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PROPAGATING SETS IN SIERPIŃSKI FRACTAL GRAPHS

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Abstract. A set S of black colored vertices in a graph G is called a zero forcing set of G if at every discrete time step, a black vertex with exactly one non-black vertex as its neighbour forces it to be colored black and by iteratively applying the procedure, all vertices in G are colored black. If S induces an independent set of edges, then S is termed an edge-forcing set of G. The minimum cardinality of edge-forcing sets of G, denoted by $\zeta_e(G)$ is called edge-forcing number of G. In this paper, we obtain the edge-forcing number for Sierpiński graphs and Sierpiński gasket graphs.

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1. Introduction

The zero forcing problem in graphs was introduced in 2008 by the AIM Special Work Group [1] to find the minimum rank and the maximum nullity of a graph [2, 3]. The zero forcing problem is efficiently used in monitoring power networks, quantum physics and logic [4], path covering [5] coding theory, integer programming and in modeling the spread of diseases and information in social networks [6]. In 2020 Anitha *et al.* [7] introduced edge-forcing in graphs and in the same year Jessey *et al.* proved that the edge-forcing problem is NP-complete [8]. In electrical power systems, where an electrical node is represented as a vertex and a transmission line joining two electrical nodes as an edge, Phase Measurement Units (PMUs) are placed at selected vertices to regularly access or monitor electrical parameters like phase and voltage. Due to the high cost of a PMU, placing them at the locations of a minimum zero forcing set of the system, helps the monitoring of the entire system at minimum cost [9, 10].

Zero forcing set is also termed "infecting set" in relation to quantum systems [4, 11]. The problem of determining the zero forcing number is NP-complete [2]. Amos *et al.* (2015) [12] proved that $\zeta(G) \leq \frac{(\Delta-k+1)n}{\Delta-k+1+min\{\delta,k\}}$ and $\zeta(G) \leq \frac{((\Delta)n+2)}{\Delta-k-2}$ for a connected graph G of order n and maximum degree $\Delta \geq k$ with equality if and only if $G \cong C_n, K_n, K_{\Delta,\Delta}$, when $k > 1$. The problem has been studied for cacti graphs [5] and graphs of large girth. For any graph G with girth $g \geq 5$, and minimum degree $\delta \geq 2$, $2\delta - 2 \leq \zeta(G)$ [13]. In generalized Petersen graphs $P(n,k)$, $\zeta(P(n,2)) = 6$ for $n \geq 10$, $\zeta(P(n,3)) = 8$ for $n \geq 12$ and $\zeta(P(2k+1,k)) = 6$ for $k \geq 5$ [14]. The problem has been extensively studied for generalised Sierpinski graphs [15], tensor product graphs and

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Cartesian product graphs [16] and propagation time for zero forcing on a graph [17]. Total forcing [18] and connected forcing [19] have been investigated by several authors.

In 2015, Davila *et al.* [18] extended forcing problem to total forcing problem in graphs by way of restricting the structure of forcing sets as an induced subgraph. In the same paper they observed that the lower bound and upper bound on the total forcing number for any graph G is $2 \leq \zeta_t(G) \leq \zeta_c(G)$ [18]. An upper bound for the total forcing number of any tree T of order $n \geq 3$ is $\zeta_t(T) \leq \frac{\Delta}{\Delta+2}n$ [18]. Coloring in Sierpiński graphs and Sierpiński gasket graphs [20] relate to forcing sets. Another propagating problem is the power domination in graphs. It is worth nothing that the problem of deciding whether a graph admits a power dominating set of a given size is NP-complete [21]. There has been extensive work on the values of the power domination and zero forcing number of families of graphs [16].

In this paper we determine the edge-forcing number and power domination number for Sierpiński graphs.

2. Preliminaries

In this section we provide the definitions with illustrations, necessary for our main results.

Definition 2.1. [16] Let G be a graph and v be a vertex in G. The open neighbourhood of v, denoted by $N(v)$, is the set of vertices adjacent to v; the closed neighbourhood of v, denoted by $N[v]$, is $N_G(v) \cup \{v\}$. Let $S \subseteq V(G)$. Then the open neighbourhood of S is defined as $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is defined as $N[S] = N(S) \cup S$.

Definition 2.2. [16] For a graph $G(V, E)$, $S \subseteq V$ is a dominating set of G if every vertex in $V \backslash S$ has at least one neighbour in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Definition 2.3. [16] For a graph G, let $S \subseteq V(G)$. We define the sets $M^j(S)$ of vertices monitored by S at level j, $j \geq 0$, inductively as follows:
1. $M^0(S) = N[S]$.
2. $M^{j+1}(S) = M^j(S) \cup \{w : \exists v \in M^j(S),\, N(v) \cap (V(G) \backslash M^j(S)) = w\}$.

If $M^\infty(S) = V(G)$, then the set S is said to be a power dominating set of G. The minimum number of a power dominating set in G is called the power domination number of G and is denoted by $\gamma_p(G)$.

Definition 2.4. [14] The color *change rule*: If u is a black vertex and exactly one neighbor w of u is non-colored, then change the color of w to black. We say u forces w and denote this by $u \to w$. A zero forcing set for G is a subset of vertices B such that when the vertices in B are colored black and the remaining vertices are non-colored initially, repeated application of the color change rule can color all vertices of G black. The zero forcing number $\zeta(G)$ of G is the cardinality of a minimum zero forcing set.

Definition 2.5. [8] If a zero forcing set induces a set of independent edges in a graph G, then the set is called an edge-forcing set of G. Edge-forcing problem is to determine the minimum cardinality of all edge-forcing sets of G, which is denoted by $\zeta_e(G)$.

See Figure 1 for illustration.

For the graph G, in Figure 1(a), the vertex labeled 1 marked in red, is a power dominating set. At the first time step, vertex 1 monitors vertex 2 and vertex 3 , at second time step, vertex 2 monitor vertex 4, at third time step vertex 3 or 4 monitors vertex 5.

