figure 5, we obtain

$$\begin{aligned} B_k^{L(1)}(p) = {} & p^{2k+2} + p^4 B_{k-1}^{B(1)}(p) + (p^2 + p^4\delta_{k\geq 2}) B_k^{L(1)}(p) + p^2 B_{k+1}^{L(1)}(p) \\ & + p^6 B_{k-2}^{B(1)}(p) + p^4 B_{k-1}^{L(1)}(p) + (p^2 + p^4\delta_{k\geq 2}) B_{k-1}^{B(2)}(p) + p^2 B_k^{B(2)}(p) \\ & + 2p^4 B_{k-2}^{B(2)}(p) + p^6 B_{k-3}^{B(2)}(p). \end{aligned} \tag{2.11}$$

Next, we describe $B_k^{L(i)}(p)$, $i = 2, 3, 4$:

- By removing an up-cell from the upper of the first column of a lower smooth convex polyiamond whose first column is of type 2 and has k edges on its right side, we have a bounded smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{L(2)}(p) = pB_k^{B(1)}(p). \tag{2.12}$$

- By adding an up-cell to the bottom of the first column of a lower smooth convex polyiamond whose first column is of type 3 and has k edges on its right side, we have a lower smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{L(3)}(p) = \frac{1}{p} B_k^{L(1)}(p). \tag{2.13}$$

- By adding an up-cell to the bottom and removing an up-cell to the upper of the first column of a lower smooth convex polyiamond whose first column is of type 4 and has k edges on its right side, we have a bounded smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{L(4)}(p) = B_k^{B(1)}(p). \tag{2.14}$$

- Note that the initial conditions are given by $B_0^{L(2)}(p) = p^3$ and $B_0^{L(i)}(p) = 0$ for all $i = 1, 3, 4$.

Define $B^{L(i)}(p;t)=\sum_{k\geq 0}B_k^{K(i)}(p)t^k$ for all $i=1,2,3,4$. Hence, by multiplying (2.11)-(2.14) by t^k and summing over $k\geq 1$, we have

$$\begin{aligned}
B^{L(1)}(p;t) &= \frac{p^4t}{1-p^2t}+p^4tB^{B(1)}(p;t)+(p^2+p^4)B^{L(1)}(p;t)\\
&\quad+\frac{p^2}{t}(B^{L(1)}(p)-tB_1^{L(1)}(p))+p^6t^2B^{B(1)}(p;t)+p^4tB^{L(1)}(p;t)\\
&\quad+(p^2+p^4)tB^{B(2)}(p;t)+p^2(B^{B(2)}(p;t)-p^3)+2p^4t^2B^{B(2)}(p;t)\\
&\quad+p^6t^3B^{B(2)}(p;t)-p^4tB_1^{L(1)}(p)-p^7t,\\
B^{L(2)}(p;t) &= p^3+pB^{B(1)}(p;t),\\
B^{L(3)}(p;t) &= \frac{1}{p}B^{L(1)}(p;t),\\
B^{L(4)}(p;t) &= B^{B(1)}(p;t).
\end{aligned}$$

Thus, by Theorem 2.1, we have

$$\frac{p^2-(1-p^2-p^4)t+p^4t^2}{t}B^{L(1)}(p;t)-p^2(1+p^2t)B_1^{L(1)}(p)=h(p,t), \tag{2.15}$$

where

$$\begin{aligned}
h(p,t) &= \frac{p^9t^3(p^6(p-1)t^2+2p^2(p^3-p^2+1)t-(p^2-p+3))}{(p^2t-1)(p^7t^3+p^5(p+2)t^2+p^3(2p+1)t+p^2-1)}\\
&\quad-\frac{p^4(p^4(3p^2-2)t^2+(p^3-p+1)(p+1)^2t+p^3)}{(p^2t-1)(p^7t^3+p^5(p+2)t^2+p^3(2p+1)t+p^2-1)}.
\end{aligned}$$

This type of functional equation can be solved systematically using the kernel method (see [32]). Here, the kernel is given by $K(p,t)=\frac{p^2-(1-p^2-p^4)t+p^4t^2}{t}$. Note that the equation $K(p,t)=0$ has two solutions. By substituting the solution

$$t=t_0=\frac{1-p^2-p^4-\sqrt{p^8-2p^6-p^4-2p^2+1}}{2p^4}$$

into (2.15), we obtain

$$B_1^{L(1)}(p)=-\frac{h(p,t_0)}{p^2(1+p^2t_0)}.$$

By (2.15), we can state the following result.

