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DECISION MAKING STRATEGY FOR $\delta - I$ EQUILIBRIUM PROBLEM IN NETWORK DESIGNING

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Abstract. The existence of fractional factors characterizes the fractional flows in a network, and hence indirectly characterizes the feasibility of data transmission. The minimum degree and isolated toughness characterize network topology from the perspectives of sparsity and stability, which serves as the theoretical conditions for fractional factors. This article reveals from a theoretical perspective that if the minimum degree condition increases, the corresponding tight isolated toughness variant bound will decrease. This infinite number of parameter combinations cause a "choice dilemma" for decision-makers. To solve this problem, we regard these two parameters as the Pareto front of the bi-objective optimization problem, and a knot point calculation approach is designed to determine the optimal combination.

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1. Introduction

Only simple and finite graphs are discussed in this work. Let G be a graph (corresponding to a network), and $\delta(G)$ be its minimum degree. For $S \subseteq V(G)$, we denote $i(G-S)$ by the number of isolated vertices in $G-S$. The notations and terminologies in graph theory used in this paper follow from the standard in Bondy and Mutry [1].

Let a, b, k be positive integers $(1 \le a \le b)$ and $h : E(G) \to [0,1]$. A *fractional* $[a,b]$*-factor* is a spanning subgraph consisting of edge set $E_h = \{e \in E(G) \mid h(e) > 0\}$ such that $a \le \sum_{x' \in N(x)} h(xx') \le b$ for any $x \in V(G)$. A graph G admits a fractional $[a,b]$-factor if such h exists. In particular, if $a = b = k$, then we call it a fractional k-factor. In computer networks, fractional factors are used to measure the existence of fractional flaws, thereby indirectly judging the practicability of data transmission in networks. Therefore, in communication network designing, the presence of fractional factors within a specific assumption is an essential feature of network topology.

Yang *et al.* [2] introduced the notion of *isolated toughness* which is stated by

$$I(G) = \min\left\{ \frac{|S|}{i(G-S)} \middle| S \subset V(G), i(G-S) \ge 2 \right\}.$$

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A unique isolated toughness variant is denoted as (Zhang and Liu [3])

$$I'(G)=\min\left\{\frac{|S|}{i(G-S)-1}\Big|S\subset V(G), i(G-S)\geq 2\right\}.$$

In particular, for a complete graph G, $I(G)=I'(G)=+\infty$ since the cut set S doesn't exist. Both $I(G)$ and $I'(G)$ are employed to appraise the stability and vulnerability of the network, where the more robust the network, the larger the corresponding parameter values.

Due to the significance of isolated toughness, the $I(G)$ or $I'(G)$ conditions for fractional factors have been intensively studied in various settings. Ma and Liu [4] proved the sharp isolated toughness bound (*i.e.*, $\delta(G)\geq k$ and $I(G)\geq k$) for a graph G to admit a fractional k-factor. However, the corresponding tight $I'(G)$ bound is much more difficult, which has been an open problem for a long time. Until recently, He *et al.* [5] solved this problem and stated that a graph G admits a fractional k-factor if $\delta(G)\geq k$ and $I'(G)>2k-1$. The counterexample in He *et al.* [5] explains the sharpness of $I'(G)$ bound. Gao *et al.* [6] determined that a graph G admits a fractional $[a,b]$-factor if $\delta(G)\geq a$ and $I(G)\geq a-1+\frac{a}{b}$, and they explained that both $\delta(G)$ and $I(G)$ bounds are sharp. Gao and Wang [7] argued that G has a fractional $[a,b]$-factor if $\delta(G)\geq a$ and $I'(G)>a-1+\frac{a}{n_{a,b}-1}$, where $(n_{a,b}-1)a\leq b\leq n_{a,b}a-1$ with $n_{a,b}\geq 2$ is an integer. More recent contributions in isolated toughness bounds can be referred to [8] and [9].

The tight isolated toughness bound has important theoretical guiding significance in network design, as it can effectively answer how to balance network cost and functionality. Specifically, the denser the graph, the larger the isolated toughness value and the stronger the data transmission capacity of the corresponding network, but the higher the cost of network construction. On the contrary, the sparser the graph, the lower the network construction costs, but the corresponding performances will decline. From a theoretical perspective, the tight isolated toughness bound for fractional factors provides a balance between network construction cost and functionality. The network obtained in terms of the tight bound requirements saves construction cost on the one hand, while making the network have certain data transmission ability and vulnerability to attacks on the other hand.

It is well acknowledged that a sufficient condition for fractional k-factors (resp. fractional $[a,b]$-factor) is the combination of minimum degree and isolated toughness. Clearly, $\delta(G)\geq k$ (resp. $\delta(G)\geq a$) is tight for a graph G admiting a fractional k-factor (resp. fractional $[a,b]$-factor) in view of its definition, and hence the previous contributions focused on the tight bounds of isolated toughness. Because minimum degree and isolated toughness are two parameters that measure the density of graphs from different perspectives, and they have the same trend (the denser the graph, the greater the corresponding minimum degree and isolated toughness; on the contrary, the sparser the graph, the lower their corresponding values), thus it naturally raises the following question:

Problem 1.1. Considering the combination bound of $\delta(G)$ and $I(G)$ (or $I'(G)$) for the existence of fractional factors. If the lower bound of the minimum degree is increased, will the tight bound of the corresponding isolated toughness (variant) decrease?

Our first theoretical conclusion answers Problem 1.1 in the affirmative for $I'(G)$.

Theorem 1.2. *Let G be a graph, $k\geq 2$ be an integer and $t\in\mathbb{N}\cup\{0\}$. If $\delta(G)\geq k+t$ and $I'(G)>k+\frac{k-1}{t+1}$, then G admits a fractional k-factor.*

Furthermore, we have the following generalization of Theorem 1.2.

Theorem 1.3. *Let G be a graph, $t\in\mathbb{N}\cup\{0\}$, and a,b be integers satisfying $2\leq a\leq b$. If $\delta(G)\geq a+t$ and $I'(G)>a-1+\frac{a+t}{\lceil\frac{b(t+1)+1}{a}\rceil-1}$, then G admits a fractional $[a,b]$-factor.*

By setting $t=0$ in Theorem 1.2 (resp. Thm. 1.3), then we immediately get the corollary which is exactly the main result in He *et al.* [5] (resp. Gao and Wang [7]). If we consider $a=b=k$ in Theorem 1.3, we immediately infer Theorem 1.2 which implies that Theorem 1.2 is only a special case of Theorem 1.3. Therefore, only the proof of Theorem 1.3 is elaborated.

However, the answer to Problem 1.1 for $I(G)$ is negative (both for fractional k-factor and fractional $[a, b]$-factor), *i.e.*, the tight isolated toughness bound will not change if we increase the minimum degree of the graph. The reason why different parameter leads to a completely different answer will be explained after the proof of Theorem 1.3 (*cf.* Sect. 4).

From the main result given in this work, we see that there are infinite combinations of $\delta(G)$ and $I'(G)$ for fractional factors. In all combinations, the values of minimum degree and isolated toughness variant develop in the opposite direction. The larger the $\delta(G)$, the smaller the corresponding $I'(G)$ value ($\frac{k-1}{t+1}$, $\frac{a+t}{\lceil \frac{b(t+1)+1}{a} \rceil - 1}$ are the decreasing function of t). It leads to a "choice dilemma" for network decision-makers, where too many parameter combinations make it unclear to decide which group to choose as the network parameters. We call this problem as *$\delta - I$ equilibrium problem*. To conquer this problem, the second contribution of this article proposes a strategy in light of knee point selection as follows: treating the resulting combinations as a Pareto front in a bi-objective optimization problem, and introducing the knee point selection strategy into the $\delta - I$ equilibrium problem. By determining the knee point, the final $(\delta(G), I'(G))$ combination is obtained.

The organization of the remaining sections is described as follows. A useful lemma and basic knowledge on multi-objective optimization will be stated in the next section. The proof of Theorem 1.3 is elaborated in the third section, and furthermore, the counterexample is showcased to state the sharpness of these bounds. Subsequently, the knee point selection strategy is proposed from decision making standpoint. Finally, two future study topics are posed in the conclusion section.