For the same graph G, in Figure 1(b), we determine a zero forcing set S as follows: Since the minimum degree of G is > 1, choosing only the vertex labeled 1 marked in red in S cannot be a zero forcing set. Hence one more vertex labeled 2 is included in the set S. At the first time step, vertex 1 forces vertex 3 and vertex 2 forces vertex 4 , at second time step, either vertex 3 or vertex 4 forces vertex 5. In Figure 1(c), the edge marked in red is an edge-forcing set. At the first time step, vertex 1 forces vertex 3 and vertex 2 forces vertex 4 , at second time step, either vertex 3 or vertex 4 forces vertex 5.

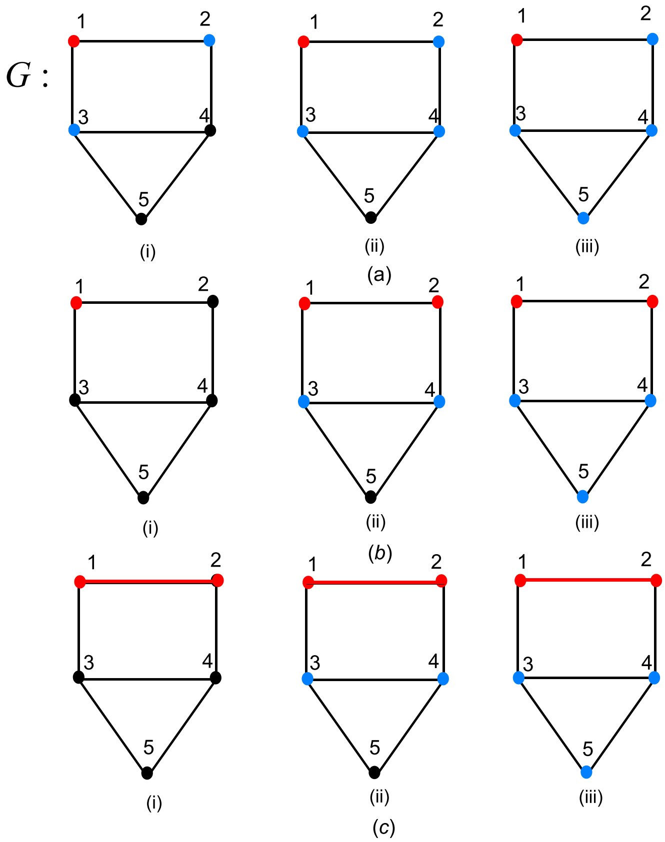

FIGURE 1. (a) Vertex in red is a power dominating set in G with $\gamma_p(G) = 1$ (b) Vertex in red yielding a zero forcing set in G with $\zeta(G) = 2$ and (c) edge $(1, 2)$ in red yielding a edge-forcing set in G with $\zeta_e(G) = 1$.

The power domination process on a graph G is choosing a set $S \subseteq V(G)$ and applying the zero forcing process to the closed neighborhood $N[S]$ of S. The set S is a power dominating set of G if and only if $N[S]$ is a zero forcing set for G [16].

Theorem 2.6. [16] *Let G be a graph. Then, $\left\lceil \frac{\zeta(G)}{\Delta(G)} \right\rceil \leq \gamma_p(G)$ and this bound is tight.*

Rao *et al.* [22] proved that the bound is tight for fully connected cubic networks.
The observation listed below are based on the following results:

Lemma 2.7. [7] *For a graph G of order $r \geq 2, \zeta_e(G) = 1$, where $G \cong P_n, C_n$.*

Lemma 2.8. [7] *For a spider graph of length $r \geq 3$, $\zeta_e(G) = r - 1$.*

Observations

The following results hold good:
1. For a graph G of order $r \geq 2$, $\gamma_p(G)) = \zeta_e(G) = 1$, where $G \cong P_n, C_n$.
2. For a graph G of order $r \geq 2$, $1 \leq \gamma_p(G)) \leq \zeta_e(G) \leq \zeta(G)$.
3. For a spider graph of length $r \geq 3$, $\zeta_e(G) = \zeta(G) = r - 1$.
4. For star graph $r \geq 3$, $\zeta_e(G)$ does not exist.

The following are some of the already proved results on zero forcing and power domination number of Sierpiński graphs.

Theorem 2.9. [15] *Let G be a graph of order n and size m. Then for any integer $t \geq 2$, $\zeta(S(C_n,t)) \geq n^{t-1}\zeta(G) - m\frac{n^{t-1}-1}{n-1}$*

Theorem 2.10. [15] *For any integers $n \geq 4$ and $t \geq 2$, $\zeta(S(C_n,t)) = \frac{n^t-2n^{t-2}+n}{n-1}$*

Theorem 2.11. [15] *For any integers n and t, $\zeta(S(K_n,t)) = \frac{n^t-2n^{t-1}+n}{2}$*

Theorem 2.12. [15] *For any positive integers n and t, $\zeta(S(K_{1,n},t)) = (n-1)(n+1)^{t-1}$*

Theorem 2.13. [15] *For any tree T and any positive integer t, $S(T,t)$ is a tree*

Theorem 2.14. [5] *For any integers $n \geq 3$ and $p \geq k+2$, $\gamma_{p,k}(S_p^n) = (p-k-1)^{n-2}$, $S_p^1 \cong K_p$*

Anitha *et al.* [7] determined the edge-forcing number for benzenoid networks and few nano tori. Jessy *et al.* [23] have obtained edge-forcing number for triangular grid networks and designed an approximation algorithm for edge-forcing of Butterfly networks [8]. But the edge-forcing problem in a large number of interconnection networks such as hypercubes, circulant networks and benes networks is yet to be explored.

In this paper, we deal with Sierpiński graphs and Sierpiński gasket graphs. These graphs are addressed as Sierpiński-like graphs [24]. Sierpiński-like graphs are extensively studied graphs of fractal nature with applications in Computer Science and Interconnection networks [25].

3. Sierpiński graphs

Fractals exhibit similar pattern at increasingly smaller scales. The most important use of fractals is in antenna technology. The self-similar property of fractal shapes means that the antenna can operate in the same way at different scales [26]. During the late 1990s and early 2000s, researchers discovered that these self-similar fractals shaped antennae delivered higher quality signals across a wider band of frequencies. In electronics tiny transistors within the electronic circuits can have fractal shapes for similar reasons as the antennas. They allow the transistor to operate over a greater range without any loss in performance [27].

