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ZAGREB INDEX IN EXPONENTIAL PLANE-ORIENTED RECURSIVE TREES

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Abstract. The Zagreb index of a graph is the sum of the squared degrees of all nodes in the graph. In this note, we study the Zagreb index of exponential plane-oriented recursive trees. We first show the convergence in L_2 for the root degree. Then we calculate the first two moments of the Zagreb index from a recurrence. Finally, the limit law for the Zagreb index of an exponential plane-oriented recursive tree is characterized by an application of the contraction method.

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1. Introduction

A *random recursive tree* of size n is grown dynamically, by starting with a root node labeled with 1, and at step i ($i = 2, \ldots, n$) a new node labeled with i is attached uniformly at random to one of the previous nodes $1, \ldots, i-1$ (see Fig. 1). Random recursive trees may serve as a model for many phenomena such as pyramid schemes [1]; the spread of epidemics [2]; and the stochastic growth of networks [3]. For only a few of variations of recursive trees, see [4, 5] and [6]; for fundamental results see the survey paper [7].

A *plane-oriented recursive tree* (PORT) is a version of recursive trees in which descendants of each node are ordered. The nodes that already exist in the tree are called *internal*. In order to identify the locations of possible insertions of a new node (internal node), a plane-oriented recursive tree is extended by having $d+1$ *external* nodes attached as children of a node (internal node) of outdegree d; see Figure 2. In an extended plane-oriented recursive tree of n nodes, the number of all external nodes is $2n-1$ (see Lem. 1 in [8]). At any step in the growth of a *random PORT*, each external node has the same probability of being selected as the location of the next node to be inserted.

Since only one node is added at each step in the growth of a PORT, then these slow-growing models cannot be suitable for fast-growing phenomena, *e.g.* the spreading of infectious diseases. An alternative fast-growing model, the exponential plane-oriented recursive tree (exponential PORT) is introduced in [9]. The exponential PORT grows by having every external node either convert into an internal node with probability p or stay as is with probability $q := 1-p$. The parameter p is called the *index* of the exponential PORT. The *age* of an exponential PORT is defined to be the number of steps it takes to grow the tree.

A *random exponential PORT of index p* grown as follows: Initially (at age 0), there is an internal node to which an external node is connected. For $n \geq 1$, at the nth step (at age n), every external node independently

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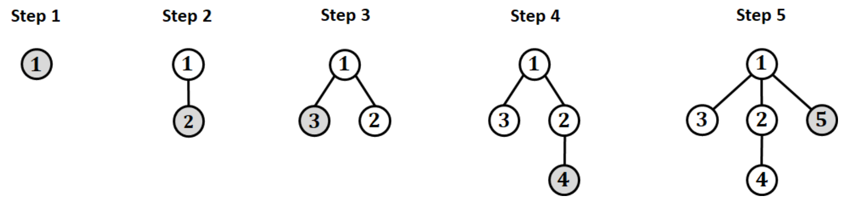

FIGURE 1. The steps of evolution of a recursive tree of size 5.

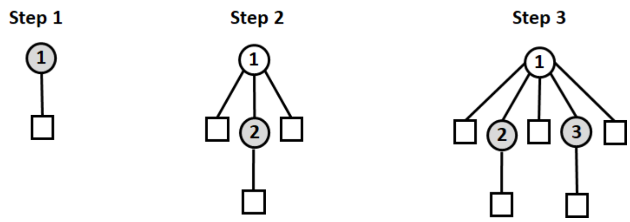

FIGURE 2. The steps of evolution of a plane-oriented recursive tree of size 3, where the internal nodes are shown as circles ◯; the external nodes are shown as squares □. At step 3, the tree has 3 internal nodes (*i.e.*, the tree has 5= $2 \times 3 - 1$ external nodes).

is changed into an internal node with probability $p \in (0, 1)$, or not to be changed with probability $q = 1 - p$. Figure 3 shows all exponential PORTs of ages 0, 1 and 2.

The number of leaves and total path length have been studied for exponential recursive trees, exponential PORTs by [10] and for exponential recursive k-ary tree (in which every node has at most k children) by [11]. The number of protected nodes (nodes that are not leaf and not all of their children are leaves) has also been investigated for exponential recursive trees by [12].