2. Preliminary

To prove Theorem 1.3, we need the assistance of the following lemma.

Lemma 2.1. (Liu and Zhang [10]) *Let G be a graph, $a, b \in \mathbb{N}$ with $a \leq b$. Then G admits a fractional $[a, b]$-factor if and only if*

$$b|S| - a|T| + \sum_{x \in T} d_{G-S}(x) \geq 0$$

holds for any disjoint subsets $S, T \subseteq V(G)$.

A standard multi-objective optimization problem (MOP) can be denoted by

$$\min \mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_r(\mathbf{x}))^T$$

$$\text{s.t.}\, \mathbf{x} \in \Omega$$

where $\mathbf{x} = (x_1, \cdots, x_d)^T$ is a decision vector (*i.e.*, a candidate solution of MOP) and Ω is a decision space (*cf.* Khalid *et al.* [11], Eichfelder and Warnow [12], Zhang and Zhu [13], Ansary [14] and Duro *et al.* [15]). If $r = 2$, then MOP is called a bi-objective optimization problem. Let $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ be two solutions, we say $\mathbf{x}^1$ dominates $\mathbf{x}^2$ (denoted by $\mathbf{x}^1 \preceq \mathbf{x}^2$) if and only if $f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$ for all $i \in \{1, \cdots, r\}$ and $f_i(\mathbf{x}^1) < f_i(\mathbf{x}^2)$ for at least one $i \in \{1, \cdots, r\}$. A solution $\mathbf{x} \in \Omega$ is Pareto optimal if there doesn't exist a solution $\mathbf{x}' \in \Omega$ such that $\mathbf{x}' \preceq \mathbf{x}$, the set of all Pareto optimal solutions is called the Pareto set (PS) and the set of their corresponding values $\{\mathbf{F}(\mathbf{x}) | \mathbf{x} \in \text{PS}\}$ is called the Pareto front (PF). Some recent advances in MOP and PF can be referred to Li *et al.* [16], Dupin [17], Cabello [18], Nagar *et al.* [19] and Bidgoli *et al.* [20].

In MOP setting, the knee point (KP) is the point on the PF with maximum effectiveness, near which a slight increase in the value of one objective will cause a large recession in at least one other objective [21]. And global

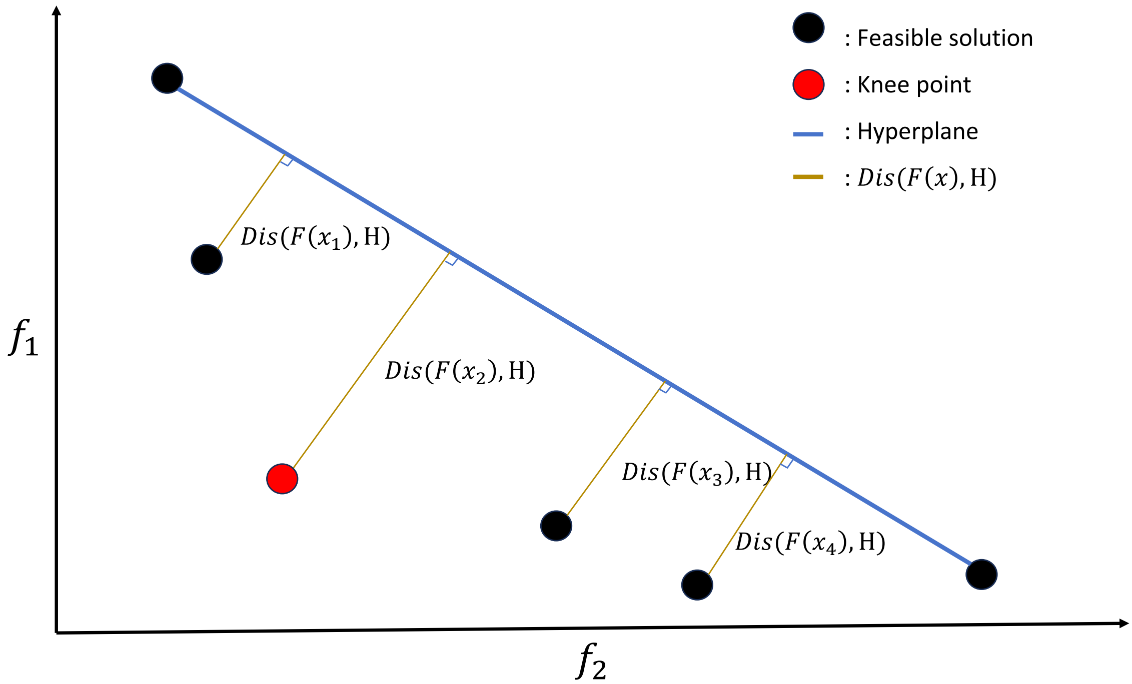


FIGURE 1. The selection process of knee point.

knee point can be denoted as

$$\mathbf{x}_{kp} = \arg\max_{\mathbf{x}\in\Omega} Dis(\mathbf{F}(\mathbf{x}), \mathbb{H}), \tag{2.1}$$

where $\mathbb{H}$ is a hyperplane on the objective space, which generally consists of extreme points on the objective space, and $Dis(\cdot)$ is the Euclidean distance from the feasible solution $\mathbf{x}$ to $\mathbb{H}$ (*cf.* Li *et al.* [22], Zhu [23], Haris *et al.* [24], Gao *et al.* [25], and Muhammadsharif and Hashim [26]). As shown in Figure 1, the point with maximum $Dis(\cdot)$ from the $\mathbb{H}$ is considered as the global knee point.

3. Proof of Theorem 1.3

If G is a complete graph, the desired conclusion is obtained in terms of $\delta(G) \geq a + t$. Assume that a non-complete graph G satisfies all conditions of Theorem 1.3, but doesn't have a fractional $[a, b]$-factor. By means of Lemma 2.1, there exist disjoint subsets $S, T \subseteq V(G)$ satisfying

$$b|S| - a|T| + \sum_{x\in T} d_{G-S}(x) = b|S| + \sum_{x\in T}(d_{G-S}(x) - a) \leq -1. \tag{3.1}$$

Select S and T with minimum $|T|$. Hence, we have $T \neq \emptyset$ and $d_{G-S}(x) \leq a - 1$ holds for each $x \in T$. Moreover, since $\delta(G) \geq a + t$, we have $S \neq \emptyset$.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_a and let $T_0 = \{x \in V(H') \,|\, d_{G-S}(x) = 0\}$. Let H be the subgraph inferred from $H' - T_0$ by deleting those l components isomorphic to K_a. Let S' be a set of vertices that contains exactly $a - 1$ vertices in each component isomorphic to K_a in H', and thus $|S'| = l(a - 1)$.

Proposition 3.1. $V(H) \neq \emptyset$.

Proof. Suppose $V(H) = \emptyset$, then in light of (3.1) and $\delta(G) \geq a + t$, we get $t + 1 \leq |S| \leq \frac{a(|T_0|+l)-1}{b}$, and hence $|T_0| + l \geq \frac{b(t+1)+1}{a}$, *i.e.*, $i(G - S \cup S') = |T_0| + l \geq \left\lceil \frac{b(t+1)+1}{a} \right\rceil \geq 2$. With the aid of the definition of $I'(G)$, we get

$$\begin{aligned} I'(G) &\leq \frac{|S \cup S'|}{i(G - S - S') - 1} \leq \frac{\left\lfloor \frac{a(|T_0|+l)-1}{b} + l(a-1) \right\rfloor}{|T_0| + l - 1} \\ &\leq \frac{\left\lfloor (a - 1 + \frac{a}{b})(|T_0| + l - 1) + a - 1 + \frac{a-1}{b} \right\rfloor}{|T_0| + l - 1} \\ &= a - 1 + \frac{a - 1 + \left\lfloor \frac{a(|T_0|+l-1)+a-1}{b} \right\rfloor}{|T_0| + l - 1}. \end{aligned}$$

