RAIRO-Theor. Inf. Appl. 59 (2025) 5
https://doi.org/10.1051/ita/2025005

RAIRO - Theoretical Informatics and Applications
www.rairo-ita.org

EXPLICIT VALUES OF A q-MULTIPLE ZETA-STAR FUNCTION AT ROOTS OF UNITY

Takao Komatsu[1,2,*]

Abstract. In this paper, we show some expressions of certain q-multiple zeta-star values at roots of unity. These explicit formulas are expressed by using the determinants or Bell polynomials. Explicit formulas for other types of values can be found from recurrence relations obtained using generating functions.

Mathematics Subject Classification. 11M32, 05A15, 05A19, 05A30, 11B37, 11B73.

Received February 5, 2025. Accepted July 11, 2025.

1. Introduction

In this paper, we consider the function

$$\mathfrak{Z}_n^\star(q;m,s):=\sum_{1\le i_1\le i_2\le\cdots\le i_m\le n-1}\frac{1}{(1-q^{i_1})^s(1-q^{i_2})^s\cdots(1-q^{i_m})^s}. \tag{1.1}$$

Then we show some fundamental ways to find the values, and give explicit expressions of the values of $\mathfrak{Z}_n^\star(\zeta_n;m,s)$ when m or s are comparatively small. When $n\to\infty$, the values in (1.1) may be called one of the q-multiple zeta-star values (see, *e.g.*, [1, 2]).

In previous papers ([3, 4]), as an application of the study of certain generalized Stirling numbers ([3, 5]), we considered a function

$$\mathfrak{Z}_n(q;m,s):=\sum_{1\le i_1< i_2<\cdots< i_m\le n-1}\frac{1}{(1-q^{i_1})^s(1-q^{i_2})^s\cdots(1-q^{i_m})^s}. \tag{1.2}$$

When $n\to\infty$, these values may be called one of the q-multiple zeta values (see, *e.g.*, [6–10]). Then when q takes the root of unity ζ_n in (1.2), some explicit expressions can be given:

$$\mathfrak{Z}_n(\zeta_n;m,1)=\frac{1}{m+1}\binom{n-1}{m}, \tag{1.3}$$

Keywords and phrases: Multiple zeta functions, q-Stirling numbers with higher level, complete homogeneous symmetric functions, Bell polynomials, determinant.

[1] Institute of Mathematics, Henan Academy of Sciences, Zhengzhou 450046 China.
[2] Department of Mathematics, Institute of Science Tokyo, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan.

[*] Corresponding author: komatsu@zstu.edu.cn; komatsu.t.al@m.titech.ac.jp

© The authors. Published by EDP Sciences, 2025

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

$$\mathfrak{Z}_n(\zeta_n; m, 2) = \frac{1}{n(m+1)}\left(\binom{n-1}{m} + (-1)^m\binom{n-1}{2m+1}\right) \tag{1.4}$$

$$= \frac{2\cdot m!}{(2m+2)!}\binom{n-1}{m}\sum_{k=0}^{m}\left[\!\!\left[\begin{matrix}2m+2\\ m+k+2\end{matrix}\right]\!\!\right]_1^{(m+1,1)}(-n)^k, \tag{1.5}$$

where $\left[\!\left[\begin{smallmatrix}n\\ k\end{smallmatrix}\right]\!\right]_1^{(r,1)}$ denotes the r-Stirling number ([11]). In [4], we also have

$$\mathfrak{Z}_n(\zeta_n; 1, s) = \left|\begin{matrix} \frac{n-1}{2} & 1 & 0 & \cdots & \\ \frac{2}{3}\binom{n-1}{2} & \frac{n-1}{2} & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ \frac{s-1}{s}\binom{n-1}{s-1} & \frac{1}{s-1}\binom{n-1}{s-2} & \cdots & \frac{n-1}{2} & 1 \\ \frac{s}{s+1}\binom{n-1}{s} & \frac{1}{s}\binom{n-1}{s-1} & \cdots & \frac{1}{3}\binom{n-1}{2} & \frac{n-1}{2} \end{matrix}\right|. \tag{1.6}$$

An expression of $\mathfrak{Z}_n(\zeta_n; m, s)$ $(s = 3, 4, \dots)$ would be possible, but it looks to become very complicated.

In addition, even though there is no direct expression of $\mathfrak{Z}_n(\zeta_n; m, s)$ $(m = 2, 3, \dots)$, its value may be calculated by using the known ones.

For example, when $m = 2$, we can obtain $\mathfrak{Z}_n(\zeta_n; 2, s)$ as

$$\mathfrak{Z}_n(\zeta_n; 2, s) = \frac{1}{2}\left(\mathfrak{Z}_n(\zeta_n; 1, s)^2 - \mathfrak{Z}_n(\zeta_n; 1, 2s)\right). \tag{1.7}$$

Similarly, for $m = 3$,

$$\mathfrak{Z}_n(\zeta_n; 3, s) = \frac{1}{6}\left(\mathfrak{Z}_n(\zeta_n; 1, s)^3 - 3\mathfrak{Z}_n(\zeta_n; 1, s)\mathfrak{Z}_n(\zeta_n; 1, 2s) + 2\mathfrak{Z}_n(\zeta_n; 1, 3s)\right). \tag{1.8}$$

It is noticed that, while many explicit formulas for the q-multiple zeta values are known, few explicit formulas for the q-multiple zeta-star values are known, even though the relationships involving the q-multiple zeta-star values are known (see, *e.g.*, [12–14]). One of the reasons is that it is more difficult to handle the complete homogeneous symmetric functions corresponding to the q-multiple zeta-star values than the elementary symmetric functions corresponding to the q-multiple zeta values.

2. q-generalized (r, s)-Stirling numbers

In [3], q-generalized (r, s)-Stirling numbers of both kinds are introduced. These Stirling numbers are closely related to our q-multiple zeta function in (1.2) and q-multiple zeta-star function in (1.1).