By *degree* of a node, we mean the number of children of that node. We can see [13] and [4] as the examples where the node degrees have been investigated.

The *Zagreb index*, a kind of graph-based topological index, is defined as the sum of the squared degrees of all nodes in a graph. The exact mean, variance and asymptotic distribution of the Zagreb index for recursive trees, PORTs and a class of networks extended from recursive trees were presented in [14], [6] and [15], respectively. Our interest here is to study the Zagreb index of exponential PORTs.

2. Root degree

In the two next sections, we require to obtain the expectation and variance of the root degree. We also need to prove the convergence in L^2 for the root degree. Let X_n be the root degree in T_n, an exponential PORT of age n. At the first step, if the initial external node does not succeed to convert into an internal node, then the tree will evolve in $n-1$ subsequent steps to have $X'_{n-1} \overset{d}{=} X_{n-1}$ root degree, where the symbol $\overset{d}{=}$ denotes the equality in distribution. Alternatively, if the initial external node succeeds to convert into an internal node at the first step (event $\mathcal{R}$), then the exponential PORT evolves to have a new internal node and three new external nodes (see the gray tree at step 1 in Fig. 3): one external node (*middle* one) appears under the new internal node and the other two external nodes (the *right* and *left* of the new internal node) appear under the root. Then, each of these three external nodes acting independently to construct its own exponential tree PORT in the next $n-1$ steps of the evolution of T_n. Consider the distributional decomposition of T_n into three exponential PORT's, one sprouting from each external node (see Sect. 6 in [10]). Let the root degree arising from the left and right subtrees be respectively $X^{(1)}_{n-1}$ and $X^{(2)}_{n-1}$. Let $\mathbb{I}$ be the indicator of event $\mathcal{R}$, that assumes the value 1,