Set $a(|T_0| + l - 1) + a - 1 = mb + c$, where $m \in \mathbb{N} \cup \{0\}$ and $c \in \{0, \cdots, b-1\}$. Then, $\frac{1}{b} \leq \frac{c+1}{b} \leq 1$, $a - 1 + \frac{a-1-c}{b} \geq 0$, and

$$\begin{aligned} & a - 1 + \max\left\{ \frac{a - 1 + \left\lfloor \frac{a(|T_0|+l-1)+a-1}{b} \right\rfloor}{|T_0| + l - 1} \right\} \\ = \ & a - 1 + \max\left\{ \frac{a - 1 + \frac{a(|T_0|+l-1)+a-1-c}{b}}{|T_0| + l - 1} \right\} \\ = \ & a - 1 + \frac{a}{b} + \max\left\{ \frac{a - 1 + \frac{a-1-c}{b}}{|T_0| + l - 1} \right\}. \end{aligned}$$

Hence, $a - 1 + \max\left\{ \frac{a-1+\left\lfloor \frac{a(|T_0|+l-1)+a-1}{b} \right\rfloor}{|T_0|+l-1} \right\}$ will increase as $|T_0| + l$ decreases, and it arrives at the maximum value when $|T_0| + l = \left\lceil \frac{b(t+1)+1}{a} \right\rceil$. Set $\left\lceil \frac{b(t+1)+1}{a} \right\rceil = \frac{b(t+1)+1+\overline{c}}{a}$, where $\overline{c} \in \{0, 1, \cdots, a-1\}$. We derive

$$\begin{aligned} I'(G) &\leq a - 1 + \frac{a - 1 + \left\lfloor \frac{a(\left\lceil \frac{b(t+1)+1}{a} \right\rceil - 1) + a - 1}{b} \right\rfloor}{\left\lceil \frac{b(t+1)+1}{a} \right\rceil - 1} \\ &= a - 1 + \frac{a - 1 + \left\lfloor \frac{a \frac{b(t+1)+1+\overline{c}}{a} - 1}{b} \right\rfloor}{\left\lceil \frac{b(t+1)+1}{a} \right\rceil - 1} \\ &= a - 1 + \frac{a + t}{\left\lceil \frac{b(t+1)+1}{a} \right\rceil - 1} \end{aligned}$$

which contradicts $I'(G) > a - 1 + \frac{a+t}{\left\lceil \frac{b(t+1)+1}{a} \right\rceil - 1}$. Therefore, $V(H) \neq \emptyset$. □

Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfy that $d_{G-S}(v) = a-1$ for each vertex $v \in V(H_1)$ and $H_2 = H - H_1$. For $i \in \{1,2\}$, denote I_i and C_i by the maximum independent set and covering set, respectively, of H_i. According to the definition of H_2, each component belonging to H_2 has at least one vertex with degree at most $a-2$ in $G-S$. If $H_2 \neq \emptyset$, then $a \geq 3$.

Set $U = S \cup S' \cup N_{G-S}(I_1) \cup N_{G-S}(I_2)$. The following discussions divided into two cases according to the value of $|T_0| + l$.

Case 1. $|T_0| + l \neq 0$.

The following three subcases are divided from one of H_1 or H_2 is empty, and both H_1 and H_2 are not empty.

Case 1.1. $I_2 = \emptyset$.

If $I_2 = \emptyset$, then $I_1 \neq \emptyset$ since $V(H) \neq \emptyset$. We divide I_1 into two subsets as follows:
I_{11}: $v \in I_{11}$ if there exist $v' \neq v$, $v' \in I_1$ such that $N_{G-S}(v) \cap N_{G-S}(v') \neq \emptyset$;
I_{12}: $v \in I_{12}$ if $N_{G-S}(v) \cap N_{G-S}(v') = \emptyset$ for any $v' \in I_1 \setminus v$.

Hence, we have

$$|S| \leq \frac{|V(H_1)| + a|T_0| + la - 1}{b} \leq \frac{(a-\frac{1}{2})|I_{11}| + (a-1)|I_{12}| + a(|T_0|+l) - 1}{b},$$

$$\begin{aligned}
|S \cup S' \cup N_{G-S}(I_1)| &\leq (\frac{a}{b} - \frac{1}{2b})|I_{11}| + \frac{a-1}{b}|I_{12}| - \frac{1}{b} + \frac{a(|T_0|+l)}{b} \\
&\quad + l(a-1) + (a-1-\frac{1}{2})|I_{11}| + (a-1)|I_{12}| \\
&\leq (a-1+\frac{a}{b})(|T_0|+l) + (a-1+\frac{a}{b}-\frac{1}{b})|I_1| - \frac{1}{b} \\
&\leq (a-1+\frac{a}{b})(|I_1|+|T_0|+l) - \frac{2}{b},
\end{aligned}$$

and

$$\begin{aligned}
I'(G) &\leq \frac{|S \cup S' \cup N_{G-S}(I_1)|}{i(G - S \cup S' \cup N_{G-S}(I_1)) - 1} \\
&\leq \frac{\lfloor (a-1+\frac{a}{b})(|I_1|+l+|T_0|-1) + a - 1 + \frac{a-2}{b} \rfloor}{|I_1|+l+|T_0|-1} \\
&= a - 1 + \frac{a-1+\left\lfloor \frac{a(|I_1|+l+|T_0|-1)+a-2}{b} \right\rfloor}{|I_1|+l+|T_0|-1}
\end{aligned}$$

In view of $t+1 \leq |S| \leq \frac{(a-\frac{1}{2})|I_1|+a(|T_0|+l)-1}{b} \leq \frac{a(|I_1|+|T_0|+l)-1}{b}$, we know that $|I_1|+|T_0|+l \geq \left\lceil \frac{b(t+1)+1}{a} \right\rceil$. On the other hand, set $a(|I_1|+l+|T_0|-1)+a-2 = m_1 b + c_1$, where $m_1 \in \mathbb{N} \cup \{0\}$ and $c_1 \in \{0,1,\cdots,b-1\}$. We deduce

$$\begin{aligned}
I'(G) &\leq a - 1 + \frac{a-1+\frac{a(|I_1|+l+|T_0|-1)+a-2-c_1}{b}}{|I_1|+l+|T_0|-1} \\
&= a - 1 + \frac{a}{b} + \frac{a-1+\frac{a-2-c_1}{b}}{|I_1|+l+|T_0|-1}
\end{aligned}$$

Since $a-1+\frac{a-2-c_1}{b}>0$ and $|I_1|+l+|T_0|\geq \left\lceil\frac{b(t+1)+1}{a}\right\rceil$, we have that

$$\begin{aligned} I'(G) &\leq a-1+\frac{a-1+\left\lfloor\frac{a(\lceil\frac{b(t+1)+1}{a}\rceil-1)+a-2}{b}\right\rfloor}{\left\lceil\frac{b(t+1)+1}{a}\right\rceil-1} \\ &\leq a-1+\frac{a+t}{\left\lceil\frac{b(t+1)+1}{a}\right\rceil-1}, \end{aligned}$$

a contradiction to the assumption of $I'(G)$. □

Case 1.2. $I_1=\emptyset$.