Theorem 2.3. *We have that* $B^{L(2)}(p;t)=p^3+pB^{B(1)}(p;t)$, $B^{L(3)}(p;t)=\frac{1}{p}B^{L(1)}(p;t)$ *and* $B^{L(4)}(p;t)=B^{B(1)}(p;t)$, *where* $B^{B(1)}(p;t)$ *is given in Theorem 2.1 and*

$$B^{L(1)}(p;t)=\frac{t((1+p^2t_0)h(p,t)-(1+p^2t)h(p,t_0))}{(1+p^2t_0)(p^2-(1-p^2-p^4)t+p^4t^2)}.$$

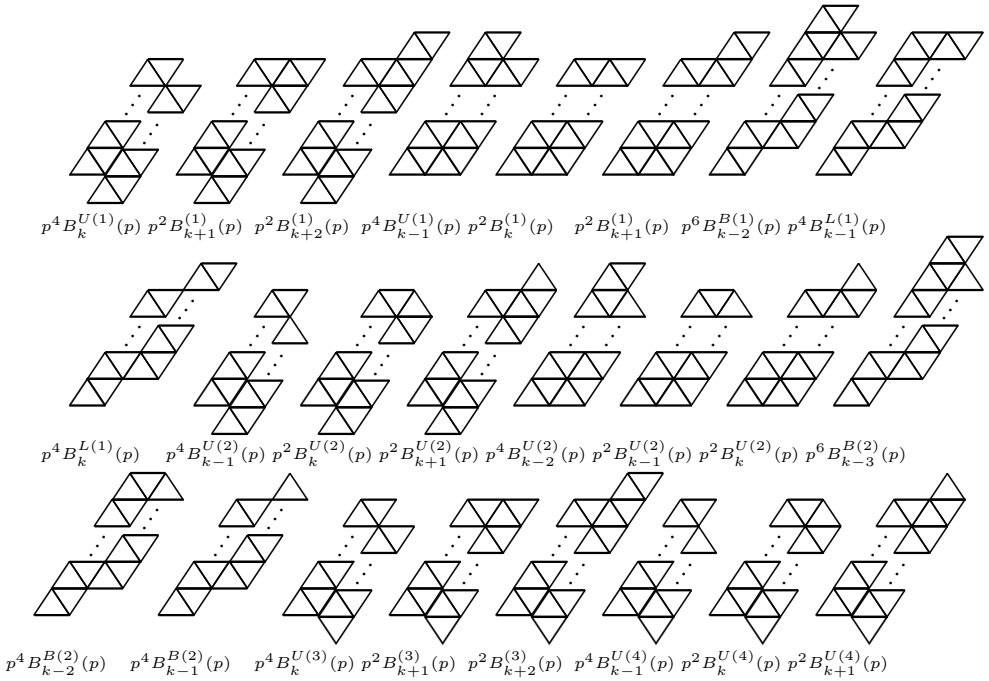

FIGURE 6. All possibilities of two columns of a smooth polyiamond such that the first column is of type 1.

Moreover, the generating function $B^L(p)$ for the number of lower smooth convex polyiamonds according to their perimeter is given by

$$\begin{aligned} B^L(p) &= 1 + p^3 - \frac{(1+p)((1+p^2t_0)h(p,1) - (1+p^2)h(p,t_0))}{p(1+p^2t_0)(1-2p^2-2p^4)} \\ &\quad + \frac{p^4(1+p+p^3-p^5-p^7)}{(1-p)(1-p^2-p^3-2p^4-2p^5-p^6-p^7)} \\ &= 1 + 2p^3 + 3p^4 + 5p^5 + 9p^6 + 18p^7 + 32p^8 + 61p^9 + 109p^{10} + \cdots . \end{aligned}$$

2.4. Smooth convex polyiamonds

Recall the types of convex columns in Figure 2. Let π be any nonempty smooth convex polyiamond with m columns such that the lower horizontal edge/vertex of the bottom cell of the first column lies on the x-axis. Let $B_k^{(i)}(p)$ be the generating function for the number of smooth convex polyiamonds whose first columns are of type i and have k down-cells according to their perimeter. It is clear that the generating function $B(p)$ for the number of smooth convex polyiamonds according to their perimeter is

$$B(p) = 1 + \sum_{k\geq 0}\sum_{i=1}^{4} B_k^{(i)}(p).$$

Let ν be any smooth convex polyiamond whose first column is of type 1 and has k down-cells. If ν has another column, then it can be described as in Figure 6.