Special self-similar shaped fractals also allowed the use of new beamforming techniques [28]. Beamforming is used in $5G$ cell towers to give the user a high-speed directed connection to the tower. Fractal-based analysis in mathematics is associated with two fundamental ideas, namely, selfsimilarity and contractivity. In applications, this involves introducing an appropriate space for contractive operators and approximating the target mathematical object by the fixed point of one of these contractions [29].

We define, a fractal graph called Sierpiński graph $S(r,k)$ with a k-sided regular polygon as a basic structure as follows:

(i). Draw a k-sided regular polygon on k-vertices representing the Sierpiński graph of dimension 1 denoted by $S(1,k)$. See Figure 2(a).

(ii). Subdivide each edge of $S(1,k)$ with two vertices and add k number of k-sided regular polygons with each polygon sharing two subdivided edges incident at a vertex of $S(1,k)$ as shown in Figure 2(b). This represents $S(2,k)$.

(iii). Repeat Step (ii) in each k-sided regular polygons of $S(2,k)$ to obtain $S(3,k)$. Repeated application of Step (ii) in each of the k-sided regular polygons in $S(r-1,k)$ yields $S(r,k)$. See Figure 2(c).

Equivalent definitions of $S(r,k)$, $k \geq 1$, with labeling of its vertices are found in [20]

3.1. Sierpiński graph $S(r,3)$, $r \geq 3$

We begin with the properties of $S(r,3)$.

The following notations and observations help us to design an algorithm to compute the edge-forcing number of $S(r,3)$.

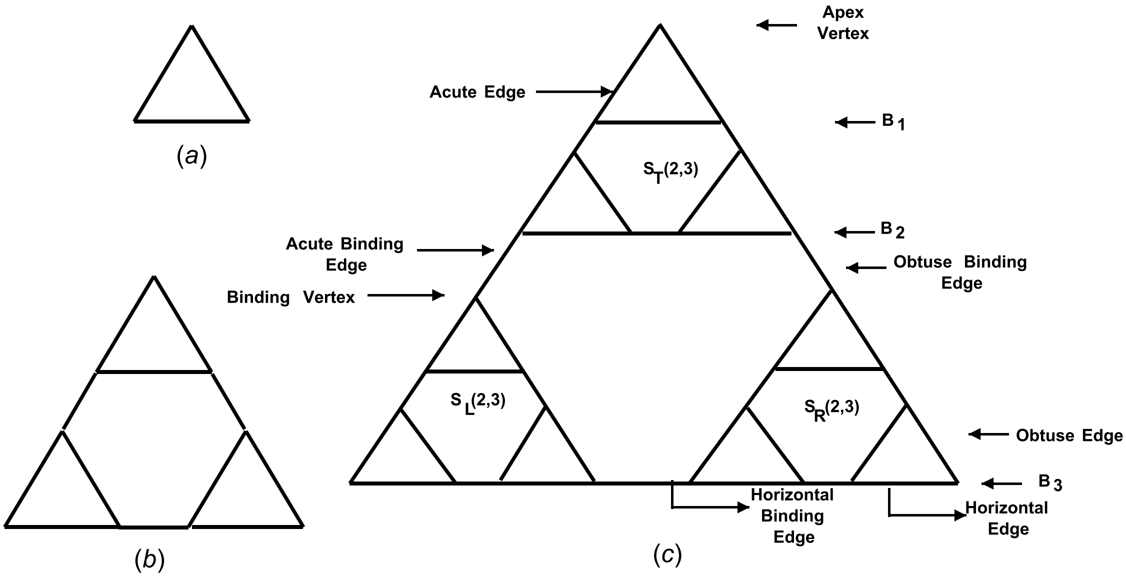

FIGURE 2. (a) $S(1,3)$, (b) $S(2,3)$ and (c) $S(3,3)$ explaining appropriate terms.

1. $S(1,3)$ is a 3-cycle. There are 3^{r-1} vertex disjoint copies of $S(1,3)$ in $S(r,3)$.
2. All the edges in $S(r,3)$ are classified as either horizontal, acute or obtuse edges.
3. There are 3 vertex disjoint copies of $S(r-1,3)$ in $S(r,3)$, $r \geq 2$, denoted by $S_T(r-1,3)$, $S_L(r-1,3)$, and $S_R(r-1,3)$ according as they are in the Top, Left or Right.
4. $S_T(r-1,3)$, $S_L(r-1,3)$, and $S_R(r-1,3)$ in $S(r,3)$ are pairwise bound by edges called binding edges. These edges are called horizontal binding edge, acute binding edge or obtuse binding edge according as they are horizontal, acute and obtuse respectively. The end vertices of the binding edges are called binding vertices.
5. There are exactly 3 vertices of degree 2 in $S(r,3)$, $r \geq 2$. The vertex of degree 2 in $S_T(r-1,3)$ is called the apex vertex of $S(r,3)$. Vertex of degree 2 in $S(r,3)$, which is in $S_L(r-1,3)$ is called the left most base vertex and the one in $S_R(r-1,3)$ is called the right most base vertex of $S(r,3)$.
6. A base line B_i is a path on 2^i vertices with all the 2^i-1 edges as horizontal edges, $2 \leq i \leq r$. If a path B is a subpath of B_i and is a base line of a Sierpiński graph of smaller dimension, then B is not considered as a base line of the smaller dimensional graph. See Figure 2(c).

Lemma 3.1. *Let H be a subgraph of $S(r,3)$isomorphic to $S(2,3)$. Then any edge-forcing set of $S(r,3)$ contains at least one edge from H.*

Proof. Let F be an edge forcing set of H. Even if all the three binding vertices of H in $S(r,3)$ are already propagated by the edges in $S(r,3)\backslash H$, the propagation cannot continue in H as each of these vertices is incident on two edges of H. Hence F should contain at least one edge from H. □

Lemma 3.2. *Let G be the Sierpiński graph $S(2,3)$. Then $\zeta_e(S(2,3)) = 2$.*

Proof. Let F be an edge-forcing set of H. Choose an edge incident with a vertex of degree 2 in F which initiates propagation. The propagation continues till it reaches 2-degree vertices of $S(1,3)$. See Figure 3(a). For further propagation, one more edge has to be selected. Choose the horizontal bridge edge of $S(2,3)$ in F. See Figure 3(b). Clearly the propagation is complete in $S(2,3)$. Hence $\zeta_e(S(2,3)) = 2$. See Figure 3(a), (b) and (c). □