If $I_1=\emptyset$, then $I_2\neq\emptyset$, $a\geq 3$, and $|S|\geq 2+t$ since $\delta(G)\geq a+t$. Let $v_1,v_2,\cdots,v_{|I_2|}$ be vertices in I_2 such that $d_{G-S}(v_1)\leq d_{G-S}(v_2)\leq\cdots\leq d_{G-S}(v_{|I_2|})$ and $d_{G-S}(v_1)\leq a-2$. We infer

$$|S|\leq\frac{a(|T_0|+l)}{b}+\frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(a-d_{G-S}(v_i))}{b}-\frac{1}{b}.$$

Let $U=S\cup S'\cup N_{G-S}(I_2)$, we know that $i(G-U)\geq 2$ and

$$\begin{aligned} |U| &\leq |S|+|S'|+|N_{G-S}(I_2)| \\ &\leq \frac{a(|T_0|+l)}{b}+\frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(a-d_{G-S}(v_i))}{b}-\frac{1}{b}+l(a-1)+\sum_{i=1}^{|I_2|}d_{G-S}(v_i) \\ &= \frac{a|T_0|}{b}+l(a-1+\frac{a}{b})+\sum_{i=1}^{|I_2|}(-\frac{(d_{G-S}(v_i))^2}{b}+\frac{a+b-1}{b}d_{G-S}(v_i)+\frac{a}{b})-\frac{1}{b} \\ &\leq \frac{a|T_0|}{b}+l(a-1+\frac{a}{b})+(-\frac{(a-2)^2}{b}+\frac{a+b-1}{b}(a-2)+\frac{a}{b}) \\ &\quad +(|I_2|-1)(-\frac{(a-1)^2}{b}+\frac{a+b-1}{b}(a-1)+\frac{a}{b})-\frac{1}{b} \\ &= \frac{a|T_0|}{b}+l(a-1+\frac{a}{b})+(a-1+\frac{a}{b})|I_2|-1+\frac{a-3}{b}. \end{aligned}$$

In light of the calculation process, we know that to maximize $|U|$ with given $|I_2|$, the structure of the extremal subgraph satisfies the following: only one vertex in I_2 has degree $a-2$ in $G-S$, and all other vertices in I_2 have degree $a-1$ in $G-S$. Therefore,

$$2+t\leq|S|\leq\frac{a(|I_2|+|T_0|+l)+a-3}{b},$$

$$|I_2|+|T_0|+l\geq\left\lceil\frac{(t+2)b-a+3}{a}\right\rceil.$$

By virtue of the definition of $I'(G)$, we obtain

$$
\begin{aligned}
I'(G) &\le \frac{|U|}{i(G-U)-1} \le \frac{\left\lfloor \frac{a|T_0|}{b} + l(a-1+\frac{a}{b}) + (a-1+\frac{a}{b})|I_2| - 1 + \frac{a-3}{b} \right\rfloor}{|I_2|+|T_0|+l-1} \\
&\le \frac{\left\lfloor (a-1+\frac{a}{b})(|T_0|+l+|I_2|-1) + (a-1+\frac{a}{b}) - 1 + \frac{a-3}{b} \right\rfloor}{|I_2|+|T_0|+l-1} \\
&= a-1+\frac{a-2+\left\lfloor \frac{a(|T_0|+l+|I_2|-1)+2a-3}{b} \right\rfloor}{|I_2|+|T_0|+l-1}.
\end{aligned}
$$

Let $\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil = \frac{(t+2)b-a+3+\overline{c'}}{a}$, where $\overline{c'} \in \{0,1,\cdots,a-1\}$. Similar to the Case 1.1, we know that $I'(G)$ reaches the maximum value when $|T_0|+l+|I_2|$ takes to its lower bound, and hence

$$
\begin{aligned}
I'(G) &\le a-1+\frac{a-2+\left\lfloor \frac{a(\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil - 1)+2a-3}{b} \right\rfloor}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil - 1} \\
&= a-1+\frac{a-2+\left\lfloor \frac{a\frac{(t+2)b-a+3+\overline{c'}}{a}+a-3}{b} \right\rfloor}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil - 1} \\
&= a-1+\frac{a+t}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil - 1},
\end{aligned}
$$

which contradicts $I'(G) > a-1+\frac{a+t}{\left\lceil \frac{b(t+1)+1}{a} \right\rceil - 1}$. □

Case 1.3. Both I_1 and I_2 are not empty.

In this case, we have $a \ge 3$. Denote $v_1, v_2, \cdots, v_{|I_2|}$ as the vertex set of I_2 defined in Case 1.2. We have

$$
b|S| \le a(|T_0|+l) + (a-\frac{1}{2})|I_{11}| + (a-1)|I_{12}| + \sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(a-d_{G-S}(v_i)) - 1,
$$

and

$$
\begin{aligned}
|S| &\le \frac{a(|T_0|+l) + (a-\frac{1}{2})|I_{11}| + (a-1)|I_{12}| + \sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(a-d_{G-S}(v_i)) - 1}{b} \\
&= \frac{a}{b}(|T_0|+l) + (\frac{a}{b}-\frac{1}{2b})|I_{11}| + \frac{a-1}{b}|I_{12}| + \frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(a-d_{G-S}(v_i))}{b} - \frac{1}{b}.
\end{aligned}
$$

Set $U = S \cup S' \cup C_1 \cup N_{G-S}(I_1) \cup N_{G-S}(I_2)$, we yield $i(G-U) \ge 3$, and the following derivation

$$
\begin{aligned}
|U| &\le |S| + |S'| + |N_{G-S}(I_1)| + |N_{G-S}(I_2)| \\
&\le \frac{a}{b}|T_0| + l(a-1+\frac{a}{b}) + (a-1+\frac{a}{b}-\frac{1}{b})|I_1| + (a-1+\frac{a}{b})|I_2| - 1 + \frac{a-3}{b} \\
&\le (|T_0|+l)(a-1+\frac{a}{b}) + (a-1+\frac{a}{b})|I_1| + (a-1+\frac{a}{b})|I_2| - 1 + \frac{a-4}{b}.
\end{aligned}
$$

Using the similar discussion in Case 1.2, to maximize $|U|$ with fixed $|I_2|$, only one vertex in I_2 has degree $a-2$ in $G-S$, and the rest vertices in I_2 have degree $a-1$ in $G-S$. Thus,

$$2+t \le |S| \le \frac{a(|I_1|+|I_2|+|T_0|+l)+a-3}{b},$$

$$|I_1|+|I_2|+|T_0|+l \ge \left\lceil \frac{(t+2)b-a+3}{a} \right\rceil.$$

We have

$$\begin{aligned} I'(G) &\le \frac{|U|}{i(G-U)-1} \le \frac{\lfloor (a-1+\frac{a}{b})(|T_0|+l+|I_1|+|I_2|-1)+(a-1+\frac{a}{b})-1+\frac{a-4}{b} \rfloor}{|I_1|+|I_2|+|T_0|+l-1} \\ &= a-1+\frac{a-2+\left\lfloor \frac{a(|I_1|+|I_2|+|T_0|+l-1)+2a-4}{b} \right\rfloor}{|I_1|+|I_2|+|T_0|+l-1}. \end{aligned}$$

Similar to the analysis in Case 1.1 and Case 1.2, $I'(G)$ reaches the maximum value when $|I_1|+|I_2|+|T_0|+l$ arrives at its lower bound $\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil$, and hence

$$\begin{aligned} I'(G) &\le a-1+\frac{a-2+\left\lfloor \frac{a(\lceil \frac{(t+2)b-a+3}{a} \rceil -1)+2a-4}{b} \right\rfloor}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil -1} \\ &= a-1+\frac{a-2+\left\lfloor \frac{a\frac{(t+2)b-a+3+c'}{a}+a-4}{b} \right\rfloor}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil -1} \\ &\le a-1+\frac{a+t}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil -1}, \end{aligned}$$

which contradicts $I'(G) > a-1+\frac{a+t}{\lceil \frac{b(t+1)+1}{a} \rceil -1}$.

Case 2. $|T_0|+l=0$.

Similar to the discussion in Case 1, we need to cope with the situations when H_1 or H_2 is empty.

Case 2.1. $I_2=\emptyset$.

If $I_2=\emptyset$, then I_1 is not empty and $b|S| \le a|T|-d_{G-S}(T)-1=|T|-1$.

If $|I_1|=1$, then $|T| \le a-1$ and $|S| \le \frac{|T|-1}{b} \le \frac{a-2}{b}$, which implies $S=\emptyset$, a contradiction. Thus, I_1 contains at least two vertices.

Let $U=S\cup C_1 \cup N_{G-S}(I_1)$, we have $i(G-U) \ge |I_1| \ge 2$. Similar to the derivation in Case 1.1, we get

$$|U| \le (a-1+\frac{a-1}{b})|I_1|-\frac{1}{b} \le (a-1+\frac{a}{b})|I_1|-\frac{3}{b}.$$

Moreover, $t+1 \le |S| \le (\frac{a}{b}-\frac{1}{2b})|I_1|-\frac{1}{b} < \frac{a}{b}|I_1|$ which implies $|I_1| > \frac{b(t+1)}{a}$. Hence, $|I_1| \ge \left\lceil \frac{b(t+1)+1}{a} \right\rceil$.