By adding up the contribution of each of these cases (see Fig. 6), we obtain

$$\begin{aligned}
B_k^{(1)}(p) &= p^{2k+2} + p^4 B_{k-1}^{U(1)}(p) + p^4\delta_{k\geq 2} B_k^{U(1)}(p) + p^2 B_k^{(1)}(p) + 2p^2 B_{k+1}^{(1)}(p) \\
&\quad + p^2 B_{k+2}^{(1)}(p) + p^6 B_{k-2}^{B(1)}(p) + p^4 B_{k-1}^{L(1)}(p) + p^4\delta_{k\geq 2} B_k^{L(1)}(p) \\
&\quad + (p^2 + p^4\delta_{k\geq 2}) B_{k-1}^{U(2)}(p) + 2p^2 B_k^{U(2)}(p) + p^2 B_{k+1}^{U(2)}(p) + p^4 B_{k-2}^{U(2)}(p) \\
&\quad + p^6 B_{k-3}^{B(2)}(p) + p^4 B_{k-2}^{B(2)}(p) + p^4\delta_{k\geq 2} B_{k-1}^{B(2)}(p) + p^4\delta_{k\geq 2} B_k^{U(3)}(p) \\
&\quad + p^2 B_{k+1}^{(3)}(p) + p^2 B_{k+2}^{(3)}(p) + p^4\delta_{k\geq 2} B_{k-1}^{U(4)}(p) + p^2 B_k^{U(4)}(p) \\
&\quad + p^2 B_{k+1}^{U(4)}(p).
\end{aligned} \tag{2.16}$$

Next, we describe the relation between $B_k^{(i)}(p)$, $i = 2, 3, 4$, and $B_k^{(1)}(p)$:

- By removing an up-cell to the upper of the first column of a smooth convex polyiamond whose first column is of type 2 and has k edges on its right side, we have an upper smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{(2)}(p) = pB_k^{U(1)}(p). \tag{2.17}$$

- By adding up-cell to the bottom of the first column of a smooth convex polyiamond whose first column is of type 3 and has k edges on its right side, we have a smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{(3)}(p) = \frac{1}{p} B_k^{(1)}(p). \tag{2.18}$$

- By adding an up-cell to the bottom and removing an up-cell to the upper of the first column of a smooth polyiamond whose first column is of type 4 and has k edges on its right side, we have an upper smooth convex polyiamond whose first column is of type 1 and have k edges on its right side; hence, for all $k \geq 1$,

$$B_k^{(4)}(p) = B_k^{U(1)}(p). \tag{2.19}$$

- Note that the initial conditions are given by $B_0^{(2)}(p) = p^3$ and $B_0^{(i)}(p) = 0$ for all $i = 1, 3, 4$.

Define $B^{(i)}(p;t) = \sum_{k\geq 0} B_k^{(i)}(p)t^k$, for all $i = 1, 2, 3, 4$. Thus, by multiplying (2.16)–(2.19) by t^k and summing over $k \geq 1$, we obtain

$$\begin{aligned}
&B^{(1)}(p;t) \\
&= \frac{p^4 t}{1-p^2 t} + p^4(1+t)B^{U(1)}(p;t) + p^2 B^{(1)}(p;t) + \frac{2p^2}{t}(B^{(1)}(p;t) - tB_1^{(1)}(p)) \\
&\quad + \frac{p^2}{t^2}(B^{(1)}(p;t) - tB_1^{(1)}(p) - t^2 B_2^{(1)}(p)) + p^6 t^2 B^{B(1)}(p;t) + p^4(1+t)B^{L(1)}(p;t) \\
&\quad + (p^2 t + p^4 t + p^4 t^2)B^{U(2)}(p;t) + 2p^2(B^{U(2)}(p;t) - p^3) \\
&\quad + \frac{p^2}{t}(B^{U(2)}(p;t) - tB_1^{U(2)}(p) - p^3) + p^4 t^2(1+p^2 t)B^{B(2)}(p;t) \\
&\quad + p^4 t(B^{B(2)}(p;t) - p^3) + p^4 B^{U(3)}(p;t) + \frac{p^2}{t}(B^{(3)}(p;t) - tB_1^{(3)}(p))
\end{aligned}$$

$$
\begin{aligned}
&+\frac{p^2}{t^2}(B^{(3)}(p;t)-t^2B_2^{(3)}(p)-tB_1^{(3)}(p))+(p^2+p^4t)B^{U(4)}(p;t)\\
&+\frac{p^2}{t}(B^{U(4)}(p;t)-tB_1^{U(4)}(p))-p^4tB_1^{U(1)}(p)-p^4tB_1^{L(1)}(p)-p^7t,\\
B^{(2)}(p;t)&=p^3+pB^{U(1)}(p;t),\\
B^{(3)}(p;t)&=\frac{1}{p}B^{(1)}(p;t),\\
B^{(4)}(p;t)&=B^{U(1)}(p;t).
\end{aligned}
$$