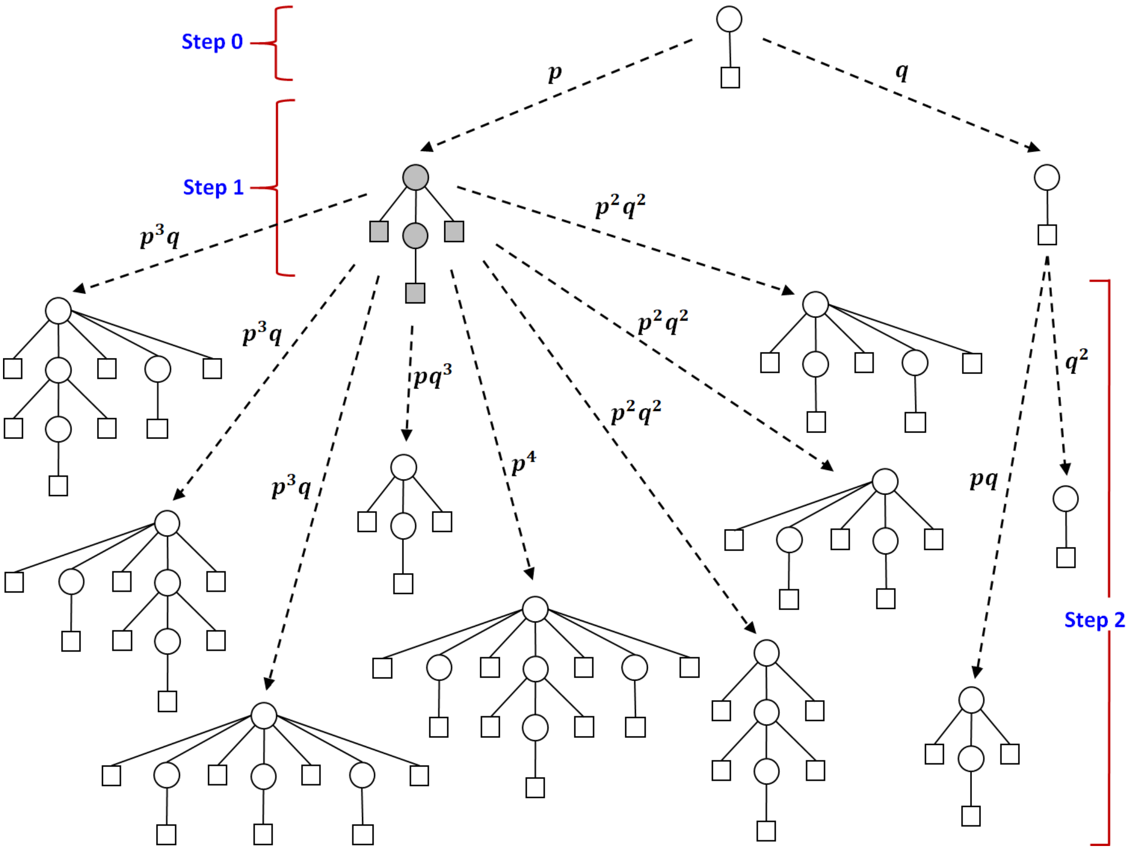

FIGURE 3. The first two steps of evolution of the exponential plane-oriented recursive tree of index p. The probability of getting each tree from the previous one appears above the arrow leading to it.

if $\mathcal{R}$ occurs, and is 0 otherwise. Hence, the root degree satisfies the distributional equation

$$X_n \stackrel{d}{=} (1 - \mathbb{I})X'_{n-1} + \mathbb{I}\big(X^{(1)}_{n-1} + X^{(2)}_{n-1} + 1\big), \tag{2.1}$$

where $X^{(1)}_{n-1}$, $X^{(2)}_{n-1}$ and X'_{n-1} are independent copies of X_{n-1} and independent of $\mathbb{I}$.

Let X^*_n be the root degree in an exponential recursive tree of age n and index p. Then we have $X^*_n \stackrel{d}{=}$ Binomial(n, p). Consequently, $\mathbb{E}[X^*_n] = np$, *i.e.*, the expected root degree is linearly increasing. However, in an exponential PORT, there are different positions (external nodes) under the root where a new internal node can be placed and joined to the root node. By the following theorem, we can see that the expected root degree in an exponential PORT of age n and index p, $\mathbb{E}[X_n] = (p + 1)^n - 1$ is exponentially increasing.

Theorem 2.1. *Let X_n be the root degree in an exponential PORT of age n and index p. We have*

$$\mathbb{E}[X_n] = (p + 1)^n - 1,$$
$$\mathrm{Var}[X_n] = (1 - p)(p + 1)^{2n-1} - 2(p + 1)^{n-1} + (p + 1)^n.$$

Proof. Take expectations both sides of the equation (2.1). By the identical distribution of X'_{n-1}, $X^{(1)}_{n-1}$ and $X^{(2)}_{n-1}$, we get

$$\mathbb{E}[X_n] = (1-p)\mathbb{E}[X_{n-1}] + 2p\mathbb{E}[X_{n-1}] + 1 = (p+1)\mathbb{E}[X_{n-1}] + p,$$

with initial condition $\mathbb{E}[X_0] = 0$. By iterating, the mean value follows:

$$\mathbb{E}[X_n] = (p+1)^n\mathbb{E}[X_0] + \sum_{j=0}^{n-1}(p+1)^{n-1-j}p = (p+1)^n - 1.$$

For obtaining the second moment, raise both sides of (2.1) to the second power. So we get

$$\begin{aligned} X_n^2 \overset{d}{=} (1-\mathbb{I})X'^2_{n-1} + \mathbb{I}\Big(\big(X^{(1)}_{n-1}\big)^2 + \big(X^{(2)}_{n-1}\big)^2 + 1\Big) \\ + \mathbb{I}\Big(2X^{(1)}_{n-1}X^{(2)}_{n-1} + 2X^{(1)}_{n-1} + 2X^{(2)}_{n-1}\Big). \end{aligned} \tag{2.2}$$