In terms of the definition of $I'(G)$, we get

$$I'(G) \leq \frac{|U|}{i(G-U)-1} \leq \frac{\lfloor (a-1+\frac{a}{b})(|I_1|-1)+(a-1+\frac{a}{b})-\frac{3}{b}\rfloor}{|I_1|-1}$$
$$= a-1+\frac{a-1+\left\lfloor \frac{a(|I_1|-1)}{b}+\frac{a-3}{b}\right\rfloor}{|I_1|-1}.$$

Using the similar discussion in Case 1.1, we know that when $|I_1|$ reaches the minimum value, then $I'(G)$ arrives at the upper bound, and hence

$$I'(G) \leq a-1+\frac{a-1+\left\lfloor \frac{a(\lceil \frac{b(t+1)+1}{a}\rceil -1)}{b}+\frac{a-3}{b}\right\rfloor}{\left\lceil \frac{b(t+1)+1}{a}\right\rceil -1}$$
$$\leq a-1+\frac{a+t}{\left\lceil \frac{b(t+1)+1}{a}\right\rceil -1},$$

which contradicts the assumption of $I'(G)$. □

Case 2.2. $I_1=\emptyset$.

If $I_1=\emptyset$, then $I_2\neq\emptyset$, $a\geq 3$ and $|S|\geq 2+t$.

If I_2 only contains one vertex, then denote $d_{min}=\min\{d_{G-S}(v)|v\in H_2\}$, $z\in V(H_2)$ satisfying $d_{G-S}(z)=d_{min}$, and hence $d_{min}\in\{1,\cdots,a-2\}$. In light of (3.1), we get

$$|S| \leq \frac{|T|(a-d_{min})-1}{b} \leq \frac{(a-1)(a-d_{min})-1}{b}$$

and

$$a+t \leq \delta(G) \leq d_{min}+|S| \leq d_{min}+\frac{(a-1)(a-d_{min})-1}{b} \leq a-2+\frac{2a-3}{b},$$

a contradiction.

Hence, I_2 contains at least two vertices. Let $v_1,v_2,\cdots,v_{|I_2|}$ be the sequence vertices in I_2 as defined in Case 1.2, we have $d_{G-S}(v_1)\leq a-2$, and

$$|S| \leq \frac{\sum_{i=1}^{|I_2|}(d_{G-S}(v_i)+1)(a-d_{G-S}(v_i))}{b}-\frac{1}{b}.$$

Let $U=S\cup N_{G-S}(I_2)$, then $i(G-U)\geq |I_2|\geq 2$. Similar to the discussion in Case 1.2, we get

$$|U| \leq |S|+|N_{G-S}(I_2)| \leq (a-1+\frac{a}{b})|I_2|-1+\frac{a-3}{b}.$$

Meanwhile, to maximize $|U|$, the extremal subgraph requires that only one vertex in I_2 has degree $a-2$ in $G-S$, and all other vertices in I_2 have degree $a-1$ in $G-S$. We re-bound $|S|$ by

$$t+2 \leq |S| \leq \frac{a|I_2|+a-3}{b},$$

$$|I_2| \geq \left\lceil \frac{(t+2)b-a+3}{a}\right\rceil.$$

According to the definition of $I'(G)$, we get

$$I'(G) \le \frac{|U|}{i(G-U)-1} \le \frac{\left\lfloor (a-1+\frac{a}{b})(|I_2|-1) + (a-1+\frac{a}{b}) - 1 + \frac{a-3}{b} \right\rfloor}{|I_2|-1}$$
$$= a-1+\frac{a-2+\left\lfloor \frac{a(|I_2|-1)+2a-3}{b} \right\rfloor}{|I_2|-1}.$$

Since $|I_2| \ge \lceil \frac{(t+2)b-a+3}{a} \rceil$, we have that

$$I'(G) \le a-1+\frac{a-2+\left\lfloor \frac{a(\lceil \frac{(t+2)b-a+3}{a} \rceil -1)+2a-3}{b} \right\rfloor}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil - 1}$$
$$= a-1+\frac{a-2+\left\lfloor \frac{a\frac{(t+2)b-a+3+\overline{c'}}{a}+a-3}{b} \right\rfloor}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil - 1}$$
$$= a-1+\frac{a+t}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil - 1},$$

which contradicts $I'(G) > a-1+\frac{a+t}{\lceil \frac{b(t+1)+1}{a} \rceil -1}$. □

Case 2.3. Both I_1 and I_2 are not empty.

In this case, $a \ge 3$ and $|S| \ge 2+t$. Let $U = S \cup C_1 \cup N_{G-S}(I_1) \cup N_{G-S}(I_2)$, then $i(G-U) \ge 2$ and

$$|U| \le |S| + |C_1| + |N_{G-S}(I_1)| + |N_{G-S}(I_2)|$$
$$\le (a-1+\frac{a-1}{b})|I_1| + (a-1+\frac{a}{b})|I_2| - 1 + \frac{a-3}{b}$$
$$\le (a-1+\frac{a}{b})(|I_1|+|I_2|) - 1 + \frac{a-4}{b}.$$

To maximize $|U|$, only one vertex in I_2 has degree $a-2$ in $G-S$, and all other vertices in I_2 have degree $a-1$ in $G-S$. Hence,

$$t+2 \le |S| \le \frac{a(|I_1|+|I_2|)+a-3}{b},$$

$$|I_1|+|I_2| \ge \left\lceil \frac{(t+2)b-a+3}{a} \right\rceil.$$

In light of the definition of isolated toughness variant, we infer

$$I'(G) \le \frac{|U|}{i(G-U)-1} \le \frac{\left\lfloor (a-1+\frac{a}{b})(|I_1|+|I_2|-1) + (a-1+\frac{a}{b}) - 1 + \frac{a-4}{b} \right\rfloor}{|I_1|+|I_2|-1}$$
$$= a-1+\frac{a-2+\left\lfloor \frac{a(|I_1|+|I_2|-1)+2a-4}{b} \right\rfloor}{|I_1|+|I_2|-1}.$$

As $|I_1|+|I_2| \geq \lceil \frac{(t+2)b-a+3}{a} \rceil$, we have that

$$\begin{aligned} I'(G) &\leq a-1+\frac{a-2+\left\lfloor \frac{a(\lceil \frac{(t+2)b-a+3}{a} \rceil -1)+2a-4}{b} \right\rfloor}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil -1} \\ &\leq a-1+\frac{a+t}{\left\lceil \frac{(t+2)b-a+3}{a} \right\rceil -1}, \end{aligned}$$

which contradicts $I'(G) > a-1+\frac{a+t}{\lceil \frac{b(t+1)+1}{a} \rceil -1}$.

In all, we get the desired result. □

4. Sharpness examples and further argument for $I(G)$

Consider $G_1 = K_{t+1} \vee ((t+2)K_k)$ where $k \geq 2$ and $t \geq 0$ are non-negative integers to show salient sharpness feature on $I'(G)$ bound in Theorem 1.2, which deduces $\delta(G_1) = k+t$ and

$$\begin{aligned} I'(G_1) &= \min_{x \in \{2,\cdots,t+2\}} \{\frac{(t+1)+x(k-1)}{x-1}\} = k-1+\min\{\frac{k+t}{x-1}\} \\ &= k-1+\frac{k+t}{(t+2)-1} = k+\frac{k-1}{t+1}. \end{aligned}$$

Let $S = V(K_{t+1})$ and $T = V((t+2)K_k)$, it yields

$$k|S|-k|T|+\sum_{x\in T} d_{G_1-S}(x) = k(t+1)-(t+2)k = -k < 0.$$

Thus, G_1 doesn't have a fractional k-factor in terms of Lemma 2.1.

