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MEANDERS AND DYCK-PATH BILLIARDS

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Abstract. We study a statistic traj on the ordered pairs (P, Q) of Dyck paths of size n, which counts the number of billiard trajectories in the grid polygon enclosed by P and $-Q$, where $-Q$ is the path obtained by reflecting Q over the ground line. In terms of grid polygon, we establish an involution on the set of such ordered pairs (P, Q) which either increases or decreases $\mathsf{traj}(P, Q)$ by 1. This proves a result by Di Francesco–Golinelli–Guitter that the numbers of semimeanders (meanders, respectively) of order n with even and odd numbers of components are equal if n is even and differ by a Catalan number (the square of a Catalan number, respectively) if n is odd. Some results about (-1)-evaluation of the generating functions for the statistic traj on restricted sets of Dyck paths are also presented.

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1. Introduction

1.1. Trajectory statistic for Dyck paths

A *Dyck path* of size n is a lattice path in the plane $\mathbb{Z}^2$ from $(0, 0)$ to (n, n) using *north steps* $\mathsf{N} = (0, 1)$ and *east steps* $\mathsf{E} = (1, 0)$ which never goes below the line $y = x$. Sometimes, we use the notation N^k (E^k, respectively) for k consecutive north steps (east steps, respectively). Let $\mathcal{D}_n$ denote the set of Dyck paths of size n. It is known that $|\mathcal{D}_n| = c_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. For any $P \in \mathcal{D}_n$, let $-P$ denote the lattice path obtained by reflecting the path P over the line $y = x$, called a *negative Dyck path*.

Let $P, Q \in \mathcal{D}_n$. We associate the ordered pair (P, Q) with a *grid polygon* in the plane $\mathbb{Z}^2$ enclosed by P and $-Q$. The boundary of the grid polygon consists of the steps of P and $-Q$, which are numbered $1, 2, \ldots, 4n$ clockwise. For example, the grid polygon associated to (P, Q), where $P = \mathsf{NNENNENEEE}$ and $Q = \mathsf{NNEENENNEE}$, is shown in Figure 1.

Motivated by the study of rainbow meanders and Cartesian billiards by Fiedler–Castañeda [1] (see also [2], Sect. 2.7), we consider billiard trajectories, which are also interpreted as trajectories of light beams, in the grid polygon (P, Q). Pick some step p on the boundary of the grid polygon, say with label i. Emit a beam of light from the midpoint of p into the interior of the polygon so that the light beam forms a 45° angle with p and travels either northeast, southeast, southwest, or northwest (depending on the orientation of p). The light beam will

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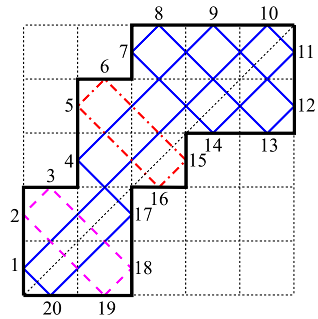

FIGURE 1. A grid polygon with three trajectories of light beams.

TABLE 1. The number of ordered pairs $(P,Q) \in \mathcal{D}_n \times \mathcal{D}_n$ with k trajectories of light beams for $1 \le k \le n \le 7$.

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	2	2					
3	8	12	5				
4	42	84	56	14			
5	262	640	580	240	42		
6	1828	5236	5894	3344	990	132	
7	13 820	45 164	60 312	42 840	17 472	4004	429

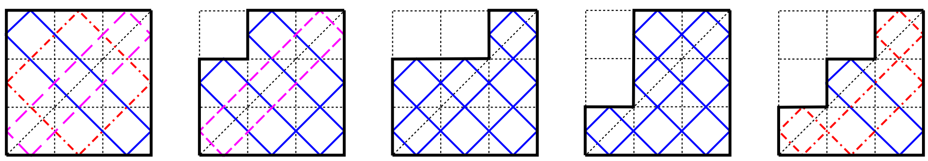

FIGURE 2. The trajectories of light beams for the paths in $\mathcal{D}_3$.

travel through the interior of the polygon until reaching the midpoint of another step, with label $\pi(i)$, meeting it at a 45° angle. This defines a permutation $\pi : [4n] \to [4n]$. As shown in Figure 1, the cycle decomposition of π of that polygon is $\pi = (1, 10, 11, 14, 7, 8, 13, 12, 9, 4, 17, 20)(2, 3, 18, 19)(5, 6, 15, 16)$. Each cycle of π corresponds to a *trajectory* of a beam of light (*cf.* [1, 2]).

Let $\mathsf{traj}(P,Q)$ denote the number of trajectories of light beams in the grid polygon associated to (P,Q). The initial values for the distribution of the ordered pairs $(P,Q) \in \mathcal{D}_n \times \mathcal{D}_n$ with respect to $\mathsf{traj}(P,Q)$ are shown in Table 1.

In particular, we define the statistic traj on Dyck paths as follows. Let L_n denote the Dyck path $\mathsf{N}^n\mathsf{E}^n$. For any $P \in \mathcal{D}_n$, let $\mathsf{traj}(P) = \mathsf{traj}(P, L_n)$ for short. For $n = 3$, the values of $\mathsf{traj}(P)$ for the five Dyck paths P of size 3 shown in Figure 2 are 3, 2, 1, 1, and 2 accordingly. The initial values for the distribution of Dyck paths with respect to the statistic traj are shown in Table 2.

1.2. Meanders/Semimeanders

Fix a straight line in the plane $\mathbb{R}^2$ and $2n$ points on it. A *meander* of order n is a collection of closed self-avoiding and noncrossing curves that intersect the line in exactly those $2n$ points. Two configurations are considered as equivalent if they are smooth deformations of one another. We consider only the inequivalent

TABLE 2. The number of Dyck path $P \in \mathcal{D}_n$ with $\mathsf{traj}(P) = k$ for $1 \le k \le n \le 7$.