Let $[n]_q$ denote the q-number, defined by

$$[n]_q = \frac{q^n - 1}{q - 1} \quad (q \neq 1).$$

Its q-factorial is given by $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ with $[0]_q! = 1$. Let r be a positive integer. The q-version of r-Stirling numbers of the first kind with higher level (level s) are denoted by $\left[\!\left[\begin{smallmatrix}n\\ k\end{smallmatrix}\right]\!\right]_q^{(r,s)}$, and appear in the coefficients in the expansion

$$(x)_{n,q}^{(r,s)} = \sum_{k=0}^{n}(-1)^{n-k}\left[\!\!\left[\begin{matrix}n\\ k\end{matrix}\right]\!\!\right]_q^{(r,s)} x^k, \tag{2.1}$$

where for $r, s \ge 1$, $(x)_{n,q}^{(r,s)}$ is defined by

$$(x)_{n,q}^{(r,s)} := x^r \prod_{i=r}^{n-1} \left(x - ([i]_q)^s\right) \quad (n > r)$$

with $(x)_{r,q}^{(r,s)} = x^r$. When $r = s = 1$, $s_q(n,k) = (-1)^{n-k} \left[\!\left[{n \atop k} \right]\!\right]_q^{(1,1)}$ are the signed q-Stirling numbers of the first kind (see, *e.g.*, [15]), and $\left[\!\left[{n \atop k} \right]\!\right]_q = \left[\!\left[{n \atop k} \right]\!\right]_q^{(1,1)}$ are the unsigned q-Stirling numbers of the first kind. When $r = 1$ and $q \to 1$, $\left[\!\left[{n \atop k} \right]\!\right]^{(s)} = \left[\!\left[{n \atop k} \right]\!\right]_1^{(1,s)}$ are the (unsigned) Stirling numbers of the first kind with higher level ([16]). When $r = s = 1$ and $q \to 1$, $\left[{n \atop k} \right] = \left[\!\left[{n \atop k} \right]\!\right]_1^{(1,1)}$ are the unsigned Stirling numbers of the first kind.

The q-version of r-Stirling numbers of the second kind with higher level are denoted by $\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)}$, and appear in the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} (x)_{k,q}^{(r,s)}. \tag{2.2}$$

When $r = 1$ and $q \to 1$, $\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}^{(s)} = \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_1^{(1,s)}$ are the Stirling numbers of the second kind with higher level, studied in [17]. When $r = s = 1$, $S_q(n,k) = \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q = (-1)^{n-k} \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(1,1)}$ are the signed q-Stirling numbers of the second kind (see, *e.g.*, [15]). When $r = s = 1$ and $q \to 1$, $\left\{ {n \atop k} \right\} = \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_1^{(1,1)}$ are the classical Stirling numbers of the second kind.

From the definitions in (2.1) and (2.2), the following recurrence relations hold:

$$\left[\!\left[{n \atop k} \right]\!\right]_q^{(r,s)} = \left[\!\left[{n-1 \atop k-1} \right]\!\right]_q^{(r,s)} + \left([n-1]_q\right)^s \left[\!\left[{n-1 \atop k} \right]\!\right]_q^{(r,s)} \tag{2.3}$$

with

$$\begin{aligned} &\left[\!\left[{n \atop k} \right]\!\right]_q^{(r,s)} = 0 \quad (0 \le k \le r,\ n \ge k), \\ &\left[\!\left[{n \atop k} \right]\!\right]_q^{(r,s)} = 0 \quad (n < k), \qquad \left[\!\left[{0 \atop 0} \right]\!\right]_q^{(r,s)} = 1\,; \end{aligned}$$

$$\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = \left\{\!\!\left\{ {n-1 \atop k-1} \right\}\!\!\right\}_q^{(r,s)} + \left([k]_q\right)^s \left\{\!\!\left\{ {n-1 \atop k} \right\}\!\!\right\}_q^{(r,s)} \tag{2.4}$$

with

$$\begin{aligned} &\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = 0 \quad (0 \le k \le r-1,\ n \ge k), \qquad \left\{\!\!\left\{ {0 \atop 0} \right\}\!\!\right\}_q^{(r,s)} = 1, \\ &\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = 0 \quad (n \le k). \end{aligned}$$

By the definitions and/or the recurrence relations, we can get expressions with combinatorial summations ([3]).

Lemma 2.1. *For $r \le m \le n-1$ and $r \ge 1$, we have*

$$\left[\!\!\left[{n \atop m} \right]\!\!\right]_q^{(r,s)} = \left(\frac{[n-1]_q!}{[r-1]_q!}\right)^s \sum_{r \le i_1 < \cdots < i_{m-r} \le n-1} \frac{1}{\big([i_1]_q \dots [i_{m-r}]_q\big)^s}.$$

For $n-m \ge r$ and $r \ge 1$, we have

$$\begin{aligned}
\left[\!\!\left[{n \atop n-m} \right]\!\!\right]_q^{(r,s)} &= \sum_{r \le i_1 < \cdots < i_m \le n-1} \big([i_1]_q \dots [i_m]_q\big)^s \\
&= \sum_{r \le i_1 \le i_2 \le \cdots \le i_m \le n-m} ([i_1]_q [i_2+1]_q \cdots [i_m+m-1]_q)^s \\
&= \sum_{i_m=r}^{n-m} ([i_m+m-1]_q)^s \sum_{i_{m-1}=r}^{i_m} ([i_{m-1}+m-2]_q)^s \cdots \sum_{i_2=r}^{i_3} ([i_2+1]_q)^s \sum_{i_1=r}^{i_2} ([i_1]_q)^s.
\end{aligned}$$

Lemma 2.2. *For $r+1 \le k \le n$ and $r \ge 1$,*

$$\begin{aligned}
\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = \sum_{i_{k-r}=0}^{n-k} \big([k]_q\big)^{(n-k-i_{k-r})s} \sum_{i_{k-r-1}=0}^{i_{k-r}} \big([k-1]_q\big)^{(i_{k-r}-i_{k-r-1})s} \\
\cdots \sum_{i_2=0}^{i_3} \big([r+2]_q\big)^{(i_3-i_2)s} \sum_{i_1=0}^{i_2} \big([r+1]_q\big)^{(i_2-i_1)s} \big([r]_q\big)^{i_1 s}.
\end{aligned}$$

For $n-k \ge r \ge 1$,

$$\left\{\!\!\left\{ {n \atop n-k} \right\}\!\!\right\}_q^{(r,s)} = \sum_{r \le i_1 \le i_2 \le \cdots \le i_k \le n-k} ([i_1]_q [i_2]_q \cdots [i_k]_q)^s.$$

From Lemma 2.1 and Lemma 2.2, q-multiple zeta functions and q-multiple zeta-star functions can be written in terms of q-generalized $(1,s)$-Stirling numbers of the first kind and of the second kind, respectively.

