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THE ANALOGUE OF OVERLAP-FREENESS FOR THE FIBONACCI MORPHISM

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Abstract. A 4^--power is a non-empty word of the form $XXXX^-$, where X^- is obtained from X by erasing the last letter. A binary word is called *faux-bonacci* if it contains no 4^--powers, and no factor 11. We show that faux-bonacci words bear the same relationship to the Fibonacci morphism that overlap-free words bear to the Thue-Morse morphism. We prove the analogue of Fife's Theorem for faux-bonacci words, and characterize the lexicographically least and greatest infinite faux-bonacci words.

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1. Introduction

We study binary words, that is words over the alphabet $\mathcal{B} = \{0, 1\}$. We use lower case letters (*e.g.*, w) to denote finite words, and we use bold-face letters to denote words with letters indexed by $\mathbb{N}$; *e.g.*,

$$\boldsymbol{w} = w_1 w_2 w_3 \cdots .$$

In the literature, words with letters indexed by $\mathbb{N}$ are variously referred to as ω-words, infinite words, one-sided infinite words, *etc.* In this article we refer to them as ω*-words*. We freely use notions from combinatorics on words and from automata theory. Thus, for example, the set of finite words over $\mathcal{B}$ is denoted by $\mathcal{B}^*$, and the set of ω-words is denoted by $\mathcal{B}^\omega$. We record morphisms inline, *i.e.*, $g = [g(0), g(1)]$.

The binary overlap-free words constitute a classical object of study in combinatorics on words. They are particularly well understood because of their intimate connection to the Thue-Morse morphism $\mu = [01, 10]$. The following was proved by Thue [1].

Theorem 1.1. *Let w be binary word. Then w is overlap-free if and only if $\mu(w)$ is overlap-free.*

Thue [1] proved that, for two-sided infinite words and for circular words, every overlap-free binary word arises as the μ image of an overlap-free word. The analysis of finite words is more complicated, but these also arise *via* iterating μ. (See, *e.g.*, Restivo and Salemi [2]).

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Theorem 1.2. *Let $w \in \mathcal{B}^*$ be overlap-free. Then we can write $w = a\mu(u)b$, where $a, b \in \{\epsilon, 0, 00, 1, 11\}$, and u is overlap-free. If $|w| \geq 7$ this factorization is unique. If $\boldsymbol{w} \in \mathcal{B}^\omega$ is overlap-free, then we can write $\boldsymbol{w} = a\mu(\boldsymbol{u})$, for some overlap-free word $\boldsymbol{u} \in \mathcal{B}^\omega$ where $a \in \{\epsilon, 0, 00, 1, 11\}$.*

Characterizations of binary overlap-free words in terms of μ have allowed sharp enumerations of these words [2–7]. These enumerations are closely connected to a classical result known as Fife's Theorem [8].

Theorem 1.3 (Fife's Theorem)**.** *Let $\boldsymbol{u} \in \mathcal{B}^\omega$. Then $\boldsymbol{u}$ is overlap-free if and only if*

$$\boldsymbol{u} = w \bullet \boldsymbol{f}$$

for some $w \in \{01, 001\}$, and some $\boldsymbol{f} \in \{\alpha, \beta, \gamma\}^\omega$ containing no factor in

$$I = (\alpha + \beta)(\gamma\gamma)^*(\beta\alpha + \gamma\beta + \alpha\gamma).$$

In this theorem, each of α, β, and γ is an operator discovered by Fife that maps a finite word ending in $\mu^n(0)$ or $\mu^n(1)$ to a finite word ending in $\mu^{n+1}(0)$ or $\mu^{n+1}(1)$. The $\bullet$ operator is then defined recursively by

$$\begin{aligned} w \bullet \epsilon &= w \\ w \bullet (\boldsymbol{f}\alpha) &= \alpha(w \bullet \boldsymbol{f}) \\ w \bullet (\boldsymbol{f}\beta) &= \beta(w \bullet \boldsymbol{f}) \\ w \bullet (\boldsymbol{f}\gamma) &= \gamma(w \bullet \boldsymbol{f}). \end{aligned}$$

The second author's thesis [9] contains a modern exposition of Fife's Theorem.

The Thue-Morse sequence $\mathbf{t}$ is a fixed point of μ, namely,

$$\mathbf{t} = \lim_{n\to\infty} \mu^n(0)$$

Due to Theorem 1.2, $\mathbf{t}$ arises naturally in any study of overlap-free binary words. For example, Berstel [10] proved (see also Allouche *et al.* [11, 12]):

Theorem 1.4. *The lexicographically greatest overlap-free binary ω-word starting with 0 is* $\mathbf{t}$.

Our rich understanding of binary overlap-free words comes from the strong connection between these words and the Thue-Morse morphism. The thesis of the present article is that there exist analogous connections between other pairs of languages and morphisms. In a recent paper, the first author [13] showed such a connection between the period-doubling morphism $\delta = [01, 00]$ and *good words*. A binary word is *good* if it doesn't contain factors 11 or 1001, and doesn't encounter pattern 0000 or 00010100. He showed that:

- Good words factorize under δ;
- Word $\delta(w)$ is good if and only if w is good;
- An analog of Fife's Theorem holds for good ω-words;
- One can exhibit the lexicographically least and greatest good ω-words.

Unfortunately, one may object that the period-doubling morphism does not give a proper 'new' example of a language/morphism connection because of the close relationship of the period-doubling sequence $\boldsymbol{d}$ to the Thue-Morse sequence $\boldsymbol{t}$; it is well-known [14] that the period-doubling sequence can be obtained from the Thue-Morse sequence as follows: Let $\boldsymbol{vtm}$ be given by

$$\boldsymbol{vtm} = g^{-1}(\boldsymbol{t}),$$

where g is the morphism on $\{0,1,2\}$ given by $g = [011, 01, 0]$. Then

$$\boldsymbol{d} = h(\boldsymbol{vtm}),$$

where h is the morphism on $\{0,1,2\}$ given by $h = [0,1,0]$. For this reason, in this article we consider another morphism, not connected to μ in the same way.

Another famous binary sequence is the Fibonacci word, which is the fixed point

$$\boldsymbol{\phi} = 0100101001001010010100100101001001\cdots$$

of the binary morphism φ, where $\varphi = [01, 0]$. We call φ the **Fibonacci morphism**. The word $\boldsymbol{\phi}$ is central to the study of Sturmian words, and has a large literature. (See [15], for example).

For a non-empty word X, the word X^- is obtained from X by erasing its last letter. The word ^-X is obtained by erasing its first letter. We define a **4^--power** (said "four minus"-power) to be a word of the form $XXXX^-$, some non-empty word X. Equivalently, a 4^--power is a word of period p, length $4p-1$ for some positive p. Extending periodically on the right or left with period p by a single letter gives a fourth power. Thus a 4^--power $XXXX^-$ can also be written as ^-YYYY, where $Y = aXa^{-1}$ and a is the last letter of X.