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	2	1				
4	4	6	3	1			
5	10	16	11	4	1		
6	24	48	37	17	5	1	
7	66	140	126	66	24	6	1

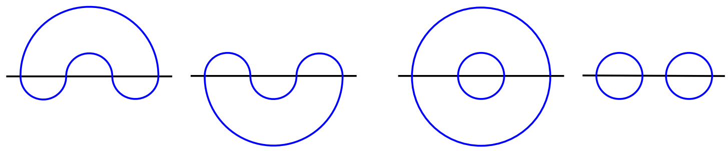

FIGURE 3. The four meanders of order 2.

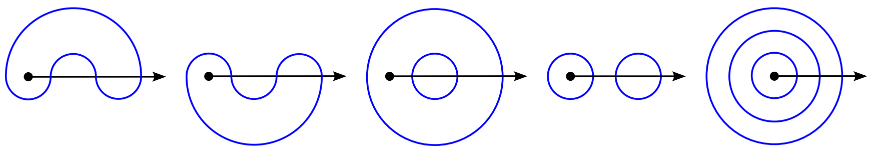

FIGURE 4. The five semimeanders of order 3.

meanders. For example, the four meanders of order 2 are shown in Figure 3. Let $\mathcal{M}_n$ denote the set of meanders of order n. Let $\mathsf{comp}(T)$ denote the number of components of a meander $T \in \mathcal{M}_n$. Note that the triangular array in Table 1 coincides with the distribution of the meanders in $\mathcal{M}_n$ with respect to the number of components [3], A008828; see also [4]. We refer the readers to [5] for historical backgrounds and recent developments of the enumeration of meanders.

Now, fix a ray in the plane $\mathbb{R}^2$ and n points on it. A *semimeander* of order n is a collection of closed self-avoiding and noncrossing curves that intersect the ray in exactly those n points. For example, the five semimeanders of order 3 are shown in Figure 4. Let $\mathcal{M}_n^*$ denote the set of semimeanders of order n. Note that the triangular array in Table 2 coincides with the distribution of the semimeanders in $\mathcal{M}_n^*$ with respect to the number of components [3], A046726; see also [4]. Recently, Cori and Hetyei [6] present a bijection between meanders (semimeanders, respectively) of order n and specialized hypermaps, called the spanning hypertrees of the reciprocal of a dipole with n parallel edges (a monopole with $n/2$ nested edges, respectively).

1.3. Main results

There is a standard bijection between ordered pairs of Dyck paths and meanders; see [7, 8]. Our first result is to describe the bijective properties in Theorem 1.1

Theorem 1.1. *The following results hold.*

(i) *There is a bijection* $(P, Q) \mapsto T$ *of* $\mathcal{D}_n \times \mathcal{D}_n$ *onto* $\mathcal{M}_n$ *with* $\mathsf{traj}(P, Q) = \mathsf{comp}(T)$.
(ii) *There is a bijection* $P \mapsto S$ *of* $\mathcal{D}_n$ *onto* $\mathcal{M}_n^*$ *with* $\mathsf{traj}(P) = \mathsf{comp}(S)$.

Our second result is to give a combinatorial proof of the (-1)-evaluations of the generating functions for the statistic traj on Dyck paths and ordered pairs of Dyck paths.

Theorem 1.2. *The following results hold.*

(i) *There is an involution* $\varphi : P \mapsto P'$ *on* $\mathcal{D}_n$ *satisfying* $\mathsf{traj}(P) - \mathsf{traj}(P') \in \{1, 0, -1\}$ *and resulting in the following identity*

$$\sum_{P \in \mathcal{D}_n} (-1)^{\mathsf{traj}(P)} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -c_k & \text{if } n = 2k+1. \end{cases} \tag{1.1}$$

(ii) *There is an involution* $\psi : (P, Q) \mapsto (P', Q')$ *on* $\mathcal{D}_n \times \mathcal{D}_n$ *satisfying* $\mathsf{traj}(P, Q) - \mathsf{traj}(P', Q') \in \{1, 0, -1\}$ *and resulting in the following identity*

$$\sum_{(P,Q) \in \mathcal{D}_n \times \mathcal{D}_n} (-1)^{\mathsf{traj}(P,Q)} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -c_k^2 & \text{if } n = 2k+1. \end{cases} \tag{1.2}$$

By Theorems 1.1 and 1.2, we prove the following result [4], equation (6.1) anew.

Theorem 1.3 (Di Francesco–Golinelli–Guitter)**.** *We have*

(i) $\displaystyle\sum_{S \in \mathcal{M}_n^*} (-1)^{\mathsf{comp}(S)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -c_k & \text{if } n = 2k+1, \end{cases}$

(ii) $\displaystyle\sum_{T \in \mathcal{M}_n} (-1)^{\mathsf{comp}(T)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -c_k^2 & \text{if } n = 2k+1. \end{cases}$

A *Motzkin path* of length n is a lattice path in the plane $\mathbb{Z}^2$ from $(0, 0)$ to the point $(n, 0)$ using *up step* $(1, 1)$, *horizontal step* $(1, 0)$ and *down step* $(1, -1)$ which never goes below the x-axis. Motzkin paths are counted by the *Motzkin numbers* $\{m_n\}_{n\geq 1} = \{1, 2, 4, 9, 21, 51, 127, \dots\}$ [3], A001006. Those Motzkin paths with no horizontal step on the x-axis are called *Riordan paths*, which are counted by the *Riordan numbers* $\{r_n\}_{n\geq 1} = \{0, 1, 1, 3, 6, 15, 36, \dots\}$ [3], A005043. For convenience, a Motzkin path or Riordan path is said to be *even* (*odd*, respectively) if it contains an even (odd, respectively) number of up steps.