Proposition 2.3. *For positive integers n, m and s, we have*

$$\begin{aligned}
\mathfrak{z}_n(q;m,s) &= \frac{1}{(1-q)^{ms}\big([n-1]_q!\big)^s} \left[\!\!\left[{n \atop m+1} \right]\!\!\right]_q^{(1,s)}, \\
\mathfrak{z}_n^\star(q;m,s) &= \frac{1}{(1-q)^{ms}} \left\{\!\!\left\{ {n+m-1 \atop n-1} \right\}\!\!\right\}_q^{(1,-s)}.
\end{aligned}$$

3. The main result

Note that $\mathfrak{z}_n^\star(q;1,s) = \mathfrak{z}_n(q;1,s)$. Hence, one can have interest in the case where $m \ge 2$. Then, in light of the result of $\mathfrak{z}_n(q;m,s)$, when q takes the root of unity, we can get the following expression using a determinant or a Bell polynomial.

Theorem 3.1. *For integers n, m with $n \ge 2$ and $m \ge 1$, we have*

$$\begin{aligned}
&\mathfrak{z}_n^\star(\zeta_n; m, s)\\
&= \frac{1}{m!}\mathbf{Y}_m\big(\mathfrak{z}_n(\zeta_n;1,s), 1!\mathfrak{z}_n(\zeta_n;1,2s), 2!\mathfrak{z}_n(\zeta_n;1,3s), 3!\mathfrak{z}_n(\zeta_n;1,4s),\dots\big)\\
&= \frac{1}{m!}\begin{vmatrix}
\mathfrak{z}_n(\zeta_n;1,s) & -1 & 0 & \cdots & \\
\mathfrak{z}_n(\zeta_n;1,2s) & \mathfrak{z}_n(\zeta_n;1,s) & -2 & & \vdots \\
\vdots & & \ddots & & 0 \\
\mathfrak{z}_n\big(\zeta_n;1,(m-1)s\big) & \mathfrak{z}_n\big(\zeta_n;1,(m-2)s\big) & \cdots & \mathfrak{z}_n(\zeta_n;1,s) & -m+1 \\
\mathfrak{z}_n(\zeta_n;1,ms) & \mathfrak{z}_n\big(\zeta_n;1,(m-1)s\big) & \cdots & \mathfrak{z}_n(\zeta_n;1,2s) & \mathfrak{z}_n(\zeta_n;1,s)
\end{vmatrix}.
\end{aligned}$$

For example, when $m = 2$, we have

$$\mathfrak{z}_n^\star(\zeta_n;2,s) = \frac{1}{2}\left(\mathfrak{z}_n(\zeta_n;1,s)^2 + \mathfrak{z}_n(\zeta_n;1,2s)\right). \tag{3.1}$$

Another way is

$$\mathfrak{z}_n^\star(\zeta_n;2,s) = \mathfrak{z}_n(\zeta_n;2,s) + \mathfrak{z}_n(\zeta_n;1,2s). \tag{3.2}$$

Then for $s = 1, 2, 3$ we have

$$\begin{aligned}
\mathfrak{z}_n^\star(\zeta_n;2,1) &= \frac{(n+1)(n-1)}{12},\\
\mathfrak{z}_n^\star(\zeta_n;2,2) &= \frac{(n+1)(n-1)(n-3)(n-7)}{2\cdot 5!},\\
\mathfrak{z}_n^\star(\zeta_n;2,3) &= -\frac{(n+1)(n-1)(n^4-650n^2+3780n-5291)}{12\cdot 7!}.
\end{aligned}$$

For example, when $m = 2$ and $s = 3$, by using the known values of $\mathfrak{z}_n(\zeta_n;2,3)$ (or $\mathfrak{z}_n(\zeta_n;1,3)$) and $\mathfrak{z}_n(\zeta_n;1,6)$, we have

$$\begin{aligned}
&\mathfrak{z}_n^\star(\zeta_n;2,3) = \frac{1}{2}\left(\mathfrak{z}_n(\zeta_n;1,3)^2 + \mathfrak{z}_n(\zeta_n;1,6)\right)\\
&= \frac{1}{2}\left(\frac{(n-1)^2(n-3)^2}{8^2} - \frac{(n-1)(2n^5+2n^4-355n^3-355n^2+11153n-19087)}{12\cdot 7!}\right)\\
&= -\frac{(n+1)(n-1)(n^4-650n^2+3780n-5291)}{12\cdot 7!}
\end{aligned}$$

or

$$\begin{aligned}
\mathfrak{z}_n^\star(\zeta_n;2,3) &= \mathfrak{z}_n(\zeta_n;2,3) + \mathfrak{z}_n(\zeta_n;1,6)\\
&= \frac{6(n-1)(n-2)(n^4+3n^3+301n^2-2883n+6898)}{9!}\\
&\quad - \frac{(n-1)(2n^5+2n^4-355n^3-355n^2+11153n-19087)}{12\cdot 7!}\\
&= -\frac{(n+1)(n-1)(n^4-650n^2+3780n-5291)}{12\cdot 7!}.
\end{aligned}$$

In order to prove Theorem 3.1, we need the following Lemmas ([18], Prop. 1, [19], Ch.I, §2). Notice that in the case of (1.2), the equivalent of elementary symmetric functions arises, whereas in the case of (1.1), the equivalent of complete homogeneous symmetric functions arises.