Thus, by the facts that $B^{(3)}(p;t)=\frac{1}{p}B^{(1)}(p;t)$, $B_1^{(3)}(p)=\frac{1}{p}B_1^{(1)}(p)$, and $B_2^{(3)}(p)=\frac{1}{p}B_2^{(1)}(p)$, we have

$$
\begin{aligned}
&\left(1-\frac{p(2p+1)}{t}-\frac{p(p+1)}{t^2}\right)B^{(1)}(p;t)\\
&=\frac{p^4t}{1-p^2t}+p^4(1+t)B^{U(1)}(p;t)+p^2B^{(1)}(p;t)-p(1+2p)B_1^{(1)}(p)\\
&\quad-\frac{p(p+1)}{t}(B_1^{(1)}(p)+tB_2^{(1)}(p))+p^6t^2B^{B(1)}(p;t)+p^4(1+t)B^{L(1)}(p;t)\\
&\quad+(p^2+p^4)tB^{U(2)}(p;t)+2p^2(B^{U(2)}(p;t)-p^3)-p^4tB_1^{U(1)}(p)-p^4tB_1^{L(1)}(p)\\
&\quad+\frac{p^2}{t}(B^{U(2)}(p;t)-tB_1^{U(2)}(p)-p^3)+p^4t^2B^{U(2)}(p;t)+p^4t^2(1+p^2t)B^{B(2)}(p;t)\\
&\quad+p^4t(B^{B(2)}(p;t)-p^3)+p^4B^{U(3)}(p;t)+p^2(1+p^2t)B^{U(4)}(p;t)\\
&\quad+\frac{p^2}{t}(B^{U(4)}(p;t)-tB_1^{U(4)}(p))-p^7t. \qquad (2.20)
\end{aligned}
$$

Let $K(p,t)=1-\frac{p(2p+1)}{t}-\frac{p(p+1)}{t^2}$ be the kernel of the above equation. Note that there are two roots of $K(p,t)=0$, namely,

$$
t_\pm=\frac{-2p^2-p\pm\sqrt{4p+5p^2}}{2(p^2-1)}.
$$

By substituting t_+,t_- into (2.20), we obtain a system of equations. By solving this system for $B_1^{(1)}(p)$ and $B_2^{(1)}(p)$, here we use Theorems 2.1, 2.2, and 2.3, we obtain