A binary word is called *faux-bonacci* if it contains no factor 11 and no 4^--power. For the remainder of this paper we abbreviate 'faux-bonacci' as 'fb'. We will show that fb words bear the same relationship to the Fibonacci morphism that overlap-free words bear to the Thue-Morse morphism. We show that:

- The fb words factorize under φ (Thm. 2.3);
- Word $\varphi(w)$ is fb if and only if w is fb (Thm. 2.4);
- An analog of Fife's Theorem holds for fb ω-words (Thm. 3.11);
- One can exhibit the lexicographically least and greatest fb ω-words (Thm. 4.5).

These results raise the question of which well-studied properties of the Thue-Morse word $\boldsymbol{t}$ may generalize to the class of all morphic fixed points.

2. Faux-bonacci words

Unless otherwise specified, our words and morphisms are over the binary alphabet $\mathcal{B} = \{0,1\}$.

Lemma 2.1. *Let $\boldsymbol{u} \in \mathcal{B}^\omega$. Suppose $\varphi(\boldsymbol{u})$ is fb. Then $\boldsymbol{u}$ is fb.*

Proof. We prove the contrapositive: Suppose $\boldsymbol{u}$ is not fb; we prove that $\varphi(\boldsymbol{u})$ is not fb.

If 11 is a factor of $\boldsymbol{u}$, then one of $\varphi(110) = 0001$ and $\varphi(111) = 000$ is a factor of $\varphi(\boldsymbol{u})$. Each of these contains the 4^--power 000.

Suppose u contains an 4^--power $XXXX^-$ where $X = xa$ for some $a \in \mathcal{B}$. If $a = 1$, then $\varphi(u)$ contains $\varphi(x1x1x1x) = \varphi(x)0\varphi(x)0\varphi(x)0\varphi(x)$, which is an 4^--power. If $a = 0$, then u contains one of $x0x0x0x0$ and $x0x0x0x1$. In either case, $\varphi(u)$ contains the 4^--power $\varphi(x)01\varphi(x)01\varphi(x)01\varphi(x)0$. □

Remark 2.2. Let $w = w_1w_2w_3\cdots w_n$ with $w_i \in \mathcal{B}$ and suppose $|w|_{11} = 0$. There is a unique word u such that $0w = \varphi(u)$ for some u. Let $\boldsymbol{w} \in \mathcal{B}^\omega$ and suppose $|\boldsymbol{w}|_{11} = 0$. Then we can write $\boldsymbol{w} = a\varphi(\boldsymbol{u})$, some $\boldsymbol{u} \in \mathcal{B}^\omega$ where $a \in \{\epsilon, 1\}$.

Theorem 2.3. *Let $\boldsymbol{w} \in \mathcal{B}^\omega$ be fb. Then we can write $\boldsymbol{w} = a\varphi(\boldsymbol{u})$, where $\boldsymbol{u}$ is fb and where $a \in \{\epsilon, 1\}$.*

Proof. This is immediate from Lemma 2.1 and Remark 2.2. □

Theorem 2.4. *Let $\boldsymbol{w} \in \mathcal{B}^\omega$. Then $\varphi(\boldsymbol{w})$ is fb if and only if $\boldsymbol{w}$ is fb.*

Proof. The only if direction is Lemma 2.1. Suppose then that $\boldsymbol{w}$ is fb. Certainly $\varphi(\boldsymbol{w})$ cannot contain 11 as a factor. Further, if 000 is a factor of $\varphi(\boldsymbol{w})$, then one of 110 and 111 is a factor of $\boldsymbol{w}$; however, $\boldsymbol{w}$ is fb, so this is impossible.

Suppose that $\varphi(\boldsymbol{w})$ has a factor $XXXx$ where $X = xa$, some $a \in \mathcal{B}$.

If $a = 1$ then $XX = x1x1$. Since 11 is not a factor of $\varphi(\boldsymbol{w})$, x must be non-empty and have first letter 0. Write $X = \varphi(Y0)$. Then $XXXx = \varphi(Y0Y0Y0Y)0$, and $\boldsymbol{w}$ contains the overlap $Y0Y0Y0Y$. This is a contradiction, since $\boldsymbol{w}$ is fb.

Assume then that $a = 0$. If the first letter of X is 1, then the factor $XXXx = x0x0x0x$ of $\varphi(w)$ must appear in the context $0x0x0x0x$. Word x cannot be empty, since 000 is not a factor of $\varphi(\boldsymbol{w})$. If the last letter of x is 1, then replacing X by $0x$ reduces to the previous case. Suppose then that the last letter of x is 0. Write $x = x'0$. Then $XXXx = x'00x'00x'00x'0$. Since 000 is not a factor of $\varphi(\boldsymbol{w})$, x' is non-empty and starts and ends with 1. This implies that $\varphi(\boldsymbol{w})$ contains the factor $0XXXx' = 0x'00x'00x'00x'$. Write $0x'0 = \varphi(Y1)$. Then $0x'00x'00x'00x' = \varphi(Y1Y1Y1Y)$, and the 4^--power $Y1Y1Y1Y$ is a factor of $\boldsymbol{w}$. This is impossible. ☐

Corollary 2.5. *The Fibonacci word $\boldsymbol{\phi}$ is fb.*

Lemma 2.6. *Suppose that $\boldsymbol{w} \in \mathcal{B}^\omega$ is fb. Then 10101 is not a factor of $^-\boldsymbol{w}$.*

Proof. Suppose 10101 is a factor of $^-\boldsymbol{w}$. Since 11 is not a factor of $\boldsymbol{w}$, extending 10101 to the left and right we find that 0101010 is a factor of $\boldsymbol{w}$. But 0101010 is a 4^--power. ☐

Lemma 2.7. *Suppose $0101\boldsymbol{w} \in \mathcal{B}^\omega$ is fb. Then $10101\boldsymbol{w}$ is fb.*

Proof. If $10101\boldsymbol{w}$ is not fb it must begin with a 4^--power with some positive period p and length $4p-1$. If $p \geq 5$, then 10101 is a factor of the fb word $^-0101\boldsymbol{w}$. This is impossible by Lemma 2.6, so that $p \leq 4$. Thus p is a period of 10101, so that p is 2 or 4. This forces 1010101 to be a prefix of $10101\boldsymbol{w}$, and again 10101 is a factor of the fb word $^-0101\boldsymbol{w}$. ☐

3. An analogue of Fife's theorem

Let U be the set of infinite fb words. For $u \in \mathcal{B}^*$, let $U_u = U \cap u\mathcal{B}^\omega$.

Lemma 3.1. *Let $\boldsymbol{v} \in \mathcal{B}^\omega$. Then*

(i) $\varphi(\boldsymbol{v}) \in U \iff \boldsymbol{v} \in U$;
(ii) $1\varphi(\boldsymbol{v}) \in U \iff 0\boldsymbol{v} \in U$ *or* $\boldsymbol{v} \in U_{00}$.

Proof. Part (i) is Theorem 2.4. For part (ii), first suppose that $0\boldsymbol{v} \in U$ or $\boldsymbol{v} \in U_{00}$. If $0\boldsymbol{v} \in U$, then by Theorem 2.4 it follows that $01\varphi(\boldsymbol{v}) = \varphi(0\boldsymbol{v})$ is fb, so in particular $1\varphi(\boldsymbol{v})$ is fb; if $\boldsymbol{v} \in U_{00}$, then $\varphi(\boldsymbol{v})$ is fb by Theorem 2.4 and has prefix 0101, so that $1\varphi(\boldsymbol{v})$ is fb by Lemma 2.7.