Take the expectation of (2.2) and observe the independence of $X^{(1)}_{n-1}$ and $X^{(2)}_{n-1}$, to write

$$\begin{aligned} \mathbb{E}[X_n^2] &= (1-p)\mathbb{E}[X_{n-1}^2] + p\Big(2\mathbb{E}[X_{n-1}^2] + 2\big(\mathbb{E}[X_{n-1}]\big)^2 + 4\mathbb{E}[X_{n-1}] + 1\Big) \\ &= (p+1)\mathbb{E}[X_{n-1}^2] + 2p(p+1)^{2n-2} - p \\ &=: (p+1)\mathbb{E}[X_{n-1}^2] + \tilde{A}_{n-1}, \end{aligned} \tag{2.3}$$

with initial condition $\mathbb{E}[X_0^2] = 0$. Iterating this formula, we find

$$\begin{aligned} \mathbb{E}[X_n^2] &= (p+1)^n\mathbb{E}[X_0^2] + \sum_{j=0}^{n-1}(p+1)^{n-1-j}\tilde{A}_j \\ &= \sum_{j=0}^{n-1}(p+1)^{n-1-j}\big(2p(p+1)^{2j} - p\big) \\ &= 2(p+1)^{2n-1} - 2(p+1)^{n-1} - (p+1)^n + 1. \end{aligned}$$

This yields the variance of X_n. ☐

Now, we use the contraction method in [16] to characterize the limiting distribution of the scaled root degree $\hat{X}_n := X_n/(p+1)^n$. By equation (2.1), we have

$$\hat{X}_n \overset{d}{=} \frac{1-\mathbb{I}}{p+1}\,\hat{X}'_{n-1} + \frac{\mathbb{I}}{p+1}\,\big(\hat{X}^{(1)}_{n-1} + \hat{X}^{(2)}_{n-1}\big) + \frac{\mathbb{I}}{(p+1)^n} \tag{2.4}$$

All the conditions of the contraction method are satisfied (Thm. 3 of [17], page 7), then

Theorem 2.2. *The scaled root degree $\hat{X}_n := X_n/((p+1)^n$ in an exponential PORT of age n and index p, as $n \to \infty$, converges in distribution and in L_2 to some random variable X with distribution, the unique solution of the following distributional equation*

$$X \overset{d}{=} \frac{1-\mathbb{I}}{p+1}X^{(0)} + \frac{\mathbb{I}}{p+1}\big(X^{(1)} + X^{(2)}\big), \tag{2.5}$$

where $X^{(i)}$, $i = 0, 1, 2$, *are independent copies of* X *and independent of* $\mathbb{I}$, *the bernoulli random variable with success probability* p. *Therefore,*

$$\mathbb{E}[X] = 1, \qquad \mathbb{E}[X^2] = \frac{2}{(p+1)^2}.$$

Proof. The proof is based on the Lemma 3.1 in [16]. Let $\mathcal{M}_2$ be the space of all distributions with mean 1 and finite absolute second moment. Consider the transformation $T : \mathcal{M}_2 \to \mathcal{M}_2$ defined by

$$T(\mu) \;=\; \frac{1-\mathbb{I}}{p+1} X_\mu^{(0)} + \frac{\mathbb{I}}{p+1}\big(X_\mu^{(1)} + X_\mu^{(2)}\big),$$

where $X_\mu^{(i)}$, $i = 0, 1, 2$ and $\mathbb{I}$ are independent and $X_\mu^{(i)}$ have μ as distribution.

At a first step we have to prove that with respect to the metric $\mathcal{L}_2^2$ defined on $\mathcal{M}_2$ by

$$\mathcal{L}_2^2\,(\mu,\,\nu) := \inf\left\{\mathbb{E}[|Y_1 - Y_2|^2],\ \text{where } Y_1 \text{ and } Y_2 \text{ have } \mu \text{ and } \nu \text{ as distribution, respectively}\right\},$$

the transformation T have a unique fixed point. In fact let μ and ν be two measures of $\mathcal{M}_2$

$$\begin{aligned}
T(\mu) \;&=\; \frac{1-\mathbb{I}}{p+1} X_\mu^{(0)} + \frac{\mathbb{I}}{p+1}\big(X_\mu^{(1)} + X_\mu^{(2)}\big),\\
T(\nu) \;&=\; \frac{1-\mathbb{I}}{p+1} X_\nu^{(0)} + \frac{\mathbb{I}}{p+1}\big(X_\nu^{(1)} + X_\nu^{(2)}\big),\\
\mathcal{L}_2^2\,(T(\mu),\,T(\nu)) &\le \mathbb{E}\big[\big|T(\mu) - T(\nu)\big|^2\big]\\
&= \frac{\mathbb{E}\big[(1-\mathbb{I})^2\big]}{(p+1)^2}\,\mathbb{E}\big[\big|X_\mu - X_\nu\big|^2\big] + \frac{2\,\mathbb{E}[\mathbb{I}^2]}{(p+1)^2}\,\mathbb{E}\big[\big|X_\mu - X_\nu\big|^2\big]\\
&= \frac{1}{p+1}\,\mathbb{E}\big[\big|X_\mu - X_\nu\big|^2\big]. \qquad (2.6)
\end{aligned}$$