To state the sharpness of $I'(G)$ bound in Theorem 1.3, we consider $G_2 = (\left\lceil \frac{b(t+1)+1}{a} \right\rceil K_a) \vee K_{t+1}$ with $\delta(G_2) = a+t$. We infer

$$\begin{aligned} I'(G_2) &= \min_{x\in\{2,\cdots,\lceil \frac{b(t+1)+1}{a} \rceil\}} \frac{t+1+x(a-1)}{x-1} \\ &= \frac{t+1+\left\lceil \frac{b(t+1)+1}{a} \right\rceil (a-1)}{\left\lceil \frac{b(t+1)+1}{a} \right\rceil -1} \\ &= a-1+\frac{a+t}{\left\lceil \frac{b(t+1)+1}{a} \right\rceil -1}. \end{aligned}$$

Let $S = V(K_{t+1})$ and $T = V(\left\lceil \frac{b(t+1)+1}{a} \right\rceil K_a)$, then we infer

$$b|S|-a|T|+\sum_{x\in T} d_{G_2-S}(x) = b(t+1)-a\left\lceil \frac{b(t+1)+1}{a} \right\rceil < 0.$$

In view of Lemma 2.1, G_2 has no fractional $[a,b]$-factor.

We further explain the inner rationale why for isolated toughness $I(G)$, the answer to Problem 1.1 becomes negative. From the proof of Theorem 1.3, we see that the increase of $\delta(G)$ will increase the lower bound $|T_0| + l + |I_1| + |I_2|$ (part of these variables become zero in some specific cases). On the other hand, during the deduction, the maximum value of $I'(G)$ reaches when $|T_0| + l + |I_1| + |I_2|$ arrives at its lower bound. This is the reason why $I'(G)$ bound will decrease if we enhance the minimum degree of the graph. However, it becomes completely different when it comes to $I(G)$. Similar to the analysis in Gao *et al.* [27], during the derivation, the maximum value of $I(G)$ reaches when $|T_0| + l + |I_1| + |I_2|$ trends to infinity. Since the increase of $\delta(G)$ doesn't change the upper bound of $|T_0| + l + |I_1| + |I_2|$, it doesn't influence the maximum value of $I(G)$. This is why for isolated toughness $I(G)$, the answer to Problem 1.1 is negative, but for isolated toughness variant $I'(G)$, the corresponding answer is positive.

5. Knee point determination

Since the fractional factor is a crucial characteristic for measuring network data transmission, $\delta(G)$ and $I'(G)$ are important and specific aspects of network topology that need to be considered during the network design phase. From Theorem 1.2 and Theorem 1.3, $\delta(G)$ and $I'(G)$ are a pair of equilibrium parameters, and an increase in one of them will cause a decrease in the other one. Their tight bounds constitute countless pairs of possible combinations, which can cause a "selection disaster" for decision-makers, making it difficult to choose the best combination from infinite combinations as a reference indicator for network designing. To solve $\delta - I$ equilibrium problem, we consider the pair of indicators $\delta(G)$ and $I'(G)$ as PF in a bi-objective optimization problem, and a KP search algorithm is applied to determine the optimal parameter combination.

It is worth noting that the search of PF in this article is different from the search for PF in standard MOP algorithms. In general MOP algorithms, each solution can be viewed as a sample point, and thus the solution space (the number of candidates) is limited. For example, using evolutionary computation tricks to solve a MOP, each solution is an individual, and during the evolution process, the population size is limited. This ensures that there must be a boundary of points in a PF, where the extreme points can be determined and then the knot point can be calculated by its distance from the hyperplane. However, for the problem considered in this work, due to the infinite increase for the value of $\delta(G)$, there are infinite pairs of $\delta(G)$ and $I'(G)$ combinations. Therefore, new technologies should be introduced to solve a $\delta - I$ equilibrium problem.

In addition, in order to better represent PF, we treat $\delta(G)$ and $I'(G)$ as continuous variables, and for Theorem 1.2 we assume that $x = k + t$. So $t = x - k$ and $f(x) = k + \frac{k-1}{x-k+1}$. In this way, we establish a functional relationship between $\delta(G)$ and $I'(G)$, where t is the independent variable and $k \in [2, 100]$. In the following, we perform knee point determination on the corresponding curves for different values of k to obtain the relationship between k and t. Knee point determination needs to find extreme points to construct the hyperplane $\mathbb{H}$. However, for the above equation, $x \in [k, +\infty]$ and $f(x)$ is monotonically decreasing. To solve the problem of having no extreme point in the x-axis direction, we define another endpoint based on the convergence value of the function. If for any $\varepsilon > 0$, there exists a positive number M satisfying $|f(x) - A| \leq \varepsilon$ when $x > M$, then denote $\lim_{x \to +\infty} f(x) = A$. Based on the above definition of limit, we take the first x that satisfies the requirement for a different ε to be the other extreme point.

In the following, we take different ε for the curves in Theorems 1.2 and 1.3 respectively to find their knee points, and then determine the dynamics of k with respect to x by varying k (resp. a and b). For

$$f(x) = k + \frac{k-1}{x-k+1}, \tag{5.1}$$

its limit is k when $x \to +\infty$ and we set ε to be 0.05, 0.04, 0.03, 0.02, and 0.01 respectively.

As illustrated in Figure 2, we present the full set of curves corresponding to different values of $k \in [2, 9]$ under various settings of ε. In each curve, the black dots denote the left extreme points, the black pentagrams indicate the right extreme points, and the red crosses ($\times$) mark the knee points, which are of particular interest in our analysis.

TABLE 1. The relationship between different values of k and the position of knee points.

$f_1(x)$	ε	$f_x^1(k)$
$k+\frac{k-1}{x-k+1}$	0.05	$x = 1.358k + 10.61$
	0.04	$x = 1.400k + 12.03$
	0.03	$x = 1.462k + 14.09$
	0.02	$x = 1.567k + 17.34$
	0.01	$x = 1.799k + 25.11$

As the parameter k increases, both the position of the left extreme points and the limiting values of the curves exhibit noticeable changes. Additionally, a decrease in ε leads to a shift in the right extreme points, which in turn increases the distance between the two extremes. This widening of the extreme point interval causes the orientation of the corresponding hyperplane $\mathbb{H}$ to become more aligned with the x-axis, effectively reducing the angle between $\mathbb{H}$ and the horizontal direction.

For each value of ε, the variation in the position of the knee points across different k values is illustrated in Figure 3. In each subplot, blue dots represent the actual data points extracted from the curves, while the yellow lines correspond to the results of polynomial fitting.

From the five subplots in Figure 3, it is evident that the distribution of knee points exhibits a similar trend across different ε values when $k \in [2, 100]$. This similarity indicates a certain regularity in how the knee points evolve with increasing k. The detailed polynomial expressions used for fitting the data in each case are listed in Table 1, providing a precise mathematical characterization of the knee point trajectories for different ε settings.

Through Tab. 1 we can find that as ε decreases the slope and bias of $f_x^1(k)$ become larger, this is due to the fact that different ε vary the distance between the extreme points and has little to do with the curve itself.

For Theorem 1.3, we use the same method to make $x = a+t$ and $f(x) = a-1+\frac{x}{\lceil\frac{b(x-a+1)+1}{a}\rceil-1}$. $\lim_{x\to+\infty} f(x) = a-1+\frac{a}{b}$, because $a-1+\frac{x}{\frac{b(x-a+a)+1}{a}+2} \le a-1+\frac{x}{\lceil\frac{b(x-a+1)+1}{a}\rceil-1} \le a-1+\frac{x}{\frac{b(x-a+a)+1}{a}+1}$. When b is large enough, $\lim_{x\to+\infty} f(x) = a-1$, so b can not be taken very large. However, if b is obtained too small, we will not be able to get enough sample points to fit a to knee points. In this paper, we take b to be 10, 15, and 18, respectively. And for each b we take ε to be 0.05, 0.03, 0.01.

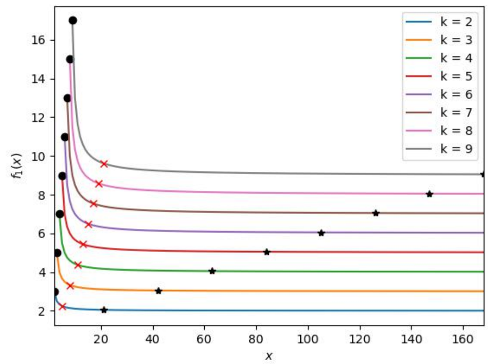
The curves for different ε and b are shown in Figure 4 ($a \in [2, 9]$), where the black dots and black pentagrams are extreme points and the red $\times$ are knee points. And Figure 5 shows the relationship between a and the position of knee points of the curves in Theorem 1.3 under different ε and b, where the blue dots are the actual correspondences and the yellow lines are the curves fitted by polynomials.