In the other direction, suppose that $1\varphi(\boldsymbol{v}) \in U$. To get a contradiction, suppose that $0\boldsymbol{v} \notin U$ and $\boldsymbol{v} \notin U_{00}$. Since $1\varphi(\boldsymbol{v}) \in U$, we must have $\varphi(\boldsymbol{v}) \in U$, so that $\boldsymbol{v} \in U$ by Theorem 2.4. From $0\boldsymbol{v} \notin U$ we deduce that a prefix of $0\boldsymbol{v}$ is not fb. Since 11 cannot be a prefix of $0\boldsymbol{v}$, we deduce that $0\boldsymbol{v}$ has a prefix of the form $XXXX^-$ for some non-empty word X. It follows that 0 is a prefix of X. Since $\boldsymbol{v} \notin U_{00}$, we conclude that $|X| \geq 2$. Since XX is a factor of the fb word $\boldsymbol{v}$, and XX has prefix $X0$, we conclude that 00 is not a suffix of X; otherwise $\boldsymbol{v}$ would have factor $000 = 0000^-$, which is impossible.

Let $\boldsymbol{v}$ have prefix $^-XXXX^-a$, where $a \in \mathcal{B}$. Letter a cannot be the last letter of X, or $\boldsymbol{v}$ would start with the 4^--power ^-XXXX. It follows that X^- is followed in $^-XXXX^-a$ variously by a and by the other letter of $\mathcal{B}$. This implies that the last letter of X^- is 0, since 1 cannot be followed by 1 in $\boldsymbol{v}$. Since 00 is not a suffix of X, word X must end in 01.

We cannot, however, have $X = 01$. In this case, word $\boldsymbol{v}$ would have prefix $^-XXXX^-0 = 1010100$, causing $1\varphi(\boldsymbol{v})$ to have prefix 100100100101, which begins with the 4^--power 10010010010. It follows that $|X| \geq 3$. Since X^- is sometimes followed by 0 in $\boldsymbol{v}$, word X^- cannot have suffix 00, since 000 is not a factor of $\boldsymbol{v}$. As the last letter of X^- is 0, this implies that X^- ends in 10. Now X ends in 101, and XX is a factor of $\boldsymbol{v}$. Thus Lemma 2.6 implies that X does not start 01. It follows that X starts 00. Write $X = 00Y101$ and $\boldsymbol{v} =^- XXXX^-0\boldsymbol{u}$. Then

$$1\varphi(\boldsymbol{v}) = 101\varphi(Y)00100101\varphi(Y)00100101\varphi(Y)00100101\varphi(Y)00101\varphi(\boldsymbol{u})$$

which begins with the 4^--power $ZZZZ^-$ where $Z = 101\varphi(Y)00100$. This is a contradiction. ☐

Consider the *finite Fibonacci words* F_n defined for non-negative integers n by

$$\begin{aligned} F_0 &= 0, \\ F_1 &= 01, \text{ and} \\ F_{n+2} &= F_{n+1}F_n \text{ for } n \geq 0. \end{aligned}$$

The following lemma is proved by induction.

Lemma 3.2. *Suppose that u is a binary word and p is a prefix of $\varphi(u)$. If p ends in F_{n+1} for some $n \geq 2$, then $p = \varphi(q)$, where q is a prefix of u ending in F_n.*

Remark 3.3. The condition $n \geq 2$ is necessary. If $u = 00$, then F_2 is a factor of $\varphi(u)$, but F_1 is not a factor of u.

We use the notation $\pi(w)$ for the *Parikh vector* of a binary word. Thus

$$\pi(w) = [|w|_0, |w|_1].$$

Suppose that w is a word of the form y_nF_n where $\pi(y_n) \leq \pi(F_n) - \pi(0)$. In particular, F_n is the longest finite Fibonacci word which is a suffix of w. We define operations on w by

$$\begin{aligned} \alpha(w) &= y_nF_{n+1} \\ \beta(w) &= y_nF_{n-1}F_{n+1}. \end{aligned}$$

One checks that F_n is a prefix of $F_{n-1}F_{n+1}$, so that w is always a prefix of $\alpha(w)$ and $\beta(w)$. Because $\pi(y_n) \leq \pi(F_n) - \pi(0)$, we have that $\pi(y_nF_{n-1}) \leq \pi(F_nF_{n-1}) - \pi(0) = \pi(F_{n+1}) - \pi(0)$, so that we can iterate the maps α and β. Let $\mathcal{O} = \{\alpha, \beta\}$. We define operators $w \bullet f$ for $f \in \mathcal{O}^*$ by

$$\begin{aligned} w \bullet \epsilon &= w \\ w \bullet (f\alpha) &= \alpha(w \bullet f) \\ w \bullet (f\beta) &= \beta(w \bullet f). \end{aligned}$$

Example 3.4. Let $w = 0010$ and $f = \beta\alpha$. We write $w = y_2F_2$ where $y_2 = 0$. Thus $\pi(y_n) = [1,0] \leq [2,1] - [1,0] = \pi(F_2) - \pi(0)$. Then

$$\beta(w) = y_2F_1F_3 = 0(01)(01001) = y_3F_3,$$

where $y_3 = 001$. As remarked earlier, $\pi(y_3) = [2,1] \leq [3,2] - [1,0] = \pi(F_3) - \pi(0)$, so that $\alpha(\beta(w))$ is defined:

$$\alpha(\beta(w)) = y_3F_4 = 001(01001010).$$

Thus

$$w \bullet \beta\alpha = \alpha(\beta(w)) = 00101001010.$$

Remark 3.5. If g is a prefix of f, then $w \bullet g$ will be a prefix of $w \bullet f$.

Lemma 3.6. *Let $\mathbf{u}$ be a fb ω-word, and for $n \geq 2$, let $w_n = y_nF_n$ be the shortest prefix of $\mathbf{u}$ ending in F_n. For $n \geq 2$, we have $\pi(y_n) \leq \pi(F_n) - \pi(0)$.*

Proof. Word w_2 is a fb word containing $F_2 = 010$ exactly once, as a suffix. The candidates are 010, 0010, 1010, and 10010, which would yield $y_2 = \epsilon$, 0, 1, and 10, respectively. Thus the result is true for $n = 2$.