Lemma 3.2. *Let n, M and K be positive integers with $M \ge K$. For $g_n := a_1^n + a_2^n + \cdots + a_M^n$ we have*

$$\sum_{1\le j_1\le j_2\le\cdots\le j_K\le M} a_{j_1}a_{j_2}\cdots a_{j_K} = \frac{1}{K!}\mathbf{Y}_K(g_1, 1!g_2, 2!g_3, 3!g_4, \dots),$$

where $\mathbf{Y}_n(x_1, x_2, \dots, x_n)$ is the (complete exponential) Bell polynomial, defined by

$$\exp\left(\sum_{m=1}^\infty x_m\frac{t^m}{m!}\right) = 1 + \sum_{n=1}^\infty \mathbf{Y}_n(x_1, x_2, \dots, x_n)\frac{t^n}{n!}.$$

That is,

$$\begin{aligned}&\mathbf{Y}_n(x_1, x_2, x_3, \dots, x_n)\\ &=\sum_{k=1}^n \sum_{\substack{i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n\\ i_1+i_2+i_3+\cdots=k}} \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{i_1}\left(\frac{x_2}{2!}\right)^{i_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{i_{n-k+1}}\end{aligned}$$

with $\mathbf{Y}_0 = 1$.

Lemma 3.3. *For two sequences $\{a_n\}_{n\ge 0}$ and $\{b_m\}_{m\ge 0}$ with $a_0 = b_0 = 1$, the following expressions are equivalent for any positive integers m and n.*

(i) $b_m = \displaystyle\sum_{\substack{i_1+2i_2+\cdots+mi_m=m\\ i_1,i_2,\dots,i_m\ge 0}} \frac{1}{i_1!i_2!\cdots i_m!}\left(\frac{a_1}{1}\right)^{i_1}\left(\frac{a_2}{2}\right)^{i_2}\cdots\left(\frac{a_m}{m}\right)^{i_m}$

(ii) $b_m = \dfrac{1}{m!}\begin{vmatrix} a_1 & -1 & 0 & \cdots & 0\\ a_2 & a_1 & -2 & & \vdots\\ \vdots & & \ddots & & 0\\ a_{m-1} & a_{m-2} & \cdots & a_1 & -m+1\\ a_m & a_{m-1} & \cdots & a_2 & a_1\end{vmatrix}$

(iii) $a_n = (-1)^{n-1}\begin{vmatrix} b_1 & 1 & 0 & \cdots & 0\\ 2b_2 & b_1 & 1 & & \vdots\\ \vdots & & \ddots & & 0\\ (n-1)b_{n-1} & b_{n-2} & \cdots & b_1 & 1\\ nb_n & b_{n-1} & \cdots & b_2 & b_1\end{vmatrix}$

(iv) $mb_m = \displaystyle\sum_{i=1}^m a_i b_{m-i}$

(v) $a_n = -\displaystyle\sum_{j=1}^{n-1} b_j a_{n-j} + nb_n$

Remark 3.4. The relations between (i) and (ii) are yielded from a simple modification of Trudi's formula ([20], [21], Vol. 3, p. 214). By applying the inversion formula (see, *e.g.*, [22], Lem. 1, [23], Thm. 1), we can find that (iv) and (v) are equivalent. Expanding the determinants (ii) and (iii) yield the relations (iv) and (v), respectively.

Proof of Theorem 3.1. Put $b_m = \mathfrak{z}_n^\star(\zeta_n; m, s)$ and $a_j = \mathfrak{z}_n(\zeta_n; 1, js)$ in Lemma 3.3 (1) and (2). By applying Lemma 3.2, we get the desired result. $\square$

4. The values $\mathfrak{z}_n^\star(\zeta_n; m, s)$ for small s

In the previous section, we considered the values $\mathfrak{z}_n^\star(\zeta_n; m, s)$ for small m. In this section we shall consider the values $\mathfrak{z}_n^\star(\zeta_n; m, s)$ for small s.

$\mathfrak{z}_n(\zeta_n; m, 1)$ is expressed in a simple form as in (1.3), but $\mathfrak{z}_n^\star(\zeta_n; m, 1)$ is not. The following is obtained by repeating the Theorem 4.1 described later.

$$\begin{aligned}
\mathfrak{z}_n^\star(\zeta_n; 1, 1) &= \frac{n-1}{2},\\
\mathfrak{z}_n^\star(\zeta_n; 2, 1) &= \frac{(n+1)(n-1)}{12},\\
\mathfrak{z}_n^\star(\zeta_n; 3, 1) &= \frac{(n+1)(n-1)}{24},\\
\mathfrak{z}_n^\star(\zeta_n; 4, 1) &= -\frac{(n+1)(n-1)(n^2-19)}{720},\\
\mathfrak{z}_n^\star(\zeta_n; 5, 1) &= -\frac{(n+1)(n-1)(n+3)(n-3)}{480},\\
\mathfrak{z}_n^\star(\zeta_n; 6, 1) &= \frac{(n+1)(n-1)(2n^4-145n^2+863)}{60480},\\
\mathfrak{z}_n^\star(\zeta_n; 7, 1) &= \frac{(n+1)(n-1)(n+5)(n-5)(2n^2-11)}{24192},\\
\mathfrak{z}_n^\star(\zeta_n; 8, 1) &= -\frac{(n+1)(n-1)(3n^6-497n^4+9247n^2-33953)}{3628800},\\
\mathfrak{z}_n^\star(\zeta_n; 9, 1) &= -\frac{(n+1)(n-1)(n+7)(n-7)(3n^4-50n^2+167)}{1036800}.
\end{aligned}$$