Take infimum on both sides of the equation (2.6), over all X_μ and X_ν, we have

$$\mathcal{L}_2^2\,(T(\mu),\,T(\nu)) \le \frac{1}{p+1}\,\mathcal{L}_2^2\,(\mu,\,\nu)\,.$$

Since $\frac{1}{p+1} < 1$, then T is a contraction and it has only one fixed point. Now, if $\hat{X}_n$ converges in distribution to some random variable X, the limit will be the unique fixed point of the transformation T. Namely, we have to prove that

$$\lim_{n\to\infty}\,\mathcal{L}_2^2\big(X,\,\hat{X}_n\big) = 0,$$

to conclude that $\hat{X}_n$ converges in distribution and in L_2 to X. From the equations (2.4) and (2.5),

$$\begin{aligned}
\mathcal{L}_2^2\big(X,\,\hat{X}_n\big) \le \frac{1}{p+1}\mathcal{L}_2^2\big(X,\,\hat{X}_{n-1}\big) &+ \frac{4p}{(p+1)^{n+1}}\mathbb{E}\big[X^{(1)} - \hat{X}_{n-1}^{(1)}\big]\\
&+ \frac{2p}{(p+1)^2}\mathbb{E}\big[X^{(1)} - \hat{X}_{n-1}^{(1)}\big]\mathbb{E}\big[X^{(2)} - \hat{X}_{n-1}^{(2)}\big] + \frac{p}{(p+1)^{2n}},
\end{aligned}$$

where $\|\cdot\|_2$ denotes the L_2-norm. Then we deduce

$$\lim_{n\to\infty}\,\mathcal{L}_2^2\big(X,\,\hat{X}_n\big) \le \frac{1}{p+1}\lim_{n\to\infty}\,\mathcal{L}_2^2\big(X,\,\hat{X}_{n-1}\big) < \lim_{n\to\infty}\,\mathcal{L}_2^2\big(X,\,\hat{X}_n\big).$$