Suppose the result has been found to be true for $n = k$, some $k \geq 2$. Write $\boldsymbol{u} = a\varphi(\boldsymbol{u'})$ where $\boldsymbol{u'}$ is fb, and $a \in \{\epsilon, 1\}$. Since the first letter of F_{k+1} is 0 and $a \in \{\epsilon, 1\}$, any occurrence of F_{k+1} in $\boldsymbol{u}$ starts in $\varphi(\boldsymbol{u'})$. By Lemma 3.2, any prefix qF_{k+1} of $\boldsymbol{u}$ has the form $a\varphi(q'F_k)$ where $q'F_k$ is a prefix of $\boldsymbol{u'}$. Thus $w_{k+1} = a\varphi(w'_k)$ where $w'_k = y'_k F_k$ is the shortest prefix of $\boldsymbol{u'}$ ending in F_k. By the induction hypothesis, $\pi(y'_k) \leq \pi(F_k) - \pi(0)$. It follows that

$$\begin{aligned}\pi(a\varphi(y'_k)) &\leq \pi(1) + \pi(\varphi(F_k)) - \pi(\varphi(0)) \\ &= \pi(1) + \pi(F_{k+1}) - \pi(01) \\ &= \pi(F_{k+1}) - \pi(0).\end{aligned}$$

□

Remark 3.7. Again, the condition $n \geq 2$ is necessary. If $w_2 = 1001$, then $y_2 = 10$, and $\pi(y_2) \not\leq \pi(F_2) - \pi(0)$.

Lemma 3.8. *Let $\boldsymbol{u}$ be a fb ω-word, and for $n \geq 2$, let $w_n = y_n F_n$ be the shortest prefix of $\boldsymbol{u}$ ending in F_n. For $n \geq 2$, we have $w_{n+1} \in \{w_n \bullet \alpha, w_n \bullet \beta\}$.*

Proof. The only fb words starting with F_2, and containing F_3 exactly once, as a suffix, are $01001 = 010 \bullet \alpha$ and $0101001 = 010 \bullet \beta$. Thus the result is true for $n = 2$.

Suppose the result has been found to be true for $n = k$, some $k \geq 2$. Write $\boldsymbol{u} = a\varphi(\boldsymbol{u'})$ where $\boldsymbol{u'}$ is fb, and $a \in \{\epsilon, 1\}$. For each n, let $w'_n = y'_n F_n$ be the shortest prefix of $\boldsymbol{u'}$ ending in F_n. By the induction hypothesis, w'_{k+1} is either $w'_k \bullet \alpha = y'_k F_{k+1}$ or $w'_k \bullet \beta = y'_k F_{k-1} F_{k+1}$. As in the previous proof, $w_{k+2} = a\varphi(w'_{k+1})$ and $y_{k+2} = a\varphi(y'_{k+1})$. Then

$$\begin{aligned}w_{k+2} &= a\varphi(w'_{k+1}) \\ &\in \{a\varphi(y'_k F_{k+1}), a\varphi(y'_k F_{k-1} F_{k+1})\} \\ &= \{y_{k+1} F_{k+2}, y_{k+1} F_k F_{k+2}\} \\ &= \{w_{k+1} \bullet \alpha, w_{k+1} \bullet \beta\}.\end{aligned}$$

□

Corollary 3.9. *Let $\boldsymbol{u}$ be a fb ω-word. There is an ω-word $\boldsymbol{f} = \Pi_{n=2}^{\infty} f_n$, $f_n \in \mathcal{O}$, and a 'seed' word $w_2 \in \{010, 0010, 1010, 10010\}$, such that*

$$\boldsymbol{u} = \lim_{n\to\infty} w_2 \bullet (f_2 f_3 \cdots f_n).$$

Proof. Letting the w_n be as in the previous lemma, choose f_n such that $w_{n+1} = w_n \bullet f_n$. □

Suppose that w is a word of the form $y_n F_n$ where $\pi(y_n) \leq \pi(F_n) - \pi(0)$. We note that

$$\begin{aligned}w \bullet \alpha &= wF_n^{-1}F_{n+1} = wF_n^{-1}\varphi^{n-2}(F_2 \bullet \alpha), \\ w \bullet \beta &= wF_n^{-1}F_{n-1}F_{n+1} = wF_n^{-1}\varphi^{n-2}(F_2 \bullet \beta).\end{aligned}$$

If $f \in \mathcal{O}^k$, write $f = f_1 f_2 \cdots f_k$, where each $f_i \in \mathcal{O}$. Suppose that w is a word of the form $y_2 F_2$ where $\pi(y_2) \leq \pi(F_2) - \pi(0)$. By induction, $w \bullet f$ is a word of the form $y_{k+2} F_{k+2}$ where $\pi(y_{k+2}) \leq \pi(F_{k+2}) - \pi(0)$. We find that

$$\begin{aligned}&w \bullet f \\ &= (w \bullet f_1 f_2 \cdots f_{k-1}) \bullet f_k\end{aligned}$$

$$
\begin{aligned}
&= (w \bullet f_1 f_2 \cdots f_{k-1}) F_{k+1}^{-1} \varphi^{k-1}(F_2 \bullet f_k) \\
&= (w \bullet f_1 f_2 \cdots f_{k-2}) F_k^{-1} \varphi^{k-2}(F_2 \bullet f_{k-1}) F_{k+1}^{-1} \varphi^{k-1}(F_2 \bullet f_k) \\
&\quad \vdots \\
&= w \Pi_{j=1}^{k} F_{j+1}^{-1} \varphi^{j-1}(F_2 \bullet f_j) \qquad (3.1)
\end{aligned}
$$

Let $\boldsymbol{f} \in \mathcal{O}^\omega$, $\boldsymbol{f} = f_1 f_2 f_3 \cdots$, where each $f_i \in \mathcal{O}$. For $w \in \{010, 0010, 1010, 10010\}$, define

$$
w \bullet \boldsymbol{f} = \lim_{n \to \infty} w \bullet (f_1 f_2 f_3 \cdots f_n).
$$

Lemma 3.10. *Suppose* $w \in \{010, 0010, 1010, 10010\}$ *and* $\boldsymbol{f} \in \mathcal{O}^\omega$ *and* $g \in \mathcal{O}^k$*. Suppose* $F_2 \bullet \boldsymbol{f} = \boldsymbol{x}$*. Then*

$$
w \bullet (g\boldsymbol{f}) = (w \bullet g) F_{k+2}^{-1} \varphi^k(\boldsymbol{x}). \qquad (3.2)
$$

Proof. Write $\boldsymbol{f} = f_1 f_2 f_3 \cdots$ and $g = g_1 g_2 \cdots g_k$ where the f_i, $g_i \in \mathcal{O}$. Suppose that $F_2 \bullet f_1 \cdots f_n = x_n$. We will show that $w \bullet (g_1 g_2 \cdots g_k f_1 f_2 \cdots f_n) = (w \bullet g) F_{k+2}^{-1} \varphi^k(x_n)$, and the result follows by taking limits. From (3.1),

$$
x_n = F_2 \bullet f_1 \cdots f_n = F_2 \Pi_{j=1}^{n} F_{j+1}^{-1} \varphi^{j-1}(F_2 \bullet f_j)
$$