Lemma 3.3. *Let G be a Sierpiński graph $S(3,3)$. Then $\zeta_e(S(3,3)) = 4$.*

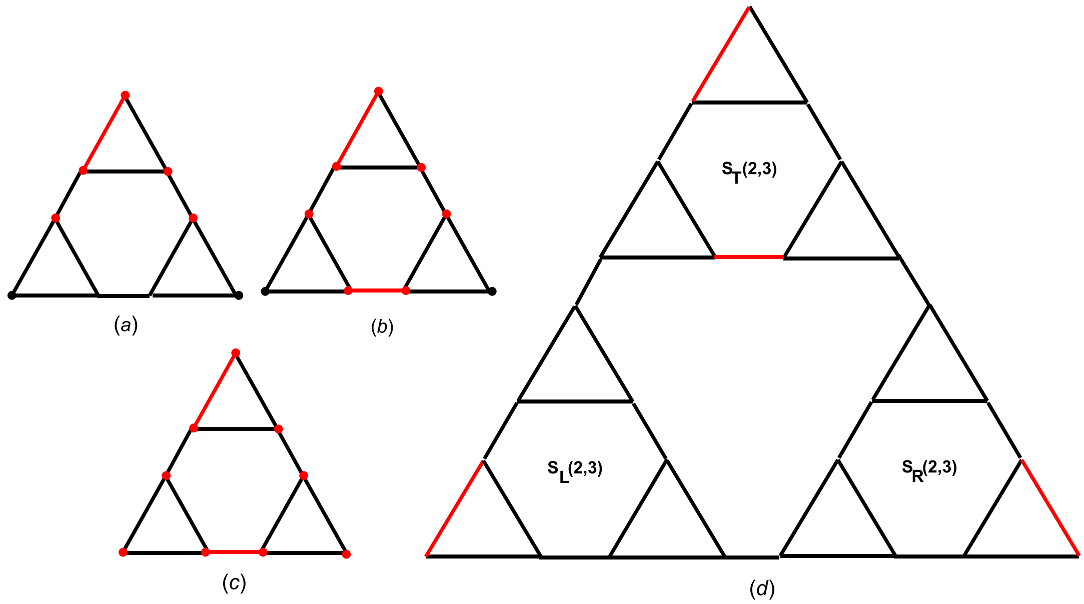

FIGURE 3. Propagation in Sierpiński graph $(a)-(c)$ $S(2,3)$ (d) $S(3,3)$.

Proof. Initiate propagation in $S_T(2,3)$ by selecting an edge incident at the apex vertex of $S(2,3)$. By Lemma 3.2, $\zeta_e(S_T(2,3))$ in $S(3,3)$ is at least 2. The propagation continues by choosing the centre edge of B_1 in $(S_T(2,3))$. Again by Lemma 3.1, $\zeta_e(S_T(2,3)) \geq 1$. Thus $\zeta_e(S_T(3,3)) \geq 4$. The edges shown in red in Figure 4 enable propagation in $S(3,3)$. Therefore, $\zeta_e(S(3,3)) = 4$. See Figure 3(d). □

Lemma 3.4. *Let G be the Sierpiński graph $S(r,3)$, $r \geq 3$. Then $\zeta_e(S(r,3)) \geq \frac{3^{r-1}-1}{2}$.*

Proof. Let F be an edge-forcing set of G. We observe that propagation has to be initiated through an edge incident at a vertex of degree 2. Without loss of generality, we assume that it begins with the apex vertex of $S_T(r-1,3)$. We take this vertex to be at Level 0. Its children are at Level 1 and iteratively, vertices at Level i are children of vertices at Level $i-1$, $1 \leq i \leq 2^r - 1$.

Let S_i be the edge cut comprising of acute and obtuse binding edges between Levels $2i-1$ and $2i$, $1 \leq i \leq 2^{r-1}-2$. See Figure 4. Even if the binding vertex at Level $2i$ of a binding edge in S_i, $1 \leq i \leq 2^{r-1}-2$ is already propagated, the propagation cannot continue at it is adjacent to two unpropagated vertices at Level $2i+1$, $1 \leq i \leq 2^{r-1}-2$. Hence every component obtained by deleting the edges in the edge cuts $S_1, S_2, \ldots, S_{2^{r-1}-2}$ except the one containing 2^{r-2} copies of $S(2,3)$ on base line B_r contributes at least 1 edge to F.

By Lemma 3.3, we have $\zeta_e(S(3,3)) = 4$. The number of components when $S_1, S_2, \ldots, S_6$ are removed in $S(4,3)$ is 9 other than the component with 4 copies of $S(2,3)$ on the base line B_3. Thus $\zeta_e(S(4,3)) \geq 9+4 = 3 \times 4 + 1 = 3\zeta_e(S(3,3)) + 1$.

Iteratively moving forward, $\zeta_e(S(3,3)) \geq 3\zeta_e(S(r-1,3)) + 1$
$= 3^2\zeta_e(S(r-2,3)) + 3 + 1$
$= 3^{r-3}\zeta_e(S(r-(r-3),3)) + 3^{r-4} + 3^{r-5} + \ldots + 1$
$= 3^{r-3} \times 4 + \frac{3^{r-3}-1}{2}.$
$= \frac{3^{r-1}-1}{2}.$
Thus $\zeta_e(S(3,3)) \geq \frac{3^{r-1}-1}{2}$, for all $r \geq 3$. □