A *peak* of a Dyck path is an occurrence of NE (*i.e.*, a north step followed by an east step) and a *valley* is an occurrence of EN. The coordinates of a peak or valley is given by the intersection point of its N and E. A peak is said to be at *height* h if the intersection point is on the line $y = x + h$. It is known that the Dyck paths of size n with all peaks at odd (even, respectively) height are counted by the Motzkin number m_{n-1} (Riordan number r_n, respectively) [9]. Our third result is about the (-1)-evaluation of the generating functions for the statistic traj on restricted sets of Dyck paths, which coincide up to a sign with the excess of the number of even Motzkin/Riordan paths over the odd ones (*cf.* [3], A343773).

Theorem 1.4. *The following results hold.*

(i) *Let* $\mathcal{A}_n \subset \mathcal{D}_n$ *be the set of Dyck paths with all peaks at odd height. Then we have*

$$\sum_{P \in \mathcal{A}_{n+1}} (-1)^{\mathsf{traj}(P)} = (-1)^{\lfloor \frac{n+2}{2} \rfloor} (e_n - d_n), \tag{1.3}$$

where e_n *(*d_n*, respectively) is the number of even (odd, respectively) Motzkin paths of length* n*.*

(ii) *Let* $\mathcal{B}_n \subset \mathcal{D}_n$ *be the set of Dyck paths with all peaks at even height. Then we have*

$$\sum_{P \in \mathcal{B}_n} (-1)^{\mathsf{traj}(P)} = (-1)^{\lfloor \frac{n+2}{2} \rfloor} (\hat{d}_n - \hat{e}_n), \tag{1.4}$$

where $\hat{e}_n$ *(*$\hat{d}_n$*, respectively) is the number of even (odd, respectively) Riordan paths of length* n*.*

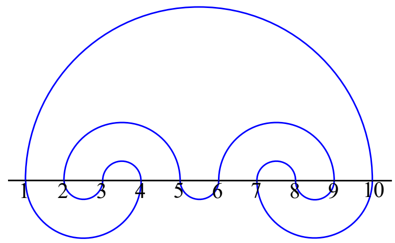

FIGURE 5. A meander is the superimposition of two arch configurations.

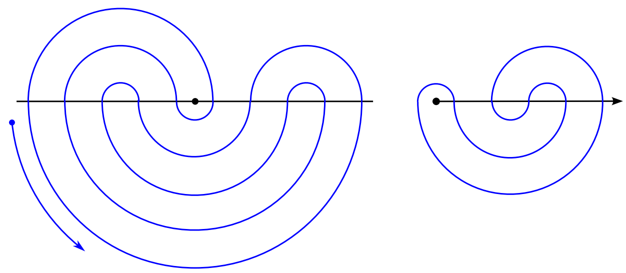

FIGURE 6. A specialized meander is folded into a semimeander.

2. Bijective properties

2.1. Arch configurations

A *noncrossing matching* of order n is a noncrossing partition of the set $\{1, \ldots, 2n\}$ into blocks of size two; see [10]. For any meander $T \in \mathcal{M}_n$, the straight line divides the plane into two half-planes. If the $2n$ points on the straight line are numbered from 1 to $2n$, the part of T in each half-plane determines a noncrossing matching of order n. Such a system of n arches in the upper (lower, respectively) half-plane is called an *arch configuration* of order n. For example, the meander shown in Figure 5 is the superimposition of two arch configurations, the upper (lower, respectively) configuration of which is given by the noncrossing matching $\{(1, 10), (2, 5), (3, 4), (6, 9), (7, 8)\}$ ($\{(1, 4), (2, 3), (5, 6), (7, 10), (8, 9)\}$, respectively).

Semimeanders of order n can be viewed as the specialization of meanders of order n to those with the prescribed noncrossing matching $\{(1, 2n), (2n, 2n-1), \ldots, (n, n+1)\}$ as their lower arch configuration, which are also called *rainbow meanders* [4]. The straight line of the meanders is interpreted as a pair of rays originating from the center of the lower arch configuration. Fold the pair of rays into one, as indicated in Figure 6, so that those $2n$ points on the straight line are identified by pairs according to the lower arch configuration. This provides a transformation to obtain the semimeander from a specialized meander; see [4, 8].

2.2. Bijective properties

A north or east step of a Dyck path P is said to be at *height* h if it is between the two lines $y = x + h - 1$ and $y = x + h$. We say that a north step p and an east step q form a *matching pair* of P if p and q are at the same height and face each other in the sense that the line connecting the midpoints of p and q stays under the path. We identify the matching pairs of $-P$ with the matching pairs of P.

In the following, we describe the bijection between $\mathcal{D}_n \times \mathcal{D}_n$ and $\mathcal{M}_n$ in terms of arch configuration.

Proof of Theorem 1.2. Given an ordered pair $(P, Q) \in \mathcal{D}_n \times \mathcal{D}_n$, let $P = p_1 p_2 \cdots p_{2n}$ and let $Q = q_1 q_2 \cdots q_{2n}$. The corresponding meander $T \in \mathcal{M}_n$ is constructed as follows.

(i) Draw a straight line in the plane $\mathbb{R}^2$ and $2n$ points on it, numbered from 1 to $2n$.

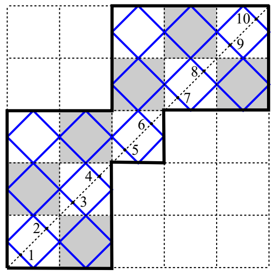

FIGURE 7. The grid polygon associated to the pair (P, Q) in Example 2.1.

(ii) Find all the matching pairs of P. For each matching pair, say (p_i, p_j), draw an arch from i to j above the points.
(iii) Find all the matching pairs of Q. For each matching pair, say (q_i, q_j), draw an arch from i to j below the points.

The inverse map can be established in reverse. To describe the billiard trajectories in the grid polygon, let $z_1, z_2, \dots, z_{2n}$ be the $2n$ points on the line $y = x$ in the grid polygon such that z_k is the intersection of the two lines $y = x$ and $x + y = \frac{2k-1}{2}$, *i.e.*, the coordinate of z_k is $(\frac{2k-1}{4}, \frac{2k-1}{4})$.