and

$$
\begin{aligned}
&w \bullet (g_1 g_2 \cdots g_k f_1 f_2 \cdots f_n) \\
&= w \Pi_{j=1}^{k} F_{j+1}^{-1} \varphi^{j-1}(F_2 \bullet g_j) \Pi_{j=1}^{n} F_{k+j+1}^{-1} \varphi^{k+j-1}(F_2 \bullet f_j) \\
&= (w \bullet g) \Pi_{j=1}^{n} \varphi^k(F_{j+1})^{-1} \varphi^k(\varphi^{j-1}(F_2 \bullet f_j)) \\
&= (w \bullet g) \varphi^k(F_2^{-1} F_2 \Pi_{j=1}^{n} F_{j+1}^{-1} \varphi^{j-1}(F_2 \bullet f_j)) \\
&= (w \bullet g) \varphi^k(F_2)^{-1} \varphi^k(F_2 \Pi_{j=1}^{n} F_{j+1}^{-1} \varphi^{j-1}(F_2 \bullet f_j))) \\
&= (w \bullet g) F_{k+2}^{-1} \varphi^k(x_n),
\end{aligned}
$$

as desired. □

Define sets F and V by

$$
F = \alpha^* \beta(\alpha\alpha + \alpha\beta + \beta\alpha\alpha + \beta\beta\beta^*\alpha\alpha)^*(\beta\alpha\beta + \beta\beta\beta^*\alpha\beta),
$$

$$
V = \mathcal{O}^\omega - \mathcal{O}^* F \mathcal{O}^\omega.
$$

Theorem 3.11. *Let* $\boldsymbol{x} \in \mathcal{B}^\omega$*. If* $\boldsymbol{x}$ *begins with* 010*, then* $\boldsymbol{x}$ *is fb if and only if* $\boldsymbol{x} = 010 \bullet \boldsymbol{f}$ *for some* $\boldsymbol{f} \in V$*.*

The set F consists of forbidden factors for words of V.

Let $W = \{\boldsymbol{f} \in \mathcal{O}^\omega : 010 \bullet \boldsymbol{f} \in U\}$. To prove Theorem 3.11 it is enough to prove that $W = V$. Let $L \subseteq \mathcal{O}^\omega$ and let $x \in \mathcal{O}^*$. We define the (left) quotient $x^{-1}L$ by $x^{-1}L = \{\boldsymbol{y} \in \mathcal{O}^\omega : x\boldsymbol{y} \in L\}$. The next lemma establishes several identities concerning quotients of the set W. They are proved using (3.2) and Lemma 3.1. The identities demonstrate that W is precisely the set of infinite labeled paths through the automaton A_{010} given in Figure 1. These are just the labeled paths omitting factors in F, so that $W = V$. Thus, proving Lemma 3.12 establishes Theorem 3.11.

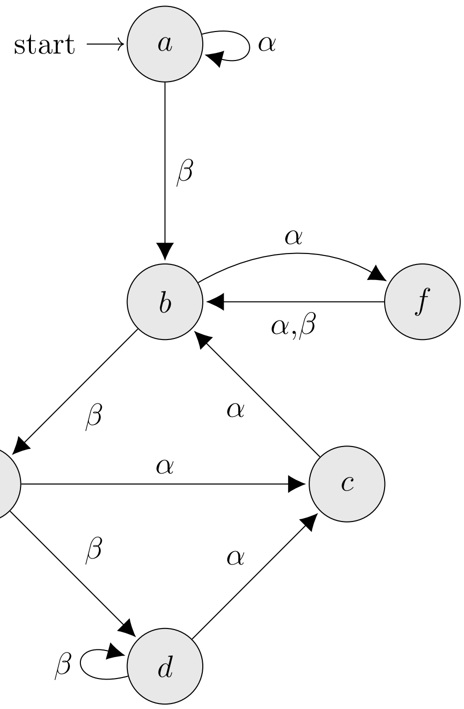

FIGURE 1. 'Fife' automaton A_{010} for U.

Lemma 3.12. *The following identities hold:*

(a) $W = \alpha^{-1}W$*;*
(b) $\beta^{-1}W = (\beta\alpha\alpha)^{-1}W = (\beta\alpha\beta)^{-1}W^{-1} = (\beta\beta\alpha\alpha)^{-1}W$*;*
(c) $(\beta\beta\alpha)^{-1}W = (\beta\beta\beta\alpha)^{-1}W$*;*
(d) $(\beta\beta\beta)^{-1}W = (\beta\beta\beta\beta)^{-1}W$*;*
(e) $(\beta\beta\alpha\beta)^{-1}W = \emptyset$*.*

Each set of identities corresponds to the state of A_{010} with the same label as the identities. There are two further states: (f), corresponding to $(\beta\alpha)^{-1}W$, and (g), corresponding to $(\beta\beta)^{-1}W$ The non-accepting sink (e) is not shown in the figure. The automaton A_{010} is not minimal; for example, (d) can be identified with (g). However, this form highlights parallels between it and the three automata we present later.

Proof. (a) We have

$$\begin{aligned}
&\alpha\boldsymbol{f} \in W \\
\iff\ & 010 \bullet \alpha\boldsymbol{f} \in U \\
\iff\ & (010 \bullet \alpha)\varphi(F_2)^{-1}\varphi(\boldsymbol{x}) \in U \text{ by (3.2)} \\
\iff\ & \underline{01001(01001)}^{-1}\varphi(\boldsymbol{x}) \in U \\
\iff\ & \boldsymbol{x} \in U \text{ by Lemma 3.1 (i)} \\
\iff\ & 010 \bullet \boldsymbol{f} \in U \\
\iff\ & \boldsymbol{f} \in W,
\end{aligned}$$

so that $\alpha^{-1}W = W$.

(b) Here

$$\begin{aligned}
&\beta \boldsymbol{f} \in W \\
\iff & 010 \bullet \beta \boldsymbol{f} \in U \\
\iff & (010 \bullet \beta)\varphi(F_2)^{-1}\varphi(\boldsymbol{x}) \in U \\
\iff & 01\ \sout{01001(01001)^{-1}}\varphi(\boldsymbol{x}) \in U \\
\iff & \varphi(0\boldsymbol{x}) \in U \\
\iff & 0\boldsymbol{x} \in U \text{ by Lemma 3.1 (i).}
\end{aligned}$$

Similarly we find that

$$\begin{aligned}
&\beta\alpha\alpha \boldsymbol{f} \in W \\
\iff & 010 \bullet \beta\alpha\alpha \boldsymbol{f} \in U \\
\iff & (010 \bullet \beta\alpha\alpha)\varphi^3(F_2)^{-1}\varphi^3(\boldsymbol{x}) \in U \\
\iff & 01\ \sout{0100101001001(0100101001001)^{-1}}\varphi^3(\boldsymbol{x}) \in U \\
\iff & \varphi(0\varphi^2(\boldsymbol{x})) \in U \\
\iff & 0\varphi^2(\boldsymbol{x}) \in U \text{ by Lemma 3.1 (i)} \\
\iff & \varphi(1\varphi(\boldsymbol{x})) \in U \\
\iff & 1\varphi(\boldsymbol{x}) \in U \text{ by Lemma 3.1 (i)} \\
\iff & 0\boldsymbol{x} \in U \text{ or } \boldsymbol{x} \in U_{00} \text{ by Lemma 3.1 (ii)} \\
\iff & 0\boldsymbol{x} \in U.
\end{aligned}$$