This implies $\lim_{n\to\infty}\mathcal{L}_2^2\big(X,\,\hat{X}_n\big) = 0$. □

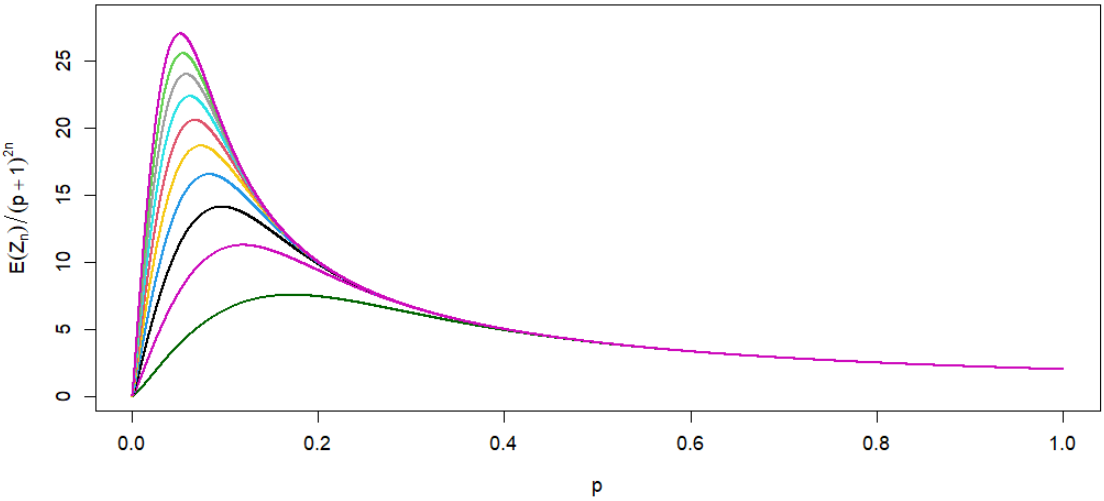

FIGURE 4. Graphs of $\frac{\mathbb{E}[Z_n]}{(p+1)^{2n}}$ versus p, for $n = 50, 100, 150, 200, \ldots, 500$. For each p, the fraction $\frac{\mathbb{E}[Z_n]}{(p+1)^{2n}}$ is increasing in n. *e.g.*, the bottom and top curves are the graphs of $\frac{\mathbb{E}[Z_{50}]}{(p+1)^{100}}$ and $\frac{\mathbb{E}[Z_{500}]}{(p+1)^{1000}}$.

3. The first two moments of Zagreb index

Let S_n be the size of T_n, an exponential PORT of age n. For $i = 1, \ldots, S_n$, let $D_{i,n}$ be the degree of node i of the tree T_n, where $D_{1,n}$ is the root degree. Furthermore, by the equation (2.1), we have $D_{1,n} \stackrel{d}{=} (1-\mathbb{I})X'_{n-1} + \mathbb{I}\big(X^{(1)}_{n-1} + X^{(2)}_{n-1} + 1\big)$. Denote by Z_n the Zagreb index of T_n, *i.e.*,

$$Z_n := \sum_{i=1}^{S_n} D^2_{i,n} \stackrel{d}{=} \Big((1-\mathbb{I})X'_{n-1} + \mathbb{I}\big(X^{(1)}_{n-1} + X^{(2)}_{n-1} + 1\big)\Big)^2 + \sum_{i=2}^{S_n} D^2_{i,n}. \tag{3.1}$$

In Section 2, we have applied a distributional decomposition of T_n into three subtrees sprouted from three external nodes: right, middle and left, at the first step, if the initial external node succeeds to convert into an internal node. Now, let $Z^{(1)}_{n-1}$, $Z^{(2)}_{n-1}$ and $Z^{(3)}_{n-1}$ be the Zagreb indices arising from the left, middle and right subtrees, respectively. Then, by the equation (3.1), the Zagreb index of T_n satisfy the distributional equation

$$Z_n \stackrel{d}{=} (1-\mathbb{I})Z'_{n-1} + \mathbb{I}\big(Z^{(1)}_{n-1} + Z^{(2)}_{n-1} + Z^{(3)}_{n-1} + 2X^{(1)}_{n-1}X^{(2)}_{n-1} + 2X^{(1)}_{n-1} + 2X^{(2)}_{n-1} + 1\big), \tag{3.2}$$

where $Z^{(i)}_{n-1}$, $i = 1, 2, 3$, and Z'_{n-1} are independent copies of Z_{n-1} and independent of $\mathbb{I}$.

In Figure 4, by the following theorem, we show that the scaled expected Zagreb index $\frac{\mathbb{E}[Z_n]}{(p+1)^{2n}}$ in an exponential PORT of age n and index p, *i.e.*,

$$\frac{\mathbb{E}[Z_n]}{(p+1)^{2n}} = \frac{2}{p} - \frac{2}{p} \cdot \frac{(2p+1)^n}{(p+1)^{2n}} - \frac{1}{2} \cdot \frac{(2p+1)^n}{(p+1)^{2n}} + \frac{1}{2} \cdot \frac{1}{(p+1)^{2n}},$$

is an increasing function of n, for $p \in [0, 1]$.

Theorem 3.1. *Let Z_n be the Zagreb index in an exponential PORT of age n and index p. We have*

$$\mathbb{E}[Z_n] = \frac{2}{p}(p+1)^{2n} - \frac{2}{p}(2p+1)^n - \frac{1}{2}(2p+1)^n + \frac{1}{2} = \frac{2}{p}(p+1)^{2n} + \mathcal{O}\big((2p+1)^n\big),$$
$$\mathrm{Var}[Z_n] = \frac{4(1-p)(p^5+9p^4+34p^3+64p^2+36p+8)}{p^2(p^3+4p^2+6p+2)(p+2)}(p+1)^{4n-2} + \mathcal{O}\big((2p+1)^n(p+1)^{2n}\big).$$