(i) For each upper arch of T, say from i to j, draw a line segment connecting z_i to the midpoint of p_i, draw a line segment connecting the midpoints of p_i and p_j, and then draw a line segment from the midpoint of p_j to z_j.
(ii) For each lower arch of T, say from i to j, draw a line segment connecting z_i to the midpoint of q_i, draw a line segment connecting the midpoints of q_i and q_j, and then draw a line segment from the midpoint of q_j to z_j.

Thus, the billiard trajectories are deformations of the components of T. The result of Theorem 1.1(i) follows.

To establish the bijection $\mathcal{D}_n \to \mathcal{M}_n^*$, for any $P \in \mathcal{D}_n$ we construct the meander corresponding to (P, L_n), whose lower arch configuration is the noncrossing matching $\{(1, 2n), (2, 2n-1), \dots, (n, n+1)\}$. The proof of Theorem 1.1 is completed. $\square$

Example 2.1. Let P = NNNEENNEEE and Q = NNEENENNEE. The grid polygon associated to (P, Q) is shown in Figure 7. Let $p_1, p_2, \dots, p_{10}$ denote the steps of P. The matching pairs of P are $(p_1, p_{10}), (p_2, p_5), (p_3, p_4), (p_6, p_9)$ and (p_7, p_8). This determines the upper arch configuration of the the meander in Figure 5. Let $q_1, q_2, \dots, q_{10}$ denote the steps of Q. The matching pairs of Q are $(q_1, q_4), (q_2, q_3), (q_5, q_6), (q_7, q_{10})$ and (q_8, q_9). This determines the lower arch configuration of the the meander in Figure 5. Note that the unique billiard trajectory shown in Figure 7 is a deformation of the meander in Figure 5.

3. PROOF OF THEOREM 1.2

For any $(P, Q) \in \mathcal{D}_n \times \mathcal{D}_n$, a trajectory of light beam in the grid polygon associated to (P, Q) can be partitioned into two families of parallel line segments, on the lines with slope 1 and -1, respectively. If a trajectory is assigned an orientation, we observe that the parallel segments of light of the trajectory, on different lines with the same slope, move alternately in the opposite directions.

Lemma 3.1. *Let ℓ_1, ℓ_2 be two parallel segments of light in a trajectory. The following properties hold.*

(i) *If ℓ_1, ℓ_2 are on the lines $y = x + \frac{4k+1}{2}$ and $y = x + \frac{4k+3}{2}$, respectively, for some integer k then ℓ_1, ℓ_2 move in the opposite directions.*

(ii) *If ℓ_1, ℓ_2 are on the lines $y = -x + \frac{4k+1}{2}$ and $y = -x + \frac{4k+3}{2}$, respectively, for some integer k then ℓ_1, ℓ_2 move in the opposite directions.*

Proof. For convenience, we color the unit squares in the grid polygon in black and white like a checkerboard. A unit square is colored white if its upper right corner (i, j) satisfies the condition that $i + j$ is even, and black otherwise, as shown in Figure 7. Choose a white square along the path P and emit a beam of light from the midpoint of its left edge to the midpoint of its top edge. We have the following observations.

(i) The light beam goes northeast-bound, hits an east step of P, which is the top edge of a white square, and turns 90° clockwise.
(ii) Then the light beam either hits a north step (the right edge of a white square) of $-Q$, turns 90° clockwise and hits its matching east step (the bottom edge of a white square), or hits an east step (the bottom edge of a black square) of $-Q$, turns 90° counter clockwise and hits its matching north step (the right edge of a black square).
(iii) In the former case, the light beam turns 90° clockwise and hits either a north step or an east step of P. In the latter case, the light beam turns 90° counter clockwise and hits either an east step or a north step of P.

Along the resulting trajectory, we observe that the segments of light within each white (black, respectively) square move clockwise (counter clockwise, respectively). The result follows. ☐

Example 3.2. For the grid polygon shown in Figure 1, consider the trajectory of the cycle $(1, 10, 11, 14, 7, 8, 13, 12, 9, 4, 17, 20)$ of that permutation π. If the trajectory is oriented in the direction from the entry 20 to the entry 1, then the light segments between the entries 1 and 10, between the entries 7 and 8, and between the entries 13 and 12 move in northeast direction. Moreover, the light segments between the entries 11 and 14, between the entries 9 and 4, and between the entries 17 and 20 move in southwest direction.

Lemma 3.3. *For $n \ge 2$, if P, P' are two Dyck paths in $\mathcal{D}_n$ such that P' is obtained from P by replacing a peak by a valley then $\mathsf{traj}(P) - \mathsf{traj}(P') = \pm 1$.*

Proof. Let $P = p_1 p_2 \cdots p_{2n}$. Suppose $p_i p_{i+1} = \mathsf{NE}$ is a peak of P at height at least 2 and P' is obtained from P by replacing $p_i p_{i+1}$ by EN. Depicted in Figures 8(i) and 9(i), let b, c be the midpoints of p_i, p_{i+1}, respectively. Then the shaded unit squares are enclosed in the grid polygon associated to (P, L_n), and the four points $\{a, b, c, d\}$ ($\{e, f, g, h\}$, respectively) are in a trajectory of a beam of light. There are two cases.

Case 1. Those eight points are in the same trajectory. By Lemma 3.1, if the trajectory is oriented from b to c then the light moves in the direction from g to f. The relative order of those eight points is shown in Figure 8(ii). If the peak $p_i p_{i+1}$ is replaced by a valley, as shown in Figure 8(iii)–(iv), then the trajectory is split into two trajectories, one passing the points a, f and e and the other passing the points h, g and d. Thus, $\mathsf{traj}(P) - \mathsf{traj}(P') = -1$.

Case 2. Those eight points are in two different trajectories. The relative order of those eight points is shown in Figure 9(ii). If the peak $p_i p_{i+1}$ is replaced by a valley, as shown in Figure 9(iii)–(iv), then these two trajectories are merged into one trajectory, passing the points a, f, e, h, g and d. Thus, $\mathsf{traj}(P) - \mathsf{traj}(P') = 1$.