Here we use the fact that 01 is a prefix of $\boldsymbol{x}$, so that $\boldsymbol{x} \notin U_{00}$.
Again,

$$\begin{aligned}
&\beta\alpha\beta \boldsymbol{f} \in W \\
\iff & 010 \bullet \beta\alpha\beta \boldsymbol{f} \in U \\
\iff & (010 \bullet \beta\alpha\beta)\varphi^3(F_2)^{-1}\varphi^3(\boldsymbol{x}) \in U \\
\iff & 0101001\ \sout{0100101001001(0100101001001)^{-1}}\varphi^3(\boldsymbol{x}) \in U \\
\iff & \varphi(0010\varphi^2(\boldsymbol{x})) \in U \\
\iff & \varphi(101\varphi(\boldsymbol{x})) \in U \\
\iff & 101\varphi(\boldsymbol{x}) \in U \\
\iff & 1\varphi(0\boldsymbol{x}) \in U \\
\iff & 00\boldsymbol{x} \in U \text{ or } 0\boldsymbol{x} \in U_{00} \\
\iff & 0\boldsymbol{x} \in U.
\end{aligned}$$

Here we use the fact that 0 is a prefix of $\boldsymbol{x}$, so that 00 is a prefix of $0\boldsymbol{x}$.
Finally we get

$$\begin{aligned}
&\beta\beta\alpha\alpha \boldsymbol{f} \in W \\
\iff & 010 \bullet \beta\beta\alpha\alpha \boldsymbol{f} \in U \\
\iff & (010 \bullet \beta\beta\alpha\alpha)\varphi^4(F_2)^{-1}\varphi^4(\boldsymbol{x}) \in U
\end{aligned}$$

$$
\begin{aligned}
&\iff 01010\cancel{010010100100101001010(010010100100101001010)^{-1}}\varphi^4(\boldsymbol{x}) \in U\\
&\iff \varphi(001\varphi^3(\boldsymbol{x})) \in U\\
&\iff 001\varphi^3(\boldsymbol{x}) \in U\\
&\iff \varphi(10\varphi^2(\boldsymbol{x})) \in U\\
&\iff 10\varphi^2(\boldsymbol{x}) \in U\\
&\iff 1\varphi(1\varphi(\boldsymbol{x})) \in U\\
&\iff 01\varphi(\boldsymbol{x}) \in U \text{ or } 1\varphi(\boldsymbol{x}) \in U_{001}\\
&\iff 01\varphi(\boldsymbol{x}) \in U\\
&\iff \varphi(0\boldsymbol{x}) \in U\\
&\iff 0\boldsymbol{x} \in U.
\end{aligned}
$$

Thus $\beta^{-1}W = (\beta\alpha\alpha)^{-1}W = (\beta\alpha\beta)^{-1}W = (\beta\beta\alpha\alpha)^{-1}W$, as desired.

(c) Here

$$
\begin{aligned}
&\beta\beta\alpha\boldsymbol{f} \in W\\
&\iff 010 \bullet \beta\beta\alpha\boldsymbol{f} \in U\\
&\iff (010 \bullet \beta\beta\alpha)\varphi^3(F_2)^{-1}\varphi^3(\boldsymbol{x}) \in U\\
&\iff 01010\ \cancel{0100101001001(0100101001001)^{-1}}\varphi^3(\boldsymbol{x}) \in U\\
&\iff \varphi(001\varphi^2(\boldsymbol{x})) \in U\\
&\iff 001\varphi^2(\boldsymbol{x}) \in U\\
&\iff 10\varphi(\boldsymbol{x}) \in U\\
&\iff 01\boldsymbol{x} \in U \text{ or } 1\boldsymbol{x} \in U_{00}\\
&\iff 01\boldsymbol{x} \in U.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\beta\beta\beta\alpha\boldsymbol{f} \in W\\
&\iff 010 \bullet \beta\beta\beta\alpha\boldsymbol{f} \in U\\
&\iff (010 \bullet \beta\beta\beta\alpha)\varphi^4(F_2)^{-1}\varphi^4(\boldsymbol{x}) \in U\\
&\iff 0101001001\cancel{010010100100101001010(010010100100101001010)^{-1}}\varphi^4(\boldsymbol{x}) \in U\\
&\iff \varphi(001010\varphi^3(\boldsymbol{x})) \in U\\
&\iff 001010\varphi^3(\boldsymbol{x}) \in U\\
&\iff \varphi(1001\varphi^2(\boldsymbol{x})) \in U\\
&\iff 1001\varphi^2(\boldsymbol{x}) \in U\\
&\iff 1\varphi(10\varphi(\boldsymbol{x})) \in U\\
&\iff 010\varphi(\boldsymbol{x}) \in U \text{ or } 10\varphi(\boldsymbol{x}) \in U_{001}\\
&\iff 010\varphi(\boldsymbol{x}) \in U\\
&\iff \varphi(01\boldsymbol{x}) \in U\\
&\iff 01\boldsymbol{x} \in U.
\end{aligned}
$$

Thus $(\beta\beta\alpha)^{-1}W = (\beta\beta\beta\alpha)^{-1}W$, as desired.

(d) We have

$$
\begin{aligned}
&\beta\beta\beta\boldsymbol{f} \in W \\
\iff{}& 010 \bullet \beta\beta\beta\boldsymbol{f} \in U \\
\iff{}& (010 \bullet \beta\beta\beta)\varphi^3(F_2)^{-1}\varphi^3(\boldsymbol{x}) \in U \\
\iff{}& 0101001001\ \cancel{0100101001001(0100101001001)^{-1}}\varphi^3(\boldsymbol{x}) \in U \\
\iff{}& \varphi(001010\varphi^2(\boldsymbol{x})) \in U \\
\iff{}& 001010\varphi^2(\boldsymbol{x}) \in U \\
\iff{}& 1001\varphi(\boldsymbol{x}) \in U \\
\iff{}& 010\boldsymbol{x} \in U \text{ or } 10\boldsymbol{x} \in U_{00} \\
\iff{}& 010\boldsymbol{x} \in U.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\beta\beta\beta\beta\boldsymbol{f} \in W \\
\iff{}& 010 \bullet \beta\beta\beta\alpha\boldsymbol{f} \in U \\
\iff{}& (010 \bullet \beta\beta\beta\alpha)\varphi^4(F_2)^{-1}\varphi^4(\boldsymbol{x}) \in U \\
\iff{}& 010100100101001010\cancel{010010100100101001010(010010100100101001010)^{-1}}\varphi^4(\boldsymbol{x}) \in U \\
\iff{}& \varphi(00101001001\varphi^3(\boldsymbol{x})) \in U \\
\iff{}& 00101001001\varphi^3(\boldsymbol{x}) \in U \\
\iff{}& \varphi(1001010\varphi^2(\boldsymbol{x})) \in U \\
\iff{}& 1001010\varphi^2(\boldsymbol{x}) \in U \\
\iff{}& 1\varphi(1001\varphi(\boldsymbol{x})) \in U \\
\iff{}& 01001\varphi(\boldsymbol{x}) \in U \text{ or } 1001\varphi(\boldsymbol{x}) \in U_{001} \\
\iff{}& 01001\varphi(\boldsymbol{x}) \in U \\
\iff{}& \varphi(010\boldsymbol{x}) \in U \\
\iff{}& 010\boldsymbol{x} \in U.
\end{aligned}
$$

Thus $(\beta\beta\beta)^{-1}W = (\beta\beta\beta\beta)^{-1}W$, as desired.