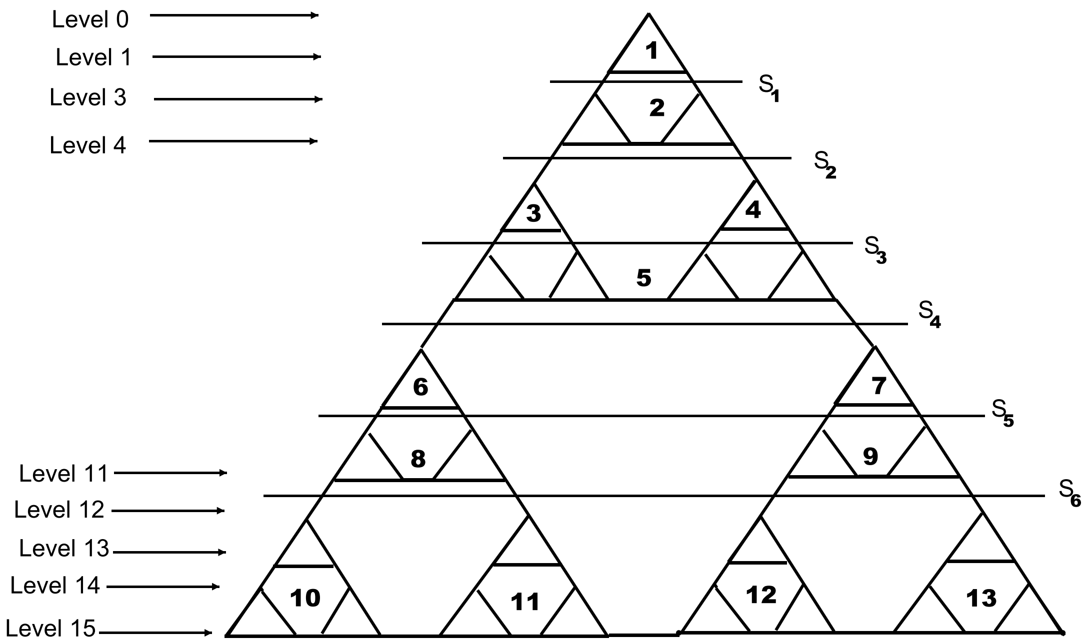

FIGURE 4. Levels of vertices in $S(4,3)$, edge-cuts $S_1, S_2, \ldots, S_6$ and the numbers $1-13$ representing its components.

We now design an algorithm to prove that $\zeta_e(G) = \frac{3^{r-1}-1}{2}$.
Algorithm Edge-Forcing $S(r,3)$
Input: $S(r,3), r \geq 3$.
Algorithm: Let an edge-forcing set of $S(r,3)$ be denoted by F.
Select the following edges in F.
(i). The acute edges incident at the apex vertex and leftmost base vertex of $S(r,3)$.
(ii). The obtuse edge incident at the rightmost base vertex of $S(r,3)$.
(iii). All acute edges incident at the apex vertex of each of the remaining copies of $S(2,3)$ in $S(r,3)$ which do not contain any of the apex vertices of degree 2 in $S(r,3)$.
(iv). The center edge of each of the base lines in $S(r,3)$ except B_r.
Output: F is an edge-forcing set of $S(r,3)$ of cardinality $\frac{3^{r-1}-1}{2}$. See Figure 5.

Proof of Correctness:
Step 1: The acute edge incident at the apex vertex of degree 2 in $S_T(r-1,3)$ initiates the propagation. The propagation terminates with the apex vertices of two copies of $S(1,3)$ with their horizontal base edges on a base line B_2. The inclusion of the centre edge of this base line in F along with the acute edges of two copies of $S(2,3)$ allows the propagation to continue till it reaches the apex vertices of two copies of $S(2,3)$.

Step 2: Repeat Step 1 in both the copies of $S(2,3)$. Remove the centre edges of the corresponding base lines and include the centre base edge of B_3 in F. The propagation continues till it reaches the apex vertices of two copies of $S(3,3)$.

Step 3: Repeat Step 1 and Step 2 successively in the two copies of $S(r,3)$, $r \geq 3$, till propagation reaches the apex vertices of the 2^{r-2} copies of $S(2,3)$ in $S_L(r-1,3)$ and $S_R(r-1,3)$ with their base lines on B_r.

Step 4: By virtue of the leftmost base vertex of $S(r,3)$ with degree 2 and the rightmost base vertex of $S(r,3)$ of degree 2, the propagation process continues till the propagation is complete in $S(r,3)$.

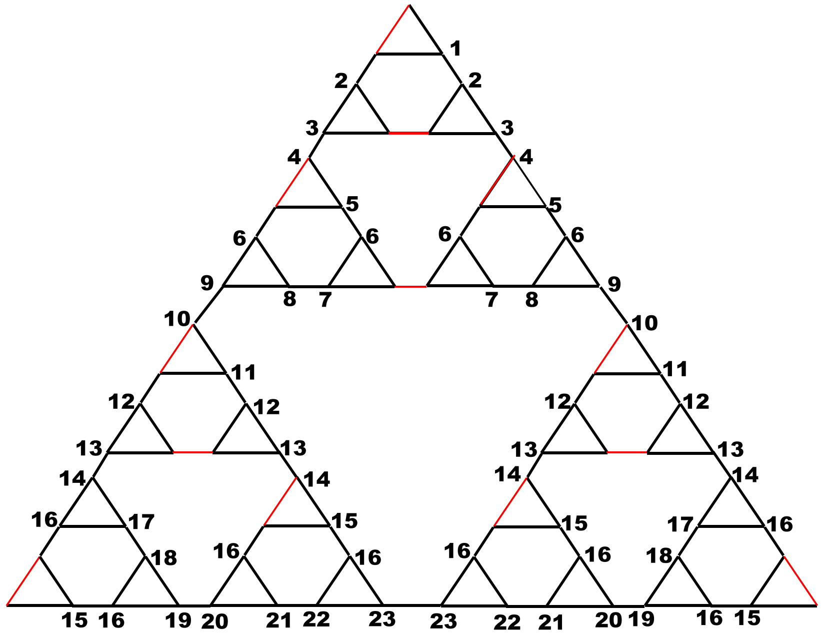

FIGURE 5. Red colored edges indicate edge-forcing set of $S(4,3)$. Numbers indicate the discrete time step in the propagation process.

In F, there are $3^{r-2}-2$ apex edges, one leftmost base edge and one rightmost base edge in B_r and $\frac{3^{r-2}-1}{2}$ number of horizontal base edges. These horizontal base edges are chosen , one each in each of the components with respect to the cuts described in Lemma 3.4. Thus the cardinality of F is $3^{r-2}+\frac{3^{r-2}-1}{2}=\frac{3\times 3^{r-2}-1}{2}=\frac{3^{r-1}-1}{2},\ r\geq 3$. Hence $\zeta_e(S(r,3))\leq\frac{3^{r-1}-1}{2},\ r\geq 3$. See Figure 5.