The result follows. ☐

Lemma 3.4. *For $n \ge 2$, if P, P' are two Dyck paths in $\mathcal{D}_n$ such that P' is obtained from P by replacing a valley by a peak then $\mathsf{traj}(P) - \mathsf{traj}(P') = \pm 1$.*

Proof. Let $P = p_1 p_2 \cdots p_{2n}$. Suppose $p_i p_{i+1} = \mathsf{EN}$ is a valley of P and P' is obtained from P by replacing $p_i p_{i+1}$ by NE. Depicted in Figures 8(iii) and 9(iii), let f, g be the midpoints of p_i, p_{i+1}, respectively. Then the shaded unit squares are enclosed in the grid polygon (P, L_n), and the three points $\{a, f, e\}$ ($\{d, g, h\}$, respectively) are in a trajectory of a beam of light. There are two cases.

Case 1. Those six points are in the same trajectory. By Lemma 3.1, if the trajectory is oriented from e to f then the light moves in the direction from g to h. The relative order of those six points is shown in

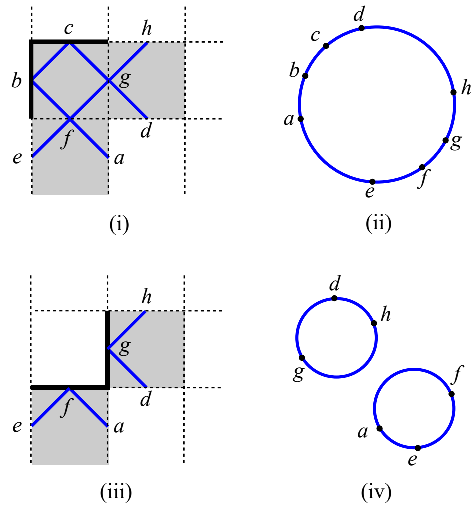

FIGURE 8. A trajectory of light beam split into two trajectories.

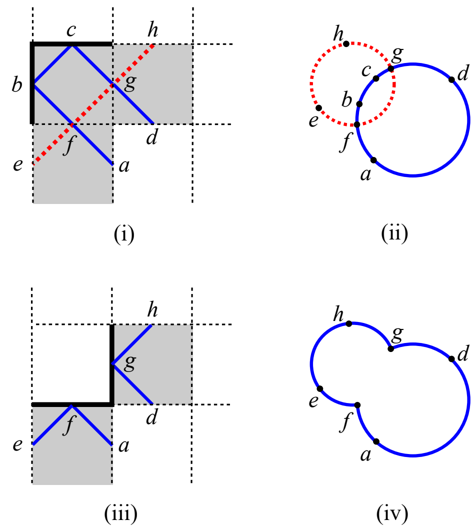

FIGURE 9. Two trajectories of light beams merged into one trajectory.

Figure 9(iv). If the valley p_ip_{i+1} is replaced by a peak, as shown in Figure 9(i)–(ii), then the trajectory is split into two trajectories, one passing the points e, f, g and h and the other passing the points a, b, c and d. Thus, $\mathsf{traj}(P) - \mathsf{traj}(P') = -1$.

Case 2. Those six points are in two different trajectories. The relative order of those six points is shown in Figure 8(iv). If the valley p_ip_{i+1} is replaced by a peak, as shown in Figure 8(i)–(ii), then these two trajectories are merged into one trajectory, passing the points a, b, c, d, h, g, f and e. Thus, $\mathsf{traj}(P) - \mathsf{traj}(P') = 1$.

The result follows. $\square$

In the following, we construct the involution $\varphi : \mathcal{D}_n \to \mathcal{D}_n$ in Theorem 1.2(i). First, we describe the fixed points of the involution. Let $\mathcal{F}_n \subset \mathcal{D}_n$ be the set of Dyck paths containing no peak with even y-coordinate and no valley with odd x-coordinate. For example, the grid polygon shown in Figure 10 is associated to an ordered

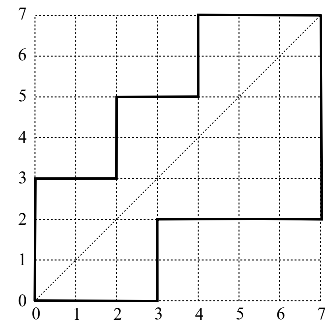

FIGURE 10. A grid polygon associated to an ordered pair in $\mathcal{F}_n \times \mathcal{F}_n$.

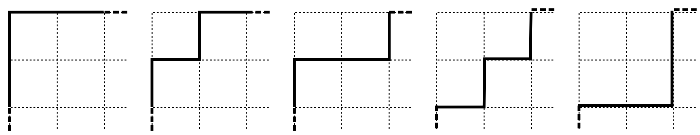

FIGURE 11. A serial operation to remove a 2×2 square from a grid polygon.

pair $(P, Q) \in \mathcal{F}_n \times \mathcal{F}_n$. (Note that the peaks (valleys, respectively) of Q become the valleys (peaks, respectively) of the negative Dyck path $-Q$.)