As $\mathfrak{z}_n^\star(\zeta_n; m, s)$ is a complete homogeneous symmetric polynomial, let us consider the generating function:

$$\begin{aligned}
\sum_{m=0}^{n-1} \mathfrak{z}_n^\star(\zeta_n; m, s) X^m &= \sum_{m=0}^{\infty} \mathfrak{z}_n^\star(\zeta_n; m, s) X^m\\
&= \sum_{m=0}^{\infty} \sum_{1\le i_1\le i_2\le\cdots\le i_m\le n-1} \frac{X^m}{(1-\zeta_n^{i_1})^s(1-\zeta_n^{i_2})^s\cdots(1-\zeta_n^{i_m})^s}\\
&= \prod_{j=1}^{n-1} \frac{1}{1-\dfrac{X}{(1-\zeta_n^j)^s}} = n^s \prod_{j=1}^{n-1} \frac{1}{(1-\zeta_n^j)^s - X}.
\end{aligned}$$

Let α_i $(i = 1, 2, \ldots, s)$ be the roots of the polynomial $(1-Y)^s - X$. Namely,

$$\prod_{i=1}^{s} (\alpha_i - Y) = (1-Y)^s - X. \tag{4.1}$$

Since by (4.1)

$$\prod_{i=1}^{s}(\alpha_i - 1) = -X,$$

we get

$$\prod_{j=1}^{n-1}\prod_{i=1}^{s}(\alpha_i - \zeta_n^j) = \prod_{i=1}^{s}\frac{\alpha_i^n - 1}{\alpha_i - 1} = \frac{-1}{X}\prod_{i=1}^{s}(\alpha_i^n - 1).$$

Hence,

$$\sum_{m=0}^{n-1}\mathfrak{Z}_n^\star(\zeta_n; m, s)X^m = -Xn^s\prod_{i=1}^{s}\frac{1}{\alpha_i^n - 1}. \tag{4.2}$$

4.1. The values $\mathfrak{Z}_n^\star(\zeta_n; m, 1)$

Let us consider the value $\mathfrak{Z}_n^\star(\zeta_n; m, s)$ when $s = 1$. When $s = 1, 2$, it is simpler and faster to calculate the values directly, though there is a general method that is described in Section 4.3.

When $s = 1$, by $\alpha_1 - Y = (1 - Y) - X$ in (4.1), we see that $\alpha_1 = 1 - X$. Thus, by (4.2) we have

$$\begin{aligned}
\sum_{m=0}^{n-1}\mathfrak{Z}_n^\star(\zeta_n; m, 1)X^m &= n\prod_{j=1}^{n-1}\frac{1}{(1 - \zeta_n^j) - X}\\
&= \frac{-Xn}{\alpha_1^n - 1} = \frac{Xn}{1 - (1 - X)^n}\\
&= \frac{n}{\sum_{\kappa=0}^{n-1}(1 - X)^\kappa} = \frac{n}{\sum_{k=0}^{n-1}\binom{n}{k+1}(-1)^k X^k}\\
&= \frac{1}{\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{k+1}X^k} = \frac{1}{1 - \sum_{k=1}^{n-1}\binom{n-1}{k}\frac{(-1)^{k+1}}{k+1}X^k}
\end{aligned}$$

Hence,

$$\begin{aligned}
1 &= \left(\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{(-1)^k}{k+1}X^k\right)\left(\sum_{m=0}^{n-1}\mathfrak{Z}_n^\star(\zeta_n; m, 1)X^m\right)\\
&= \sum_{l=0}^{n-1}\left(\sum_{j=0}^{l}\frac{(-1)^{l-j}}{l-j+1}\binom{n-1}{l-j}\mathfrak{Z}_n^\star(\zeta_n; j, 1)\right)X^l\\
&\quad + \sum_{l=n}^{2n-2}\left(\sum_{j=l-n+1}^{n-1}\frac{(-1)^{l-j}}{l-j+1}\binom{n-1}{l-j}\mathfrak{Z}_n^\star(\zeta_n; j, 1)\right)X^l.
\end{aligned}$$

Comparing the coefficients on both sides, we have the recurrence formula when $s = 1$. By applying this formula repeatedly, we can get explicit expressions of $\mathfrak{Z}_n^\star(\zeta_n; m, 1)$ $(m = 1, 2, \ldots, 9)$ as seen above.

Theorem 4.1. *For $1 \leq m \leq n-1$, we have*

$$\mathfrak{z}_n^\star(\zeta_n; m, 1) = \sum_{j=0}^{m-1} \frac{(-1)^{m-j+1}}{m-j+1} \binom{n-1}{m-j} \mathfrak{z}_n^\star(\zeta_n; j, 1)$$

with $\mathfrak{z}_n^\star(\zeta_n; 0, 1) = 1$.

4.2. The values $\mathfrak{z}_n^\star(\zeta_n; m, 2)$

Let us consider the value $\mathfrak{z}_n^\star(\zeta_n; m, s)$ when $s=2$. When $s=2$, by (4.1), we see that $\alpha_1+\alpha_2=2$ and $\alpha_1\alpha_2 = 1-X$. Thus, by (4.2) we have

$$\begin{aligned}
\sum_{m=0}^{n-1} \mathfrak{z}_n^\star(\zeta_n; m, 2) X^m &= n^2 \prod_{j=1}^{n-1} \frac{1}{(1-\zeta_n^j)^2 - X} \\
&= \frac{-Xn^2}{(\alpha_1^n - 1)(\alpha_2^n - 1)} = \frac{-Xn^2}{1 - \left((1+\sqrt{X})^n + (1-\sqrt{X})^n\right) + (1-X)^n} \\
&= \frac{-Xn^2}{1 - 2\sum_{\kappa=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\kappa} X^\kappa + \sum_{\kappa=0}^{n} \binom{n}{\kappa} (-X)^\kappa} \\
&= \frac{-n^2}{-2\sum_{\kappa=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\kappa} X^{\kappa-1} + \sum_{\kappa=1}^{n} \binom{n}{\kappa} (-1)^\kappa X^{\kappa-1}} \\
&= \frac{n^2}{2\sum_{\kappa=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n}{2\kappa+2} X^\kappa + \sum_{\kappa=0}^{n-1} \binom{n}{\kappa+1} (-1)^\kappa X^\kappa}.
\end{aligned}$$