It is found through Table 2 that for the curves in Theorem 1.3, the effect of b on knee points selection is small, so it is only necessary to determine the appropriate b (if it is too large, the curve is approximated as a straight line). For the curves in Theorem 1.3, the effect of ε on knee points selection is consistent with the above, so it will not be analyzed.

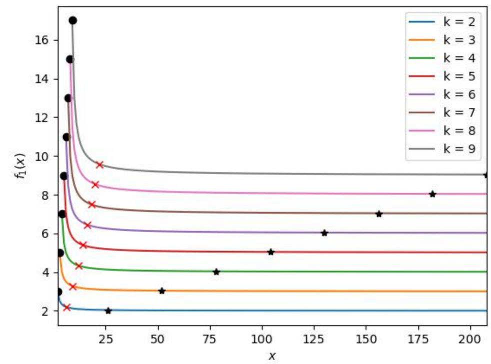
6. CONCLUSION

The contribution of this work is two-fold: (1) We answer Problem 1.1 and generalize the existing results, where Theorem 1.2 and Theorem 1.3 are the extensions of the main results in He *et al.* [5] and Gao and Wang [7], respectively. (2) By regarding the $\delta(G)$-$I'(G)$ curve as the PF in bi-objective optimization problem, we propose a novel approach to determine the KP in this setting, and hence solve the $\delta - I$ equilibrium problem for decision maker. This trick, to some extent, alleviates the "choice dilemma" and has potential theoretical guidance significance for network designing.

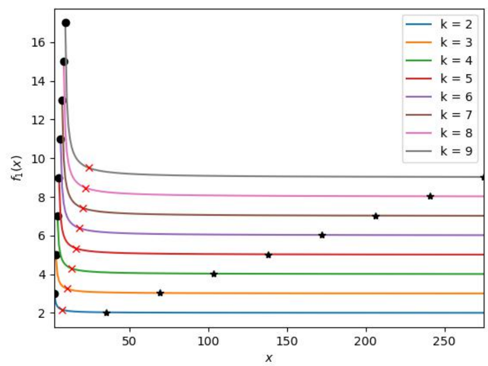
In real network construction, the number of sites is pre-planned, *i.e.*, $|V(G)|$ is known. Therefore, the selection of the minimum degree is from k to $\frac{|V(G)|-1}{2}$, because once the minimum degree reaches $\frac{|V(G)|}{2}$, according to the classical results of the fractional factor, there must be a fractional factor in the graph (assuming $|V(G)|$



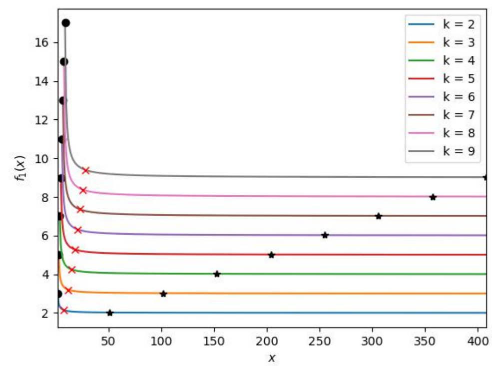
(a) $\varepsilon = 0.05$



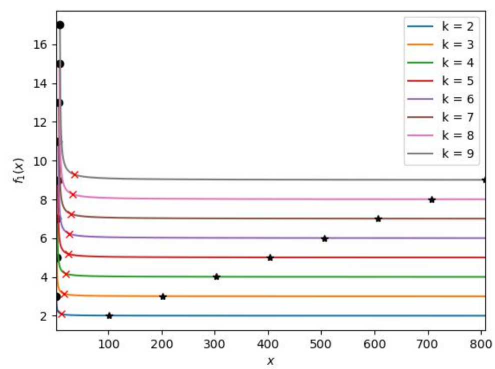
(b) $\varepsilon = 0.04$



(c) $\varepsilon = 0.03$

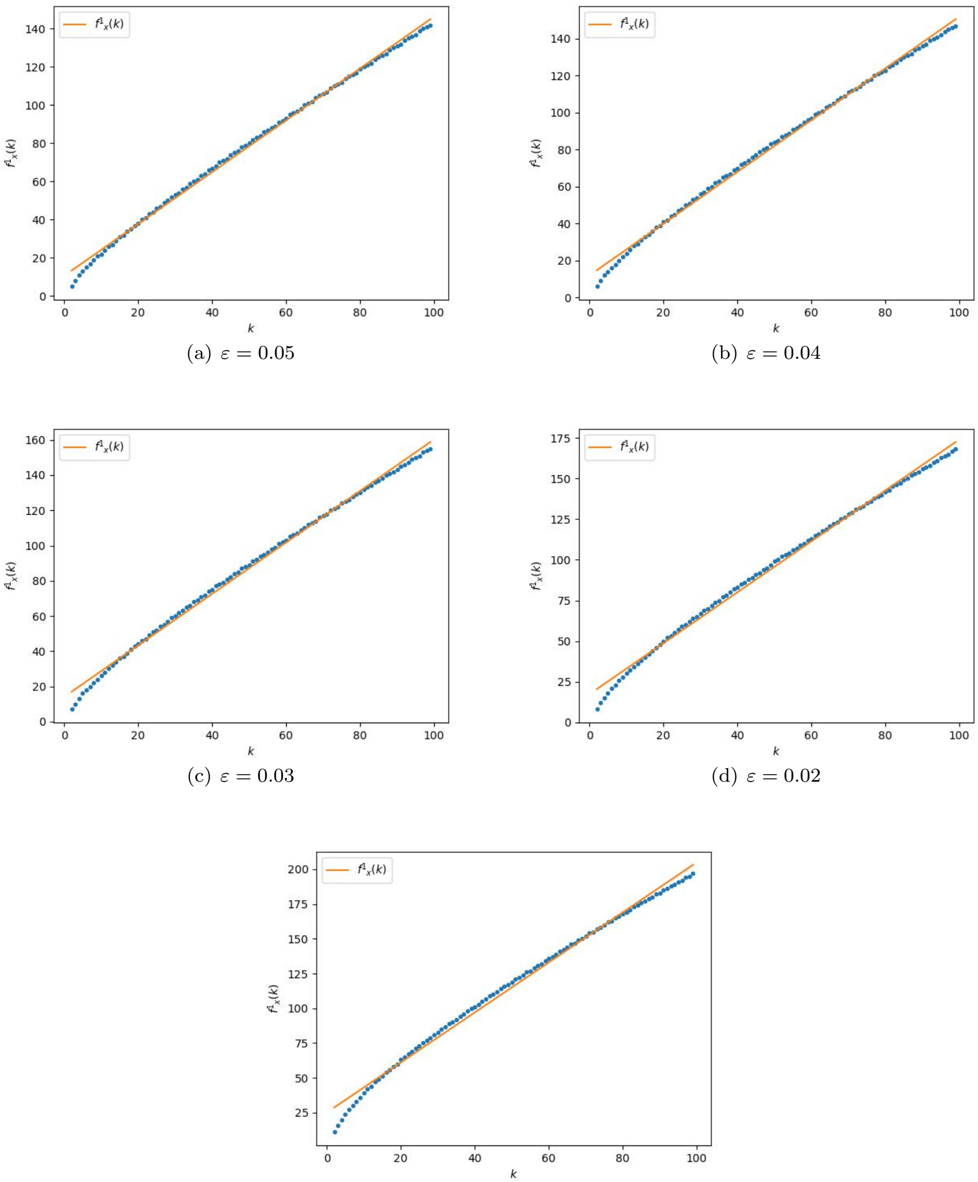


(d) $\varepsilon = 0.02$



(e) $\varepsilon = 0.01$

FIGURE 2. Knee points of the curves in Theorem 1.2 under different ε, where the black circle and the pentagram represent the extreme points of the curve, and the red $\times$ is the knee point on the curve.