Lemma 3.5. *For $n \geq 1$, we have*

$$|\mathcal{F}_n| = \begin{cases} 0 & \text{if } n \text{ is even,} \\ c_k & \text{if } n = 2k+1. \end{cases} \tag{3.1}$$

Proof. Note that $|\mathcal{F}_1| = 1$. Let $n \geq 2$. We observe that for some $t \geq 1$, every path $P \in \mathcal{F}_n$ is of the form

$$P = \mathsf{N}^{2a_1+1}\mathsf{E}^{2b_1}\mathsf{N}^{2a_2}\mathsf{E}^{2b_2}\cdots\mathsf{N}^{2a_t}\mathsf{E}^{2b_t+1} \tag{3.2}$$

for some positive integers $a_1, \dots, a_t$ and $b_1, \dots, b_t$ with $a_1 + \cdots + a_t = b_1 + \cdots + b_t = n - 1$. Thus, every path in $\mathcal{F}_n$ has odd size, say $n = 2k + 1$. Moreover, the paths P in (3.2) are in one-to-one correspondence to the paths $P' = \mathsf{N}^{a_1}\mathsf{E}^{b_1}\mathsf{N}^{a_2}\mathsf{E}^{b_2}\cdots\mathsf{N}^{a_t}\mathsf{E}^{b_t}$ in $\mathcal{D}_k$. The result follows. $\square$

Lemma 3.6. *For all $(P, Q) \in \mathcal{F}_{2k+1} \times \mathcal{F}_{2k+1}$, $\mathsf{traj}(P, Q)$ is odd.*

Proof. Let $L = \mathsf{N}^{2k+1}\mathsf{E}^{2k+1}$. Note that $L \in \mathcal{F}_{2k+1}$ and $\mathsf{traj}(L, L) = 2k + 1$. For any $(P, Q) \in \mathcal{F}_{2k+1} \times \mathcal{F}_{2k+1}$, we define a serial operation to remove a 2×2 square from the grid polygon associated to (P, Q) while preserving the parity of traj. Suppose P contains a peak at height at least 4. Starting from such a peak, we create four Dyck paths $P_0 = P, P_1, P_2, P_3, P_4$ such that P_i is obtained from P_{i-1} by replacing a peak by a valley, by the serial replacements shown in Figure 11. Note that P_4 is in $\mathcal{F}_{2k+1}$. By Lemma 3.3, we have $\mathsf{traj}(P_4, Q) \equiv \mathsf{traj}(P, Q) \pmod 2$. (A symmetric operation can apply to the path Q.) Moreover, the grid polygon of (P, Q) can be obtained from that of (L, L) by a sequence of such operations. Thus, $\mathsf{traj}(P, Q) \equiv \mathsf{traj}(L, L) \pmod 2$, which is odd. The result follows. $\square$

For any Dyck path P, we say that a peak is *good* (*bad*, respectively) if its y-coordinate is odd (even, respectively), and that a valley is *good* (*bad*, respectively) if its x-coordinate is even (odd, respectively). By a *corner* of a Dyck path we mean a peak or a valley. If P is of even size, we observe that the last peak of P is always a bad corner.

Now, we describe a simple construction of the involution $\varphi : P \mapsto P'$ on $\mathcal{D}_n$. Let $P = p_1p_2\cdots p_{2n}$. If P contains no bad corner then n is odd and we set $\varphi(P) = P$. Otherwise, find the first bad corner in P, say p_ip_{i+1}. The corresponding path $P' = p_1\cdots p_{i-1}p'_ip'_{i+1}p_{i+2}\cdots p_{2n}$ is obtained from P by interchanging p_i, p_{i+1}, *i.e.*, $p'_ip'_{i+1} = p_{i+1}p_i$.

Lemma 3.7. *The map φ is an involution. If $P \neq P'$ then $\mathsf{traj}(P) - \mathsf{traj}(P') = \pm 1$.*

Proof. Suppose $P \neq P'$. Let p_ip_{i+1} be the first bad corner of P. Then in the prefix $p_1\cdots p_i$ of P, the x-coordinate and y-coordinate of each corner p_jp_{j+1} $(j < i)$ have the opposite parities. There are two cases.

Case 1. p_ip_{i+1} is a peak. Then both of its x-coordinate and y-coordinate are even. Thus, $p'_ip'_{i+1}$ is a valley with odd x-coordinate and y-coordinate.

Case 2. p_ip_{i+1} is a valley. Then both of its x-coordinate and y-coordinate are odd. Thus, $p'_ip'_{i+1}$ is a peak with even x-coordinate and y-coordinate.

Note that in either case $p'_ip'_{i+1}$ is the first bad corner of P'. Thus, the map φ is an involution. By Lemmas 3.3 and 3.4, we have $\mathsf{traj}(P) - \mathsf{traj}(P') = \pm 1$. □

Proof of Theorem 1.2. By Lemmas 3.5, 3.6 and 3.7, we have

$$\begin{aligned}\sum_{P\in\mathcal{D}_n}(-1)^{\mathsf{traj}(P)} &= \sum_{P\in\mathcal{F}_n}(-1)^{\mathsf{traj}(P)}\\ &= -|\mathcal{F}_n|\\ &= \begin{cases}0 & \text{if } n \text{ is even,}\\ -c_k & \text{if } n = 2k+1.\end{cases}\end{aligned}$$

The assertion (i) of Theorem 1.2 follows.

Now, we describe the involution $\psi : (P,Q) \mapsto (P',Q')$ on $\mathcal{D}_n \times \mathcal{D}_n$. If (P,Q) is an ordered pair in $\mathcal{F}_n \times \mathcal{F}_n$ then n is odd and we set $\psi((P,Q)) = (P,Q)$. Otherwise, at least one of P and Q contains a bad corner. The corresponding ordered pair (P',Q') is constructed as follows. If P contains a bad corner, find the corresponding path $\varphi(P)$ and set the ordered pair $(P',Q') = (\varphi(P),Q)$. Otherwise, $P \in \mathcal{F}_n$ and Q contains a bad corner. Find the corresponding path $\varphi(Q)$ and set the ordered pair $(P',Q') = (P,\varphi(Q))$.

By Lemmas 3.5, 3.6 and 3.7, we have

$$\begin{aligned}\sum_{(P,Q)\in\mathcal{D}_n\times\mathcal{D}_n}(-1)^{\mathsf{traj}(P,Q)} &= \sum_{(P,Q)\in\mathcal{F}_n\times\mathcal{F}_n}(-1)^{\mathsf{traj}(P,Q)}\\ &= -|\mathcal{F}_n\times\mathcal{F}_n|\\ &= \begin{cases}0 & \text{if } n \text{ is even,}\\ -c_k^2 & \text{if } n = 2k+1.\end{cases}\end{aligned}$$

The assertion (ii) of Theorem 1.2 follows. □

4. Proof of Theorem 1.4

In this section, we study the (-1)-evaluation of the generating function for traj on the set $\mathcal{A}_{n+1}$ ($\mathcal{B}_n$, respectively) of Dyck paths with all peaks at odd (even, respectively) heights.