Hence,

$$\begin{aligned}
1 &= \left(\frac{2}{n^2} \sum_{\kappa=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n}{2\kappa+2} X^\kappa \right) \left(\sum_{m=0}^{n-1} \mathfrak{z}_n^\star(\zeta_n; m, 2) X^m \right) \\
&\quad + \left(\frac{1}{n^2} \sum_{\kappa=0}^{n-1} \binom{n}{\kappa+1} (-1)^\kappa X^\kappa \right) \left(\sum_{m=0}^{n-1} \mathfrak{z}_n^\star(\zeta_n; m, 2) X^m \right) \\
&= \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\sum_{j=0}^{l} \frac{2}{n^2} \binom{n}{2l-2j+2} \mathfrak{z}_n^\star(\zeta_n; j, 2) \right) X^l \\
&\quad + \sum_{l=0}^{n-1} \left(\sum_{j=0}^{l} \frac{(-1)^{l-j}}{n^2} \binom{n}{l-j+1} \mathfrak{z}_n^\star(\zeta_n; j, 2) \right) X^l \\
&\quad + R(n, l),
\end{aligned}$$

where $R(n,l)$ implies the terms for $l \geq \lfloor \frac{n}{2} \rfloor$ and for $l \geq n$, respectively. Comparing the coefficients on both sides, we have for $1 \leq l \leq \lfloor \frac{n}{2} \rfloor - 1$

$$\sum_{j=0}^{l} \frac{1}{n^2} \left(2\binom{n}{2l-2j+2} + (-1)^{l-j} \binom{n}{l-j+1} \right) \mathfrak{z}_n^\star(\zeta_n; j, 2) = 0.$$

Hence, we have the following.

Theorem 4.2. *For $1 \le m \le \lfloor \frac{n}{2} \rfloor - 1$, we have*

$$\mathfrak{Z}_n^\star(\zeta_n; m, 2) = -\sum_{j=0}^{m-1} \frac{1}{n^2}\left(2\binom{n}{2m-2j+2} + (-1)^{m-j}\binom{n}{m-j+1}\right)\mathfrak{Z}_n^\star(\zeta_n; j, 2)$$

with $\mathfrak{Z}_n^\star(\zeta_n; 0, 2) = 1$.

By applying this formula repeatedly, we can get explicit expressions of $\mathfrak{Z}_n^\star(\zeta_n; m, 2)$ $(m = 1, 2, 3, 4, 5)$ as follows.

$$\begin{aligned}
\mathfrak{Z}_n^\star(\zeta_n; 1, 2) &= -\frac{(n-1)(n-5)}{12},\\
\mathfrak{Z}_n^\star(\zeta_n; 2, 2) &= \frac{(n+1)(n-1)(n-3)(n-7)}{240},\\
\mathfrak{Z}_n^\star(\zeta_n; 3, 2) &= -\frac{(n+1)(n-1)(10n^4 - 126n^3 + 241n^2 + 1134n - 2699)}{12 \cdot 7!},\\
\mathfrak{Z}_n^\star(\zeta_n; 4, 2) &= \frac{(n+1)(n-1)(21n^6 - 300n^5 + 181n^4 + 9150n^3 - 21071n^2 - 41250n + 103669)}{10!}\\
\mathfrak{Z}_n^\star(\zeta_n; 5, 2) &= -\frac{(n+1)(n-1)(n-3)}{4 \cdot 11!} \times (30n^7 - 372n^6 - 1889n^5\\
&\quad + 24671n^4 + 18790n^3 - 346648n^2 - 57251n + 1088429).
\end{aligned}$$

4.3. The cases for $s \ge 3$

It is not easy to find an explicit expression of $\mathfrak{Z}_n^\star(\zeta_n; m, s)$ when $s \ge 3$ though it is possible theoretically.

The following method was hinted by Henrik Bachmann directly when the author visited his university in 2024. He kindly provided me his papers [12, 13]. In particular, [13], Theorem 1.1 is useful to achieve the result. In order to calculate $\prod_{i=1}^s (\alpha_i^n - 1)^{-1}$, we consider the function

$$\begin{aligned}
f(X, Y) &:= \sum_{n=1}^\infty \frac{Y^n}{n} \prod_{i=1}^s (\alpha_i^n - 1)\\
&= \sum_{n=1}^\infty \frac{Y^n}{n} \sum_{l=0}^s (-1)^{s-l} \sum_{1 \le i_1 < \cdots < i_l \le s} (\alpha_{i_1} \cdots \alpha_{i_l})^n\\
&= (-1)^{s-1} \sum_{l=0}^s (-1)^{l-1} \sum_{1 \le i_1 < \cdots < i_l \le s} \sum_{n=1}^\infty \frac{(\alpha_{i_1} \cdots \alpha_{i_l} Y)^n}{n}\\
&= (-1)^{s-1} \sum_{l=0}^s (-1)^l \sum_{1 \le i_1 < \cdots < i_l \le s} \log(1 - \alpha_{i_1} \cdots \alpha_{i_l} Y)\\
&= (-1)^{s-1} \log\left(\prod_{l=0}^s F_{s,l}(X, Y)^{(-1)^l}\right),
\end{aligned}$$

where

$$F_{s,l}(X,Y) := \prod_{1\le i_1<\cdots<i_l\le s}(1-\alpha_{i_1}\cdots\alpha_{i_l}Y) \quad (1\le l\le s) \tag{4.3}$$

with $F_{s,0}(X,Y)=1-Y$. Here, $F_{s,l}(X,Y)$'s include the elementary symmetric functions but can be determined from (4.1) as the complete homogeneous symmetric functions.

For example, we can find that

$$\begin{aligned}
F_{1,1}(X,Y) &= 1-Y+XY,\\
F_{2,1}(X,Y) &= (1-Y)^2-XY^2,\\
F_{2,2}(X,Y) &= 1-Y+XY,\\
F_{3,1}(X,Y) &= (1-Y)^3+XY^3,\\
F_{3,2}(X,Y) &= (1-Y)^3-3XY^2-(X^2-2X)Y^3,\\
F_{3,3}(X,Y) &= 1-Y+XY.
\end{aligned}$$

Though it becomes more complicated, using $F_{1,1}(X,Y)$, or $F_{2,1}(X,Y)$ and $F_{2,2}(X,Y)$, we can find the expression $\prod_{i=1}^{s}(\alpha_i^n-1)$ for $s=1$ or $s=2$, respectively.