Lemma 3.4 and the Algorithm Edge-Forcing $S(r,3),\ r\geq 3$ yield the following result.

Theorem 3.5. *Let G be the Sierpiński graph $S(r,3),\ r\geq 3$. Then $\zeta_e(S(r,3))=\frac{3^{r-1}-1}{2},\ r\geq 3$.*

3.2. Sierpiński graph $S(r,k),\ r\geq 3,\ k\geq 4$

In this section, we determine the edge-forcing number for Sierpiński graphs $S(r,k),\ r\geq 3,\ k\geq 3$. We obtain a lower bound for $\zeta_e(S(r,k)),\ r\geq 3,\ k\geq 4$ and prove that the bound obtained is sharp.

There are k^{r-1} copies of $S(1,k)$ in $S(r,k)$ where $S(1,k)$ is isomorphic to a k-cycle. Again $S(r,k)$ contains k copies of $S(r-1,k),\ r\geq 3, k\geq 4$. If each copy of $S(r-1,k)$ is contracted into a single vertex, the resulting graph is a k-cycle whose edges are called binding edges. See Figure 6(a).

Lemma 3.6. *Let G be the Sierpiński graph $S(r,k),\ r\geq 3,\ k\geq 4$. Then $\zeta_e(S(r,k))\geq(k-1)k^{r-2}$.*

Proof. We prove the result by induction on r. Let F be an edge-forcing set of $S(3,k)$, $k\geq 4$. We note that each k-cycle in $S(3,k)$ contains either a path of length 2 such that the end vertices of the path are of degree 3 and the middle vertex is of degree 2 which we call as $X-$ path or all the three vertices of the path are of degree 3, which we call as Y-path. There are $k-2$ number of X-Paths and exactly two Y-Paths in each subgraph $S(2,k)$ of $S(3,k)$. The middle vertex of each Y-Path is a binding vertex. Further, each binding edge connects the middle vertices of two Y-Path. See Figure 6(a).

Consider an X-Path. Even if both vertices of degree 3 are already propagated, the propagation cannot continue as each of them is adjacent to 2 unpropagated vertices. Hence each $S(2,k)$ containing an X-path contributes at least one edge to F.

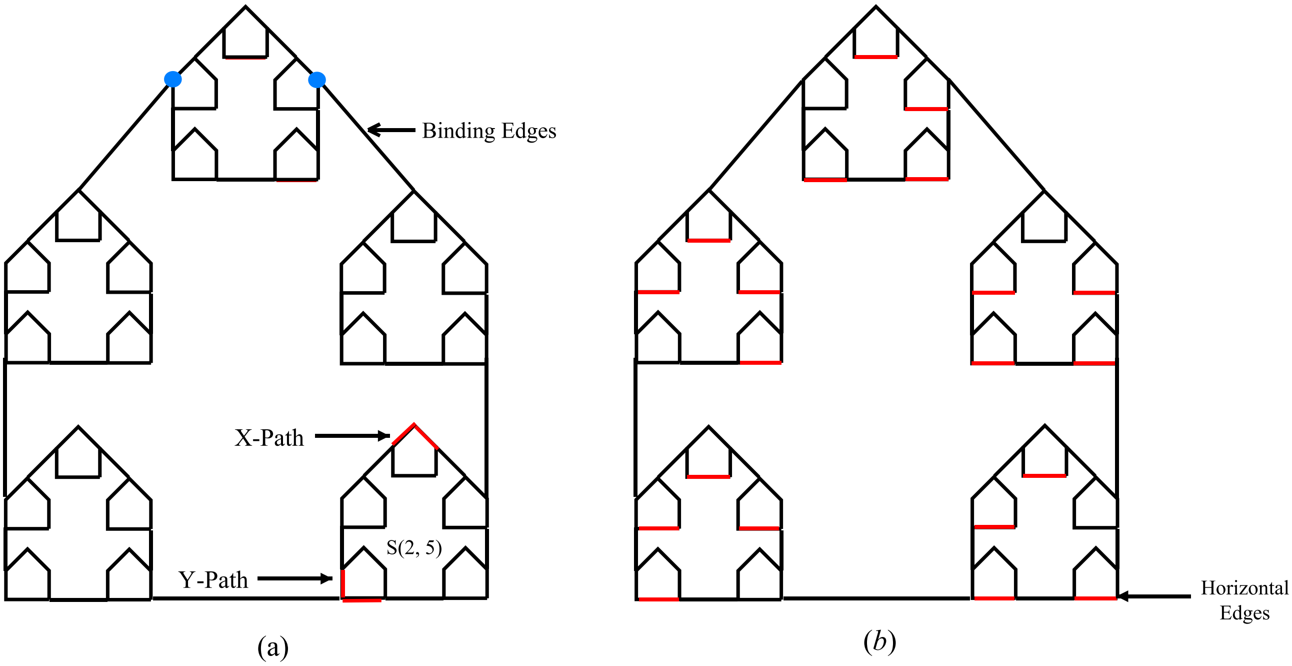

FIGURE 6. (a) X-Paths and Y-Paths in $S(5,3)$ (b) Red colored edges indicate edge-forcing set of $S(3,5)$.

Next consider a subgraph $S(2,k)$ containing two Y-Paths. Again, even if both binding vertices of the two Y-paths are propagated, the propagation cannot continue. Hence this $S(2,k)$ contributes at least one edge to F. Thus $|F| \geq (k-2)+1 = k-1$. Further, binding edges in F do not minimize the cardinality of F. Hence $|F| \geq k(k-1)$. Thus the lower bound $(k-1)k^{r-2}$ holds good for $\zeta_e(S(r,k))$ when $r=3$. Assume the result to be true for $S(r-1,k),\ r \geq 3, k \geq 4$. That is $\zeta_e(S(r-1,k)) \geq (k-1)k^{r-3}$. Now consider $S(r,k)$. $S(r,k)$ contains k vertex disjoint copies of $S(r-1,k)$. By induction hypothesis, $\zeta_e(S(r,k)) \geq k(k-1)k^{r-3} = (k-1)k^{r-2}$. □

Theorem 3.7. *Let $S(r,k),\ r \geq 3,\ k \geq 4$ be the Sierpiński graph. Then $\zeta_e(S(r,k)) = (k-1)k^{r-2}$.*

Proof. By the construction of $S(r,k),\ r \geq 3,\ k \geq 4$, $\zeta_e S(r,k)$ is a multiple of $\zeta_e(S(r-1,k))$, which in turn is multiple of $\zeta_e(S(r-1,k))$. Binding edges in a forcing set do not minimize the cardinality of a forcing set. Therefore it is enough to find the cardinality of a minimum forcing set of $S(3,k), k \geq 4$. We proceed as follows. In F, select all the horizontal edges from each copy of $S(1,k)$ which in $S(2,k)$ except one copy of $S(1,k)$ which has a Y- path. Do this for all the k copies of $S(2,k)$ in $S(3,k)$. See Figure 6(b).