For any $P \in \mathcal{D}_n$, let $\mathsf{area}(P)$ denote the number of unit squares under the Dyck path P and above the line $y = x$. For example, on the left hand side of Figure 12 is a Dyck path P in $\mathcal{G}_7$ with $\mathsf{area}(P) = 14$. In particular,

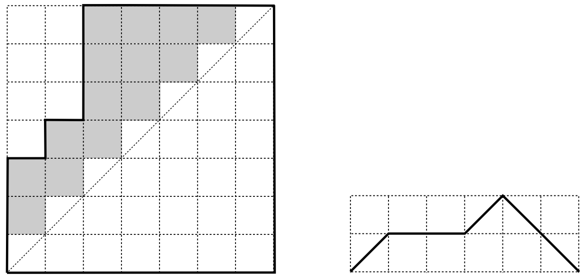

FIGURE 12. A Dyck path $P \in \mathcal{A}_7$ and its corresponding Motzkin path $\phi_A(P)$.

we have $\mathsf{area}(L_n) = \binom{n}{2}$, where $L_n = \mathsf{N}^n\mathsf{E}^n$. Those $\binom{n}{2}$ unit squares are partitioned into $n-1$ rows such that the ith row from top has $n-i$ squares.

Let $Z_n = (\mathsf{NE})^n$. Note that Z_n is a Dyck path in $\mathcal{A}_n$ with $\mathsf{area}(Z_n) = 0$. Using Lemmas 3.3 and 3.4, we determine the parity of $\mathsf{traj}(P)$ from the values $\mathsf{area}(Z_n)$ and $\mathsf{traj}(Z_n)$.

Lemma 4.1. *For $n \geq 1$, $\mathsf{traj}(Z_{n+1})$ is odd if $n \equiv 0, 1 \pmod 4$ and even if $n \equiv 2, 3 \pmod 4$.*

Proof. Let T be the meander corresponding to the ordered pair (Z_{n+1}, L_{n+1}). Note that the upper (lower, respectively) arch configuration of T is the noncrossing matching $\{(1,2),(3,4),\ldots,(2n+1,2n+2)\}$ ($\{(1,2n+2),(2,2n+1),\ldots,(n+1,n+2)\}$, respectively). Thus, $\mathsf{traj}(Z_{n+1}) = 1 + \lfloor \frac{n}{2} \rfloor$. The result follows. □

We describe a bijection ϕ_A that carries a Dyck path of size $n+1$ with all peaks at odd height to a Motzkin path of length n, which is due to J. Tirrell mentioned in [9]. Let $\mathsf{U}, \mathsf{H}, \mathsf{D}$ denote an up step, a horizontal step and a down step, respectively.

Let $P = p_1 p_2 \cdots p_{2n+2} \in \mathcal{A}_{n+1}$. Removing p_1 and p_{2n+2}, the corresponding Motzkin path $\phi_A(P) = v_1 \cdots v_n$ is defined by setting

$$v_j = \begin{cases} \mathsf{U} & \text{if } p_{2j}p_{2j+1} = \mathsf{NN} \\ \mathsf{H} & \text{if } p_{2j}p_{2j+1} = \mathsf{EN} \\ \mathsf{D} & \text{if } p_{2j}p_{2j+1} = \mathsf{EE} \end{cases} \tag{4.1}$$

for all j. For example, the Dyck path $P \in \mathcal{A}_7$ on the left hand side of Figure 12 is carries to the Motzkin path $\phi_A(P)$ on the right hand side.

Lemma 4.2. $\mathsf{traj}(P) \equiv \mathsf{traj}(Z_{n+1}) \pmod 2$ *if and only if $\phi_A(P)$ contains an even number of up steps.*

Proof. By Lemmas 3.3 and 3.4, we have $\mathsf{traj}(P) \equiv \mathsf{traj}(Z_{n+1}) \pmod 2$ if and only if $\mathsf{area}(P) \equiv \mathsf{area}(Z_{n+1}) \pmod 2$. Using (4.1), we determine the contribution to $\mathsf{area}(P)$ by each step v_j $(1 \leq j \leq n)$ as follows.

- $v_j = \mathsf{U}$. Then $p_{2j}p_{2j+1} = \mathsf{NN}$, where p_{2j}, p_{2j+1} are the left edges of two consecutive rows of unit squares above the line $y = x$. Thus, v_j contributes an odd value to $\mathsf{area}(P)$.
- $v_j = \mathsf{H}$. Then $p_{2j}p_{2j+1} = \mathsf{EN}$, where p_{2j+1} is the left edges of a row of an even number of unit squares above the line $y = x$. Thus, v_j contributes an even value to $\mathsf{area}(P)$.
- $v_j = \mathsf{D}$. Then $p_{2j}p_{2j+1} = \mathsf{EE}$, where neither of p_{2j}, p_{2j+1} is the left edge of a row of unit squares. Thus, v_j contributes nothing to $\mathsf{area}(P)$.

Since $\mathsf{area}(Z_{n+1}) = 0$ and the Motzkin path $\phi_A(Z_{n+1})$ contains no up step, we have $\mathsf{area}(P) \equiv \mathsf{area}(Z_{n+1}) \pmod 2$ if and only if $\phi_A(P)$ contains an even number of up steps. The result follows. □

Now, let $Z_n^* = \mathsf{N}(\mathsf{NE})^{n-1}\mathsf{E}$. Note that Z_n^* is a Dyck path in $\mathcal{B}_n$ with $\mathsf{area}(Z_n^*) = n-1$. For example, the path represented by dashed lines on the left hand side of Figure 13 is the Dyck path Z_7^*.