Let us find the expression $\prod_{i=1}^{s}(\alpha_i^n-1)$ for $s=3$ by using $F_{3,1}(X,Y)$, $F_{3,2}(X,Y)$ and $F_{3,3}(X,Y)$.

$$\begin{aligned}
f(X,Y) &= (-1)^{3-1}\log\left(\prod_{l=0}^{3}F_{3,l}(X,Y)^{(-1)^l}\right)\\
&= \log\frac{(1-Y)\big((1-Y)^3-3XY^2-(X^2-2X)Y^3\big)}{\big((1-Y)^3+XY^3\big)(1-Y+XY)}\\
&= \log\left(1-\frac{3XY^2+(X^2-2X)Y^3}{(1-Y)^3}\right)-\log\left(1+\frac{XY^3}{(1-Y)^3}\right)-\log\left(1+\frac{XY}{1-Y}\right)\\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3XY^2+(X^2-2X)Y^3}{(1-Y)^3}\right)^n-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{XY^3}{(1-Y)^3}\right)^n-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{XY}{1-Y}\right)^n\\
&= -\sum_{n=1}^{\infty}\frac{X^nY^{2n}\big(3+(X-2)Y\big)^n}{n}\sum_{j=0}^{\infty}\binom{3n+j-1}{3n-1}Y^j\\
&\quad+\sum_{n=1}^{\infty}\frac{(-1)^nX^nY^{3n}}{n}\sum_{j=0}^{\infty}\binom{3n+j-1}{3n-1}Y^j+\sum_{n=1}^{\infty}\frac{(-1)^nX^nY^n}{n}\sum_{j=0}^{\infty}\binom{n+j-1}{n-1}Y^j\\
&= -\sum_{n=1}^{\infty}\frac{X^nY^{2n}}{n}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}(X-2)^kY^k\sum_{j=0}^{\infty}\binom{3n+j-1}{3n-1}Y^j\\
&\quad+\sum_{n=1}^{\infty}\frac{(-1)^nX^n}{n}\sum_{\nu=3n}^{\infty}\binom{\nu-1}{3n-1}Y^\nu+\sum_{n=1}^{\infty}\frac{(-1)^nX^n}{n}\sum_{\nu=n}^{\infty}\binom{\nu-1}{n-1}Y^\nu\\
&= -\sum_{\nu=2}^{\infty}Y^\nu\sum_{\mu=1}^{\infty}\sum_{k=0}^{\mu}\frac{X^\mu}{\mu}\binom{\mu}{k}3^{\mu-k}\sum_{l=0}^{k}\binom{k}{l}X^j(-2)^{k-l}\binom{\nu+\mu-k-1}{3\mu-1}\\
&\quad+\sum_{\nu=3}^{\infty}Y^\nu\sum_{m=1}^{\lfloor\frac{\nu}{3}\rfloor}\binom{\nu-1}{3m-1}\frac{(-1)^mX^m}{m}+\sum_{\nu=1}^{\infty}Y^\nu\sum_{m=1}^{\nu}\binom{\nu-1}{m-1}\frac{(-1)^mX^m}{m}
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{\nu=2}^{\infty} Y^{\nu} \sum_{m=1}^{\infty} X^m \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{k=l}^{m-l} \frac{3^{m-l-k}(-2)^{k-l}}{m-l} \binom{m-l}{k}\binom{k}{l}\binom{\nu+m-l-k-1}{3m-3l-1} \\
&\quad + \sum_{n=3}^{\infty} Y^n \sum_{m=1}^{\lfloor \frac{n}{3} \rfloor} \binom{n-1}{3m-1} \frac{(-1)^m X^m}{m} + \sum_{n=1}^{\infty} Y^n \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{(-1)^m X^m}{m}.
\end{aligned}
$$

Comparing the coefficients, we have

$$
\begin{aligned}
&\prod_{i=1}^{3}(\alpha_i^n - 1) \\
&= n \sum_{m=0}^{\lfloor \frac{n}{3} \rfloor - 1} \binom{n-1}{3m+2} \frac{(-1)^{m+1} X^{m+1}}{m+1} + n \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{(-1)^{m+1} X^{m+1}}{m+1} \\
&\quad - n \sum_{m=0}^{\infty} X^{m+1} \sum_{l=0}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=l}^{m-l+1} \frac{3^{m-l-k+1}(-2)^{k-l}}{m-l+1} \binom{m-l+1}{k}\binom{k}{j}\binom{n+m-l-k}{3m-3l+2}.
\end{aligned}
$$

Thus, by (4.2) we have

$$
\begin{aligned}
n^2 &= \left(\sum_{m=0}^{n-1} \mathfrak{z}_n^\star(\zeta_n; m, 3) X^m \right) \left(\sum_{m=0}^{\lfloor \frac{n}{3} \rfloor - 1} \binom{n-1}{3m+2} \frac{(-1)^m X^m}{m+1} + \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{(-1)^m X^m}{m+1} \right. \\
&\quad \left. + \sum_{m=0}^{\infty} X^m \sum_{l=0}^{\lfloor \frac{m+1}{2} \rfloor} \sum_{k=l}^{m-l+1} \frac{3^{m-l-k+1}(-2)^{k-l}}{m-l+1} \binom{m-l+1}{k}\binom{k}{l}\binom{n+m-l-k}{3m-3l+2} \right).
\end{aligned}
$$

Hence, we have the following.