(a) $\varepsilon = 0.05$ (b) $\varepsilon = 0.04$

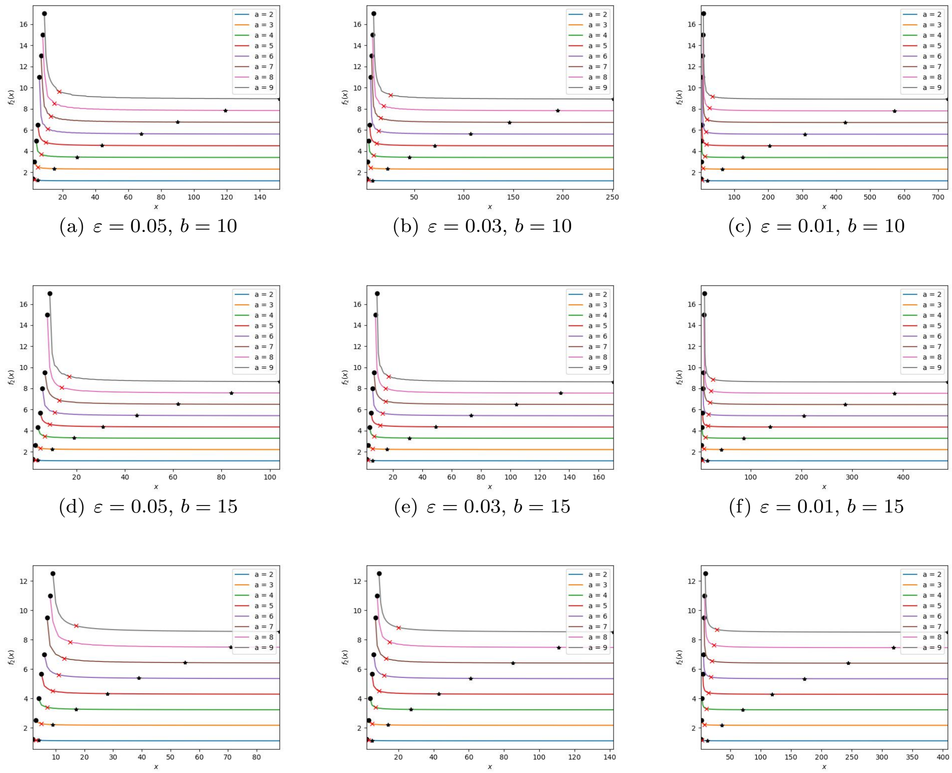
(c) $\varepsilon = 0.03$ (d) $\varepsilon = 0.02$

(e) $\varepsilon = 0.01$

FIGURE 3. The relationship between k and position of knee points of the curves in Theorem 1.2 under different ε, where the blue dots are the actual correspondences and the yellow lines are the curves fitted by polynomials

is large enough). At this point, the condition of isolated toughness no longer plays a role, and degenerates into a single minimum degree condition. Therefore, in fact, the isolated toughness condition is valuable only when the minimum degree is within the range of $[k, \frac{|V(G)|-1}{2}]$. In this way, the extremum value in PF and the resulting hyperplane are determined, and thus the knee point is unique calculated. We refer to $\frac{|V(G)|}{2}$ as the phase transition point of isolated toughness regarding the existence of fractional factors.

The following interesting questions can serve as further study topics.



(a) $\varepsilon = 0.05$, $b = 10$ (b) $\varepsilon = 0.03$, $b = 10$ (c) $\varepsilon = 0.01$, $b = 10$

(d) $\varepsilon = 0.05$, $b = 15$ (e) $\varepsilon = 0.03$, $b = 15$ (f) $\varepsilon = 0.01$, $b = 15$

(g) $\varepsilon = 0.05$, $b = 18$ (h) $\varepsilon = 0.03$, $b = 18$ (i) $\varepsilon = 0.01$, $b = 18$

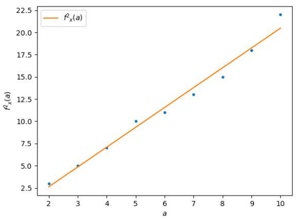
FIGURE 4. Knee points of the curves in Theorem 1.3 under different ε and b, where the black circle and the pentagram represent the extreme points of the curve, and the red $\times$ is the knee point on the curve.

TABLE 2. The relationship between different values of k and the position of knee points.

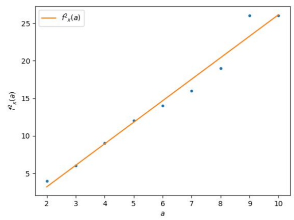
$f_2(x)$	ε	b	$f_x^2(a)$
$a - 1 + \frac{x}{\lceil \frac{b(x-a+1)+1}{a} \rceil - 1}$	0.05	10	$x = 2.233a - 1.844$
		15	$x = 2.046a - 1.464$
		18	$x = 1.983a - 1.240$
	0.03	10	$x = 2.867a - 2.533$
		15	$x = 2.453a - 2.420$
		18	$x = 2.348a - 2.892$
	0.01	10	$x = 4.117a - 4.478$
		15	$x = 3.420a - 3.068$
		18	$x = 3.086a - 2.034$

• There are various definitions of Pareto domination in MOP, while only the simplest version of domination is used in our article. Assuming the mathematical setting of Pareto domination is changed, how will the KP determination algorithm be modified to meet various frameworks?

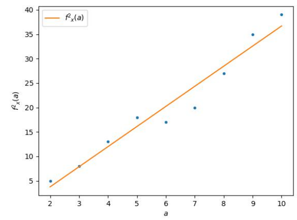
• The existence of fractional factors in a specific geometric space (with genus g) remains an open problem. Embedding a graph into a geometric space with a given number of holes, how can the fractional factor of such a specific embedded graph be characterized by specific parameters?



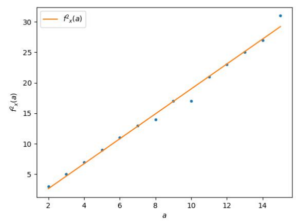
(a) $\varepsilon = 0.05,\ b = 10$



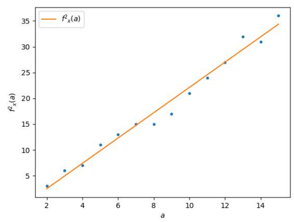
(b) $\varepsilon = 0.03,\ b = 10$



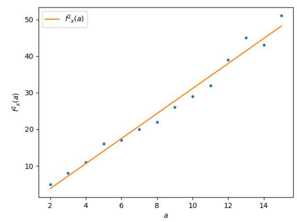
(c) $\varepsilon = 0.01,\ b = 10$



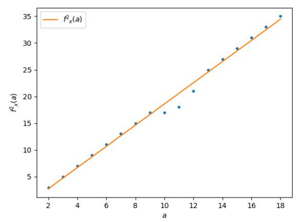
(d) $\varepsilon = 0.05,\ b = 15$



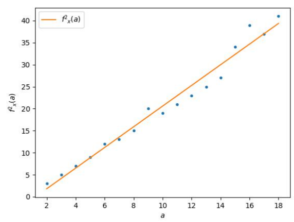
(e) $\varepsilon = 0.03,\ b = 15$



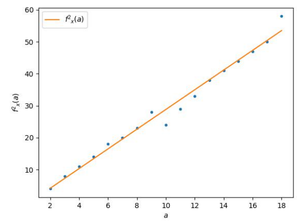
(f) $\varepsilon = 0.01,\ b = 15$



(g) $\varepsilon = 0.05,\ b = 18$



(h) $\varepsilon = 0.03,\ b = 18$



(i) $\varepsilon = 0.01,\ b = 18$

Figure 5. The relationship between a and the position of knee points of the curves in Theorem 1.3 under different ε and b, where the blue dots are the actual correspondences and the yellow lines are the curves fitted by polynomials.

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Data availability statement

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Competing interests

We declare that none of the authors have any competing interests in the manuscript.

Author contribution statement

Yaojun Chen wrote the manuscript, Hainan Zhang completed the experimental code, Wei Gao revised and made improvements on the manuscript. The authors have worked equally when writing this paper. All authors read and approved the final manuscript.

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