It can be checked manually that $\zeta_e(S(3,k)) = k(k-1)$.

Thus $\zeta_e(S(r,k)) = k\zeta_e(S(r-1,k)) = k^2\zeta_e(S(r-2,k))\ldots = k^{r-3}\zeta_e(S(3,k)) = k^{r-3}k(k-1) = (k-1)k^{r-2}$ □

Using similar arguments we arrive at the power domination number of Sierpiński graph, the proof of which is omitted. See Figure 7.

Theorem 3.8. *Let $S(r,k),\ r \geq 3,\ k \geq 4$ be the Sierpiński graph. Then $\gamma_p(S(r,k)) = (k-1)k^{r-2}$.*

3.3. Sierpiński Gasket

In this section, we determine the edge-forcing number for Sierpiński gasket graphs.

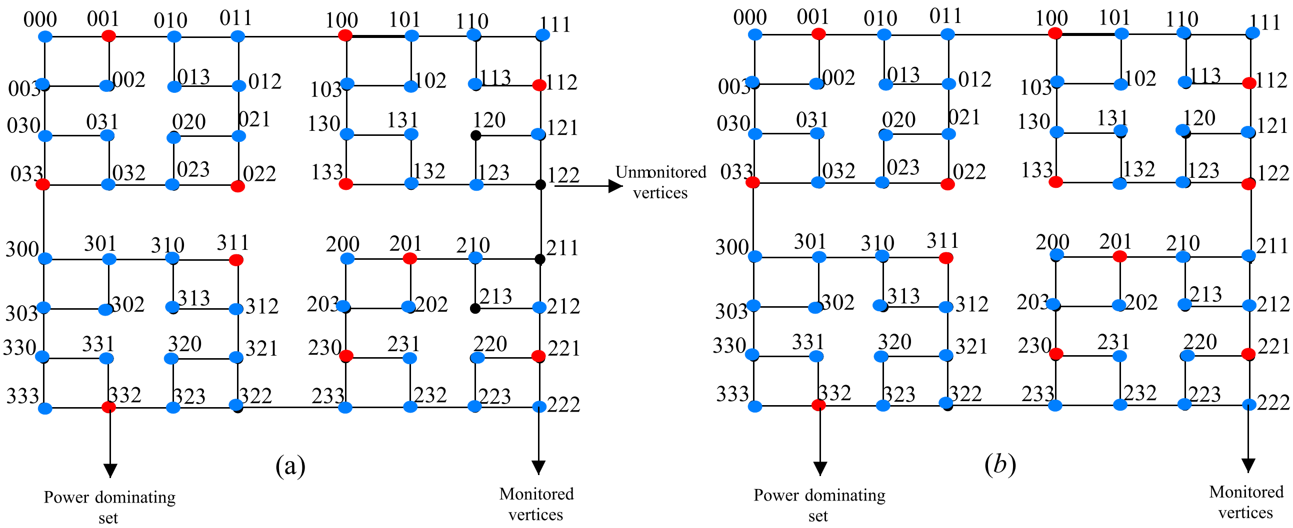

FIGURE 7. Red colored vertices indicate power dominating set of $S(4, 3)$.

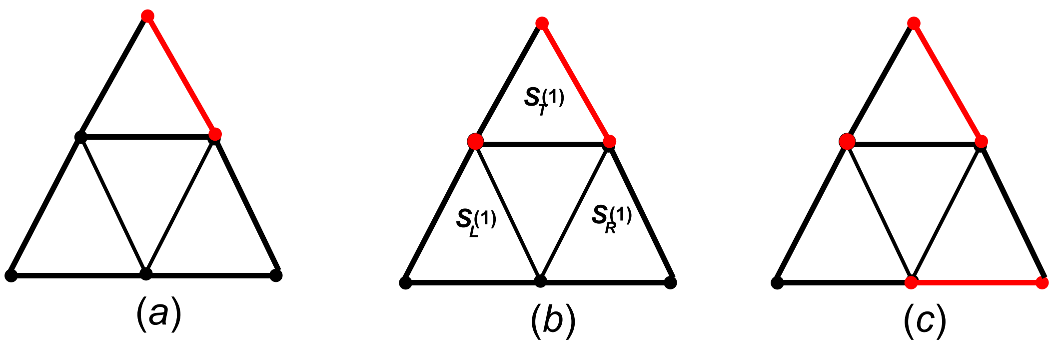

FIGURE 8. Red colored edges indicate edge forcing set of $S(2)$.

Sierpiński gasket graph $S(r)$, $r \geq 1$, is obtained from $S(r, 3)$ by contracting all the edges of $S(r, 3)$ that do not lie on any 3-cycle.

The following notations and observations help us to design an algorithm to compute the edge forcing number of $S(r)$.

1. $S(1)$ is a 3-cycle. There are 3^{r-1} edge disjoint copies of $S(1)$ in $S(r)$.
2. All the edges in $S(r)$ are classified as either horizontal, acute or obtuse edges.
3. There are 3 edge disjoint copies of $S(r-1)$ in $S(r)$, $r \geq 2$, denoted by $S_T(r-1)$, $S_L(r-1)$, and $S_R(r-1)$ according as they are in the Top, Left or Right.
4. There are exactly 3 vertices of degree 2 in $S(r)$, $r \geq 2$. The vertex of degree 2 is $S_T(r-1)$ is called its apex vertex.