Theorem 4.3. *For $1 \le m \le \lfloor \frac{n}{3} \rfloor - 1$, we have*

$$
\begin{aligned}
&\mathfrak{z}_n^\star(\zeta_n; m, 3) \\
&= -\sum_{j=0}^{m-1} \frac{1}{n^2} \left(\sum_{l=0}^{\lfloor \frac{m-j+1}{2} \rfloor} \sum_{k=l}^{m-j-l+1} \frac{3^{m-j-l-k+1}(-2)^{k-l}}{m-j-l+1} \binom{m-j-l+1}{k}\binom{k}{l}\binom{n+m-j-l-k}{3m-3j-3l+2} \right. \\
&\quad \left. + \left(\binom{n-1}{3m-3j+2} + \binom{n-1}{m-j} \right) \frac{(-1)^{m-j}}{m-j+1} \right) \mathfrak{z}_n^\star(\zeta_n; j, 3)
\end{aligned}
$$

with $\mathfrak{z}_n^\star(\zeta_n; 0, 3) = 1$.

By applying this formula repeatedly, we can get explicit expressions of $\mathfrak{z}_n^\star(\zeta_n; m, 3)$ $(m = 1, 2, 3, 4)$ as follows.

$$
\begin{aligned}
\mathfrak{z}_n^\star(\zeta_n; 1, 3) &= -\frac{(n-1)(n-3)}{8}, \\
\mathfrak{z}_n^\star(\zeta_n; 2, 3) &= -\frac{(n+1)(n-1)(n^4 - 650n^2 + 3780n - 5291)}{12 \cdot 7!}, \\
\mathfrak{z}_n^\star(\zeta_n; 3, 3) &= \frac{(n+1)(n-1)(n-3)(8n^5 + 4n^4 - 2005n^3 + 6985n^2 + 11357n - 36509)}{60 \cdot 8!},
\end{aligned}
$$

$$\mathfrak{z}_n^\star(\zeta_n;4,3)=\frac{(n+1)(n-1)}{2\cdot 15!}(703n^{10}-1109042n^8+4324320n^7+165698599n^6-1101079980n^5 \\ -545336011n^4+15460525080n^3-18165129202n^2-47055628620n+76385318153).$$

Acknowledgments

The author thanks the referees for carefully reading the manuscript and for giving constructive comments. This work was partly done when the author stayed at Tongji University in September 2024. He is grateful for the generous hospitality of Professor Haigang Zhou and the fellowship and discussion by faculty members and students, in particular Professor Zhong-Hua Li. This work was partly supported by JSPS KAKENHI Grant Number 24K22835.

References

[1] Y. Ohno, J. Okuda and W. Zudilin, Cyclic q-MZSV sum. *J. Number Theory* **132** (2012) 144–155.

[2] Y. Takeyama, A q-analogue of non-strict multiple zeta values and basic hypergeometric series. *Proc. Amer. Math. Soc.* **137** (2009) 2997–3002.

[3] T. Komatsu, On q-generalized (r,s)-Stirling numbers. *Aequationes Math.* **98** (2024) 1281–1304.

[4] T. Komatsu, Some explicit values of a q-multiple zeta function at roots of unity. arXiv:2505.09357 (2025).

[5] T. Komatsu, On s-Stirling transform and poly-Cauchy numbers of the second kind with level 2. *Aequationes Math.* **97** (2023) 31–61.

[6] D.M. Bradley, Multiple q-zeta values. *J. Algebra* **283** (2005) 752–798.

[7] K.-G. Schlesinger, Some remarks on q-deformed multiple polylogarithms. arXiv:math/0111022 (2001).

[8] K. Tasaka, Finite and symmetric colored multiple zeta values and multiple harmonic q -series at roots of unity. *Sel. Math., New Ser.* **27** (2021) Paper No. 21, 34.

[9] J. Zhao, Multiple q-zeta functions and multiple q-polylogarithms. *Ramanujan J.* **14** (2007) 189–221.

[10] W. Zudilin, Algebraic relations for multiple zeta values (in Russian). *Uspekhi Mat. Nauk* **58** (2003) 3–32; translation in *Russian Math. Surv.* **58** (2003) 1–29.

[11] A.Z. Broder, The r-Stirling numbers. *Discrete Math.* **49** (1984) 241–259.

[12] H. Bachmann, Y. Takeyama and K. Tasaka, Cyclotomic analogues of finite multiple zeta values. *Compositio Math.* **154** (2018) 2701–2721.

[13] H. Bachmann, Y. Takeyama and K. Tasaka, Special values of finite multiple harmonic q-series at roots of unity. Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA), vol. 2. IRMA Lect. Math. Theor. Phys., vol. 32. EMS Publishing House, Berlin (2020) 1–18.

[14] Z. Li and E. Pan, Sum of interpolated finite multiple harmonic q-series. *J. Number Theory* **201** (2019) 148–175.

[15] T. Ernst, q-Stirling numbers, an umbral approach. *Adv. Dyn. Syst. Appl.* **3** (2008) 251–282.

[16] T. Komatsu, J.L. Ramirez and D. Villamizar, A combinatorial approach to the Stirling numbers of the first kind with higher level. *Stud. Sci. Math. Hung.* **58** (2021) 293–307.

[17] T. Komatsu, J.L. Ramirez and D. Villamizar, A combinatorial approach to the generalized central factorial numbers. *Mediterr. J. Math.* **18** (2021) Article 192, 14.

[18] M.E. Hoffman, Harmonic-number summation identities, symmetric functions, and multiple zeta values. *Ramanujan J.* **42** (2017) 501–526.

[19] I.G. MacDonald, Symmetric Functions and Hall Polynomials, 2nd edn. Clarendon Press, Oxford (1995).

[20] T. Muir, The Theory of Determinants in the Historical Order of Development, 4 vols. Dover Publications, New York (1960).

[21] N. Trudi, Intorno ad alcune formole di sviluppo. *Rendic. dell' Accad. Napoli* (1862) 135–143.

[22] T. Komatsu, Bernoulli numbers with level 2. *Aequationes Math.* **99** (2025) 71–87.

[23] T. Komatsu and J.L. Ramirez, Some determinants involving incomplete Fubini numbers. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **26** (2018) 143–170.

Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at https://edpsciences.org/en/subscribe-to-open-s2o.