(e) We have

$$
\begin{aligned}
&\beta\beta\alpha\beta\boldsymbol{f} \in W \\
\iff{}& 010 \bullet \beta\beta\alpha\beta\boldsymbol{f} \in U \\
\iff{}& (010 \bullet \beta\beta\alpha\beta)\varphi^4(F_2)^{-1}\varphi^4(\boldsymbol{x}) \in U \\
\iff{}& 01010010010100\cancel{010010100100101001010(010010100100101001010)^{-1}}\varphi^4(\boldsymbol{x}) \in U \\
\iff{}& \varphi(00101001\varphi^3(\boldsymbol{x})) \in U \\
\iff{}& 00101001\varphi^3(\boldsymbol{x}) \in U \\
\iff{}& \varphi(10010\varphi^2(\boldsymbol{x})) \in U \\
\iff{}& 10010\varphi^2(\boldsymbol{x}) \in U \\
\iff{}& 1\varphi(101\varphi(\boldsymbol{x})) \in U \\
\iff{}& 0101\varphi(\boldsymbol{x}) \in U \text{ or } 101\varphi(\boldsymbol{x}) \in U_{001}
\end{aligned}
$$

$$\iff 0101\varphi(\boldsymbol{x}) \in U$$
$$\iff \varphi(00\boldsymbol{x}) \in U$$
$$\iff 00\boldsymbol{x} \in U.$$

However, $\boldsymbol{x}$ has prefix 0, so $00\boldsymbol{x}$ has the 4^{-}-power 000 as a prefix. Therefore, $00\boldsymbol{x} \notin U$. It follows that $(\beta\beta\alpha\beta)^{-1}W = \emptyset$.

□

Remark 3.13. We mention without proof that

$$\beta\alpha \bullet \boldsymbol{f} \in W \iff 1\boldsymbol{f} \in V,$$

and

$$\beta\beta \bullet \boldsymbol{f} \in W \iff 10\boldsymbol{f} \in V.$$

We do not need these equivalences to formulate the automaton.

Remark 3.14. As Fife's Theorem features a forbidden factor characterization, we have given such a characterization for V. In fact, however, all information about V is captured in the automaton A_{010}. For a finite string $g \in \mathcal{O}^*$, word $010 \bullet g$ is fb exactly when g can be walked on the automaton A_{010}; such a string g never encounters the (undepicted) non-acepting sink (e), which can only be reached *via* (c) on input β. If $\boldsymbol{f} \in \mathcal{O}^\omega$, word $010 \bullet \boldsymbol{f}$ is fb exactly when $010 \bullet f$ is fb for finite prefix f of $\boldsymbol{f}$.

It is routine to write down an expression for the regular language of finite words arriving at the sink, which is $F\mathcal{O}^*$. It happens F will take us from any given state to the sink. For this reason, $010 \bullet g$ is fb exactly when g has no *factor* in F.

For $u \in \{010, 0010, 1010, 10010\}$, let $W_u = \{\boldsymbol{f} \in \mathcal{O}^\omega : u \bullet \boldsymbol{f} \in U\}$. Using the same method as in Lemma 3.12, one shows that W_u is the subset of $\mathcal{O}^\omega$ which can be walked on A_u, where the additional automata are depicted in Figures 2, 3, and 4.

4. Lexicographically extremal FB words

We use the usual lexicographic order and binary words. Note that the morphism φ is order-reversing: Let u and v be non-empty binary words so that $u < v$. Write $u = u'0u''$, $v = u'1v''$ where u' is the longest common prefix of u and v. Then $\varphi(u')01$ is a prefix of $\varphi(u)$, while $\varphi(u')00$ is a prefix of $\varphi(v)$, so that $\varphi(u) > \varphi(v)$.

For each non-negative integer n, let ℓ_n (resp., m_n) be the lexicographically least (resp., greatest) word of length n such that ℓ_n (resp., m_n) is the prefix of an fb ω-word.

Lemma 4.1. *Let n be a non-negative integer. Word ℓ_n is a prefix of ℓ_{n+1}. Word m_n is a prefix of m_{n+1}.*

Proof. We prove the result for the ℓ_n; the proof for the m_n is similar. Let $\ell_n\mathbf{r}$ be an fb ω-word. Let p be the length $n+1$ prefix of $\ell_n\mathbf{r}$, and let q be the length n prefix of ℓ_{n+1}. We need to show that $q = \ell_n$. Both p and q are prefixes of fb ω-words. By definition we have $\ell_{n+1} \le p$ and $\ell_n \le q$. If $\ell_n < q$, then $p^- = \ell_n < q = \ell_{n+1}^-$, so that $p < \ell_{n+1}$. This is a contradiction. Therefore $\ell_n = q$, as desired. □

Let $\boldsymbol{\ell} = lim_{n\to\infty}\ell_n$, $\mathbf{m} = lim_{n\to\infty}m_n$.

Lemma 4.2. *Word $\boldsymbol{\ell}$ is the lexicographically least fb ω-word. Word $\mathbf{m}$ is the lexicographically greatest fb ω-word.*

Proof. We show that $\boldsymbol{\ell}$ is lexicographically least. The proof that $\mathbf{m}$ is lexicographically greatest is similar. Let $\mathbf{w}$ be an fb ω-word. For each n let w_n be the length n prefix of $\mathbf{w}$, so that $\mathbf{w} = \lim_{n\to\infty} w_n$.

If for some n we have $w_n > \ell_n$, then $\mathbf{w} > \boldsymbol{\ell}$.

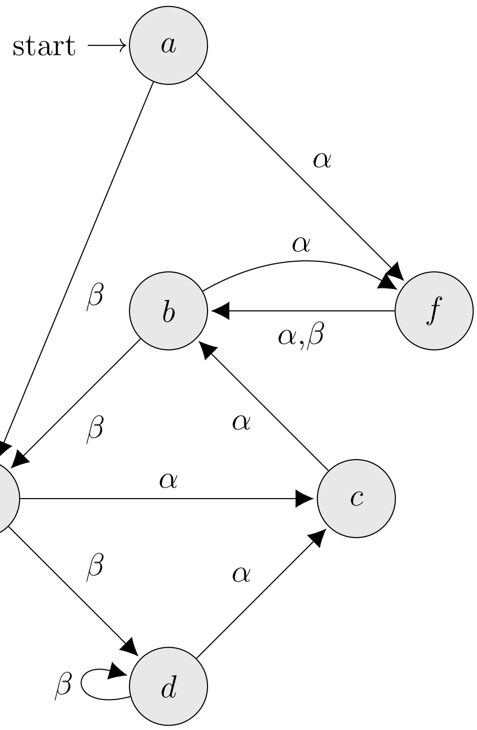

FIGURE 2. 'Fife' automaton A_{0010} for W_{0010}.