Lemma 3.9. *Let G be the Sierpiński gasket graph $S(2)$. Then $\zeta_e(G) = 2$.*

Proof. Choosing an edge incident with a vertex of degree 2 initiates propagation. See Figure 8(a) and (b). The propagation continues till it reaches the apex vertices of $S_L(1)$ and $S_R(1)$ in $S(2)$. For further propagation, one more edge has to be selected. Select an edge as shown in Figure 8(c). This completes the propagation in $S(2)$. Therefore, $\zeta_e(G) = 2$. □

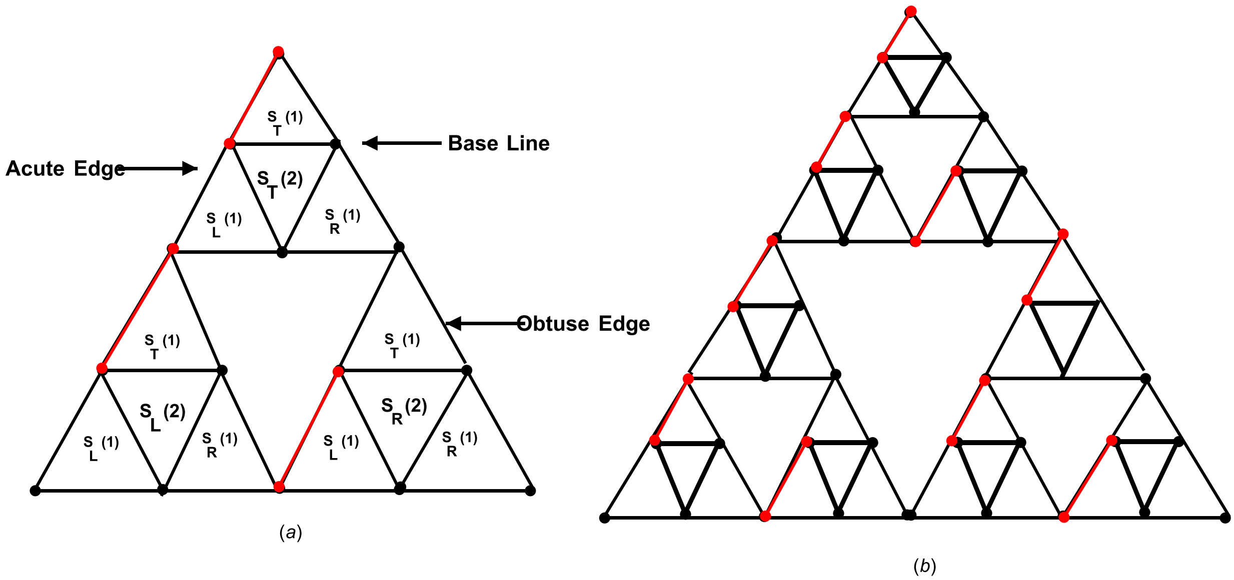

FIGURE 9. Red colored edges indicate edge forcing set of (a) $S(3)$ (b) $S(4)$.

Lemma 3.10. *Let H be a subgraph of $S(r)$isomorphic to $S(2)$. Then any edge forcing set of $S(r)$ contains at least one edge from H.*

Proof. Let F be an edge-forcing set of H. Even if all the three vertices of H in $S(r)$ are propagated by the edges in $S(r)\backslash H$, the propagation cannot continue as each of these vertices is incident on two edges of H. Hence F should contain at least one edge from H. □

Lemma 3.11. *Let G be a Sierpiński gasket graph $S(r)$, $r \geq 3$. Then $\zeta_e(G) \geq 3^{r-2}$.*

Proof. Let F be an edge forcing set of $S(r)$. There are 3^{r-2} edge disjoint copies of $S(2)$ in $S(r)$. By Lemma 3.10, the propagating set consists of at least 3^{r-2} number of edges, one from each copy of $S(2)$.Thus $\zeta_e(S(r)) \geq 3^{r-2}$. □

We now give an algorithm to prove that $\zeta_e(G) = 3^{r-2}$.
Algorithm Edge-Forcing $S(r)$
Input: $S(r), r \geq 3$.
Algorithm: Let an edge forcing set of $S(r)$ be denoted by F.
Select the following edges in F, to propagate to all vertices in $S(3)$
(i). The acute edge incident at $S_T(1)$ in $S_T(2)$ of $S(3)$.
(ii). The acute edge incident at $S_T(1)$ in $S_L(2)$ of $S(3)$.
(iii). The acute edge incident at $S_L(1)$ in $S_R(2)$ of $S(3)$. See Figure 9(a).
(iv). Repeat the Steps (i), (ii) and (iii) in every copy of $S(3)$ in $S(r)$.
Output: F is the edge forcing set of $S(r)$ of cardinality $3 \times 3^{r-3} = 3^{r-2}$. See Figure 9.

Proof of Correctness:
Step 1: The acute edge incident at the apex vertex of degree 2 in $S_T(r-1)$ initiates the propagation. The propagation terminates with the apex vertices of two copies of $S(1)$ in $S_T(2)$. The inclusion of the acute edge of $S_T(1)$ in $S_L(2)$ and $S_L(1)$ in $S_R(2)$ of $S(3)$ allow the propagation to continue till it reaches base line of $S(3)$.
Step 2: Repeat Step 1 successively in each copy of $S(3)$ in $S(r)$ and continue the propagation process till the propagation is complete in $S(r)$.

In F, there are 3^{r-2} acute edges. Thus the cardinality of F is 3^{r-2}. Hence $\zeta_e(S(r)) \leq 3^{r-2},\ r \geq 3$.

Lemma 3.11 and the Algorithm Edge-Forcing $S(r)$ yield the following result:

Theorem 3.12. *Let G be the Sierpiński gasket graph $S(r)$ of dimension r, $r \geq 3$. Then $\zeta_e(S(r)) = 3^{r-2}$.*

4. Conclusion

In this paper, we have obtained the edge-forcing number for Sierpiński graphs and Sierpiński gasket graphs. Similar pattern at increasingly smaller scales in the Sierpiński fractal graphs was instrumental in obtaining sharp bounds for the edge forcing number. It would be an interesting line of research to compute the edge-forcing number of several other fractal graphs.

Conflict of interest

The authors declare no conflict of interest.

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