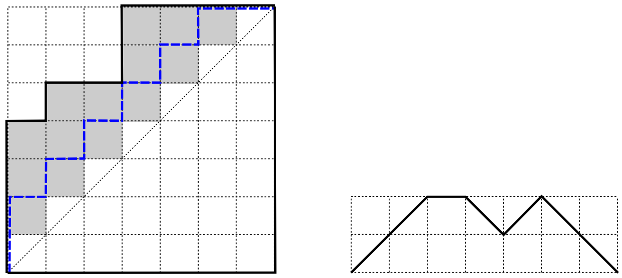

FIGURE 13. A Dyck path $P \in \mathcal{B}_7$ and its corresponding Riordan path $\phi_B(P)$.

Lemma 4.3. *For $n \geq 2$, $\mathsf{traj}(Z_n^*)$ is odd if $n \equiv 0, 1 \pmod 4$ and even if $n \equiv 2, 3 \pmod 4$.*

Proof. Let T be the meander corresponding to the ordered pair (Z_n^*, L_n). Note that the upper (lower, respectively) arch configuration of T is the noncrossing matching $\{(1, 2n)\} \cup \{(2, 3), (4, 5), \dots, (2n-2, 2n-1)\}$ ($\{(1, 2n), (2, 2n-1), \dots, (n, n+1)\}$, respectively). Thus, we have $\mathsf{traj}(Z_n^*) = 1 + \lfloor \frac{n}{2} \rfloor$. The result follows. □

We describe a bijection ϕ_B that carries a Dyck path of size n with all peaks at even height to a Riordan path of length n [9].

Let $P = p_1 p_2 \cdots p_{2n} \in \mathcal{B}_n$. The corresponding Riordan path $\phi_B(P) = u_1 \cdots u_n$ is defined by setting

$$u_j = \begin{cases} \mathsf{U} & \text{if } p_{2j-1}p_{2j} = \mathsf{NN} \\ \mathsf{H} & \text{if } p_{2j-1}p_{2j} = \mathsf{EN} \\ \mathsf{D} & \text{if } p_{2j-1}p_{2j} = \mathsf{EE}. \end{cases} \tag{4.2}$$

for all $j \in [n]$. For example, the path represented by solid lines on the left hand side of Figure 13 is carries to the Riordan path $\phi_B(P)$ on the right hand side.

Lemma 4.4. $\mathsf{traj}(P) \equiv \mathsf{traj}(Z_n^*) \pmod 2$ *if and only if $\phi_B(P)$ contains an odd number of up steps.*

Proof. By Lemmas 3.3 and 3.4, we have $\mathsf{traj}(P) \equiv \mathsf{traj}(Z_n^*) \pmod 2$ if and only if $\mathsf{area}(P) \equiv \mathsf{area}(Z_n^*) \pmod 2$. Using (4.2), we determine the contribution to $\mathsf{area}(P) - \mathsf{area}(Z_n^*)$ by each step u_j $(2 \leq j \leq n)$ as follows.

- $u_j = \mathsf{U}$. Then $p_{2j-1}p_{2j} = \mathsf{NN}$, where p_{2j}, p_{2j+1} are the left edges of two consecutive rows of unit squares above the path Z_n^*. Thus, u_j contributes an odd value to $\mathsf{area}(P) - \mathsf{area}(Z_n^*)$.
- $u_j = \mathsf{H}$. Then $p_{2j-1}p_{2j} = \mathsf{EN}$, where p_{2j} is the left edges of a row of an even number of unit squares above the path Z_n^*. Thus, u_j contributes an even value to $\mathsf{area}(P) - \mathsf{area}(Z_n^*)$.
- $v_j = \mathsf{D}$ is a down step. Then $p_{2j-1}p_{2j} = \mathsf{EE}$, where neither of p_{2j-1}, p_{2j} is the left edge of a row of unit squares. Thus, u_j contributes nothing to $\mathsf{area}(P) - \mathsf{area}(Z_n^*)$.

Since the Riordan path $\phi_B(Z_n^*)$ contains exactly one up step, we have $\mathsf{area}(P) \equiv \mathsf{area}(Z_n^*) \pmod 2$ if and only if $\phi_B(P)$ contains an odd number of up steps. The result follows. □

Proof of Theorem 1.4. Let $P \in A_{n+1}$. Suppose the Motzkin path $\phi_A(P)$ contains an even number of up steps. By Lemmas 4.1 and 4.2, $\mathsf{traj}(P)$ is odd if $n \equiv 0, 1 \pmod 4$, and even if $n \equiv 2, 3 \pmod 4$. On the other hand, suppose the Motzkin path $\phi_A(P)$ contains an odd number of up steps, it follows that $\mathsf{traj}(P)$ is even if $n \equiv 0, 1 \pmod 4$, and odd if $n \equiv 2, 3 \pmod 4$. Thus, the expression on the right hand side of (1.3) is the excess of the number of Dyck paths $P \in A_{n+1}$ with even $\mathsf{traj}(P)$ over the ones with odd $\mathsf{traj}(P)$, which equals the left hand side of (1.3). The assertion (i) of Theorem 1.4 follows.

Let $P \in B_n$. Suppose the Riordan path $\phi_B(P)$ contains an odd number of up steps. By Lemmas 4.3 and 4.4, $\mathsf{traj}(P)$ is odd if $n \equiv 0, 1 \pmod 2$, and even if $n \equiv 2, 3 \pmod 2$. On the other hand, suppose the Riordan path $\phi_B(P)$ contains an even number of up steps, it follows that $\mathsf{traj}(P)$ is even if $n \equiv 0, 1 \pmod 4$, and odd if $n \equiv 2, 3 \pmod 4$. Thus, the expression on the right hand side of (1.4) is the excess of the number of Dyck paths $P \in B_n$ with even $\mathsf{traj}(P)$ over the ones with odd $\mathsf{traj}(P)$, which equals the left hand side of (1.4). The assertion (ii) of Theorem 1.4 follows. $\square$

Acknowledgments

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