Otherwise $w_n \leq \ell_n$ for all n. By the definition of the ℓ_n we have $w_n \geq \ell_n$, so that $w_n = \ell_n$ for all n. Thus $\mathbf{w} = \lim_{n\to\infty} w_n = \lim_{n\to\infty} \ell_n = \boldsymbol{\ell}$.

In all cases we find $\mathbf{w} \geq \boldsymbol{\ell}$. □

Lemma 4.3. *We have* $\boldsymbol{\ell} = \varphi(\mathbf{m})$.

Proof. Since the Fibonacci word $\boldsymbol{\phi}$ has suffixes beginning with 00, $\ell_2 = 00$, and we can write $\boldsymbol{\ell} = \varphi(\mathbf{m}')$ for some $\mathbf{m}'$ by Theorem 2.3. Since $\boldsymbol{\ell}$ is fb, $\mathbf{m}'$ is fb by Lemma 2.1. It follows that $\mathbf{m}' \leq \mathbf{m}$. However if $\mathbf{m}' < \mathbf{m}$ then $\varphi(\mathbf{m}) < \varphi(\mathbf{m}') = \boldsymbol{\ell}$ since φ is order-reversing. This is impossible, since $\boldsymbol{\ell}$ is least. Therefore $\mathbf{m}' = \mathbf{m}$, and $\boldsymbol{\ell} = \varphi(\mathbf{m})$. □

Lemma 4.4. *We have* $\mathbf{m} = 1\varphi(\boldsymbol{\ell})$.

Proof. Since 00 is a prefix of $\boldsymbol{\ell}$ we see that 0101 is a prefix of $\varphi(\boldsymbol{\ell})$. It follows from Lemma 2.7 that $1\varphi(\boldsymbol{\ell})$ is fb. Since 10101 is the lexicographically greatest fb word of length 5, $m_5 = 10101$. It follows that we can write $\boldsymbol{m} = 1\varphi(\boldsymbol{\ell}')$ for some fb word $\boldsymbol{\ell}'$. However, φ is order reversing, so that if $\boldsymbol{\ell}' > \boldsymbol{\ell}$, then $1\varphi(\boldsymbol{\ell}) > 1\varphi(\boldsymbol{\ell}') = \boldsymbol{m}$, contradicting the maximality of $\boldsymbol{m}$. Thus $\boldsymbol{\ell}' = \boldsymbol{\ell}$ and $\mathbf{m} = 1\varphi(\boldsymbol{\ell})$. □

Theorem 4.5. *Word* $\boldsymbol{m}$ *satisfies*

$$\boldsymbol{m} = 1\varphi^2(\boldsymbol{m}). \tag{4.1}$$

Word $\boldsymbol{\ell}$ *satisfies*

$$\boldsymbol{\ell} = 0\varphi^2(\boldsymbol{\ell}). \tag{4.2}$$

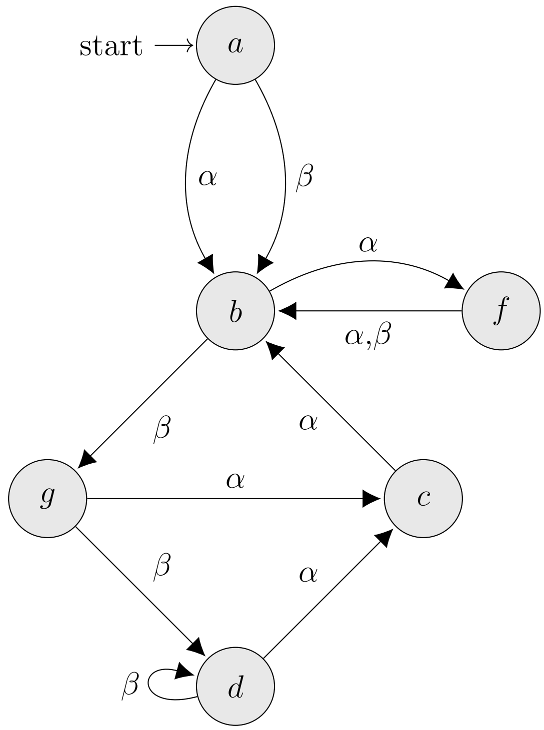

FIGURE 3. 'Fife' automaton A_{1010} for U_{1010}.

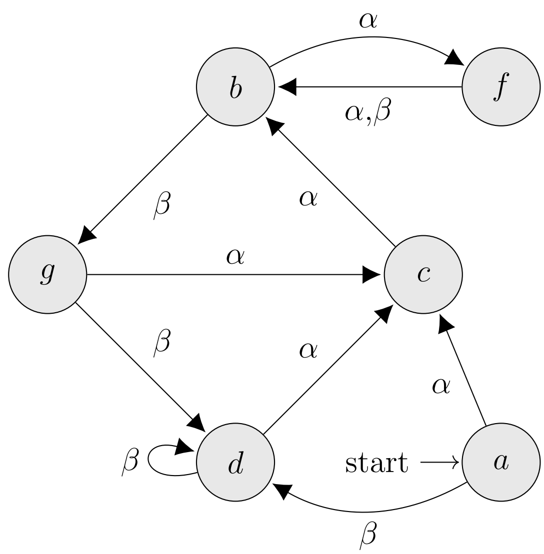

FIGURE 4. 'Fife' automaton A_{10010} for U_{10010}.

Proof. This follows from Lemma 4.3 and Lemma 4.4. □

Lemma 4.6. *Neither of $\boldsymbol{m}$ and $\boldsymbol{\ell}$ is the fixed point of a binary morphism. Every factor of $\boldsymbol{\phi}$ is a factor of $\boldsymbol{m}$ and $\boldsymbol{\ell}$, but there are infinitely many factors of $\boldsymbol{m}$ (resp., $\boldsymbol{\ell}$) which are not factors of $\boldsymbol{\phi}$ or $\boldsymbol{\ell}$ (resp., $\boldsymbol{m}$).*

Proof. Word $\boldsymbol{m}$ has prefix 10101, but by Lemma 2.6, the word 10101 is not a factor of $^{-}\boldsymbol{m}$. It follows that $\boldsymbol{m}$ cannot be the fixed point of a binary morphism. Similarly, $\varphi(10101)$ is a prefix of $\boldsymbol{\ell}$, but not a factor of $^{-}\boldsymbol{\ell}$, so that $\boldsymbol{\ell}$ is not a fixed point of a binary morphism.

Every factor of $\boldsymbol{\phi}$ is a factor of $\varphi^{2k}(0)$ for some k, and is therefore a factor of

$$\boldsymbol{m} = 1\varphi^2(\boldsymbol{m}) = 1\varphi^2(1\varphi^2(\boldsymbol{m})) = \cdots = 1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2k-2}(1)\varphi^{2k}(\boldsymbol{m})$$

Similarly, every factor of $\boldsymbol{\phi}$ is a factor of $\boldsymbol{\ell}$. However, none of factors $\varphi^{2k}(10101)$ of $\boldsymbol{m}$ (resp., $\varphi^{2k+1}(10101)$ of $\boldsymbol{\ell}$) is a factor of $\boldsymbol{\phi}$ or $\boldsymbol{\ell}$ (resp, $\boldsymbol{m}$). □

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