$$
\begin{aligned}
&B_1^{(1)}(p)\\
&=-\frac{p^2(p^{13}-4p^{11}+2p^{10}+5p^9-p^8-2p^7+2p^6+3p^5+p^4+p^2+2p+1)B_1^{U(1)}(p)}{(p^2-1)(p^{12}+p^{11}-3p^{10}-2p^9+p^8+4p^7+3p^6-2p^5-p^4+5p^3+5p^2-1)}\\
&\quad\frac{p^6(p+1)(p^5+p^4-p^3-p-1)(p^{14}-2p^{13}+4p^{12}-7p^{11}+4p^{10}-2p^9}{}\\
&-\frac{-5p^8+8p^7-2p^6+2p^5+5p^4-p^3+p^2-1)\sqrt{((p^2+1)^2-p^2)((p^2-1)^2-p^2)}}{2(p^{12}-p^{11}-2p^{10}+2p^9-2p^8+p^7+3p^5+2p^4-p^3+2p^2-1)}\\
&\qquad\cdot(2p^4+2p^2-1)(p^9+p^8-2p^7-2p^6-2p^5-p^4-2p^3-2p^2+p+1)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\begin{array}{l}p^4(9p^{48}+22p^{47}-27p^{46}-104p^{45}-108p^{44}-9p^{43}+345p^{42}+525p^{41}+399p^{40}\\
+24p^{39}-1110p^{38}-1607p^{37}-1725p^{36}-874p^{35}+1488p^{34}+3374p^{33}+5091p^{32}\\
+4586p^{31}+1733p^{30}-1426p^{29}-5797p^{28}-8990p^{27}-9717p^{26}-9137p^{25}\\
-6776p^{24}-4034p^{23}-1580p^{22}+936p^{21}+2375p^{20}+2451p^{19}+1651p^{18}+1086p^{17}\\
+790p^{16}-396p^{15}-1124p^{14}-263p^{13}+343p^{12}-163p^{11}-264p^{10}+314p^9\\
+218p^8-277p^7-138p^6+150p^5+69p^4-40p^3-19p^2+4p+2)\end{array}}{\begin{array}{l}2(p^6+p^5+2p^4+p^3+p^2-1)(p^9+p^8-2p^7-2p^6-2p^5-p^4-2p^3-2p^2+p+1)\\
\cdot(p^{12}+p^{11}-3p^{10}-2p^9+p^8+4p^7+3p^6-2p^5-p^4+5p^3+5p^2-1)(p^2-1)^2\\
\cdot(2p^4+2p^2-1)(p^{12}-p^{11}-2p^{10}+2p^9-2p^8+p^7+3p^5+2p^4-p^3+2p^2-1)\end{array}}\\
&=p^4+p^5+2p^6+6p^7+13p^8+32p^9+73p^{10}+\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&B_2^{(1)}(p)\\
&=-\frac{p(p^{12}+2p^{11}-3p^{10}-5p^9+p^8+6p^7-5p^5+5p^3+3p^2-p-1)B_1^{U(1)}(p)}{(p^{12}+p^{11}-3p^{10}-2p^9+p^8+4p^7+3p^6-2p^5-p^4+5p^3+5p^2-1)}\\
&+\frac{\begin{array}{l}p^7(p+1)^2(p^3-p+1)(p^{14}-2p^{13}+4p^{12}-7p^{11}+4p^{10}-2p^9-5p^8\\
+8p^7-2p^6+2p^5+5p^4-p^3+p^2-1)\sqrt{((p^2+1)^2-p^2)((p^2-1)^2-p^2)}\end{array}}{\begin{array}{l}(2p^{12}-p^{11}-2p^{10}+2p^9-2p^8+p^7+3p^5+2p^4-p^3+2p^2-1)(2p^4+2p^2-1)\\
\cdot(p^9+p^8-2p^7-2p^6-2p^5-p^4-2p^3-2p^2+p+1)\end{array}}\\
&-\frac{\begin{array}{l}p^5(5p^{45}+10p^{44}-18p^{43}-46p^{42}-46p^{41}-10p^{40}+190p^{39}+250p^{38}+187p^{37}\\
-26p^{36}-690p^{35}-695p^{34}-798p^{33}-295p^{32}+1078p^{31}+1656p^{30}+2478p^{29}\\
+1894p^{28}+262p^{27}-1058p^{26}-3218p^{25}-4207p^{24}-4342p^{23}-4202p^{22}\\
-2606p^{21}-1193p^{20}-55p^{19}+1401p^{18}+2288p^{17}+2319p^{16}+1456p^{15}\\
+620p^{14}+194p^{13}-679p^{12}-1148p^{11}-518p^{10}-42p^9-27p^8+154p^7\\
+275p^6+90p^5-76p^4-59p^3-3p^2+8p+2)\end{array}}{\begin{array}{l}2(p^2-1)(2p^4+2p^2-1)(p^{12}-p^{11}-2p^{10}+2p^9-2p^8+p^7+3p^5+2p^4-p^3+2p^2-1)\\
\cdot(p^6+p^5+2p^4+p^3+p^2-1)(p^9+p^8-2p^7-2p^6-2p^5-p^4-2p^3-2p^2+p+1)\\
\cdot(p^{12}+p^{11}-3p^{10}-2p^9+p^8+4p^7+3p^6-2p^5-p^4+5p^3+5p^2-1),\end{array}}\\
&=p^6+3p^7+6p^8+16p^9+37p^{10}+92p^{11}+222p^{12}+\cdots,
\end{aligned}
$$

where $B_1^{U(1)}(p)$ is given in (2.10). After we plug the expression of $B_1^{(1)}(p)$ and $B_2^{(1)}(p)$ and solve for $B^{(1)}(p;t)$, we obtain an explicit formula for $B^{(1)}(p;1)$ (we omit presenting an explicit formula because the expression is too long). Then by above we have that $B^{(2)}(p;1)=p^3+pB^{U(1)}(p;1)$, $B^{(3)}(p;1)=\frac{1}{p}B^{(1)}(p;1)$, and $B^{(4)}(p;1)=B^{U(1)}(p;1)$, where $B^{U(1)}(p;1)$ is given in Theorem 2.2. Hence, we have an explicit formula for $B(p)=1+\sum_{i=1}^{4}B^{(i)}(p;1)$, see Theorem 1.1.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Author contribution statement

The author confirms being the sole contributor of this work and has approved it for publication.

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