Proof. Taking the expectation of (3.2), we obtain

$$\begin{aligned}\mathbb{E}[Z_n] &= (2p+1)\mathbb{E}[Z_{n-1}] + 2p\big(\mathbb{E}[X_{n-1}]\big)^2 + 4p\mathbb{E}[X_{n-1}] + p \\ &= (2p+1)\mathbb{E}[Z_{n-1}] + 2p(p+1)^{2n-2} - p =: (2p+1)\mathbb{E}[Z_{n-1}] + \tilde{A}_{n-1},\end{aligned}$$

with $\mathbb{E}[Z_0] = 0$. Again, the equation is of the form (2.3). So, the expectation follows.

By multiplying (2.1) and (3.1), we get

$$\begin{aligned}Z_nX_n \overset{d}{=} &(1-\mathbb{I})Z'_{n-1}X'_{n-1} + \mathbb{I}\big(Z^{(1)}_{n-1} + Z^{(2)}_{n-1} + Z^{(3)}_{n-1}\big)\big(X^{(1)}_{n-1} + X^{(2)}_{n-1} + 1\big) \\ &+ \mathbb{I}\big(2X^{(1)}_{n-1}X^{(2)}_{n-1} + 2X^{(1)}_{n-1} + 2X^{(2)}_{n-1} + 1\big)\big(X^{(1)}_{n-1} + X^{(2)}_{n-1} + 1\big).\end{aligned}$$

By simplifying,

$$\begin{aligned}Z_nX_n \overset{d}{=} &(1-\mathbb{I})Z'_{n-1}X'_{n-1} + \mathbb{I}\Big(Z^{(1)}_{n-1} + Z^{(2)}_{n-1} + Z^{(3)}_{n-1} + 3X^{(1)}_{n-1} + 3X^{(2)}_{n-1} + 1 \\ &+ Z^{(1)}_{n-1}X^{(1)}_{n-1} + Z^{(2)}_{n-1}X^{(2)}_{n-1} + Z^{(2)}_{n-1}X^{(1)}_{n-1} + Z^{(3)}_{n-1}X^{(1)}_{n-1} + Z^{(1)}_{n-1}X^{(2)}_{n-1} + Z^{(3)}_{n-1}X^{(2)}_{n-1} \\ &+ 6X^{(1)}_{n-1}X^{(2)}_{n-1} + 2\big(X^{(1)}_{n-1}\big)^2X^{(2)}_{n-1} + 2X^{(1)}_{n-1}\big(X^{(2)}_{n-1}\big)^2 + 2\big(X^{(1)}_{n-1}\big)^2 + 2\big(X^{(2)}_{n-1}\big)^2\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}[Z_nX_n] &= (p+1)\mathbb{E}[Z_{n-1}X_{n-1}] + 4p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}] + 3p\mathbb{E}[Z_{n-1}] \\ &\quad + 6p\big(\mathbb{E}[X_{n-1}]\big)^2 + 4p\mathbb{E}[X_{n-1}]\mathbb{E}[X^2_{n-1}] + 4p\mathbb{E}[X^2_{n-1}] + 6p\mathbb{E}[X_{n-1}] + p.\end{aligned}$$

From substituting $\mathbb{E}[Z_n]$ and $\mathbb{E}[X_n]$,

$$\begin{aligned}\mathbb{E}[Z_nX_n] &= (p+1)\mathbb{E}[Z_{n-1}X_{n-1}] + \frac{8(2p+1)}{p+1}(p+1)^{3n-3} + \frac{4+p}{2}(2p+1)^{n-1} + \frac{1}{2}p \\ &\quad - \frac{2(1+4p-p^2)}{p+1}(p+1)^{2n-2} - (8+2p)(p+1)^{n-1}(2p+1)^{n-1},\end{aligned}$$

with the initial value $\mathbb{E}[Z_0X_0] = 0$. This gives the solution

$$\begin{aligned}\mathbb{E}[Z_nX_n] = &\frac{8(2p+1)}{p(p+2)}(p+1)^{3n-2} - \frac{2}{p}(1+4p-p^2)(p+1)^{2n-2} - \frac{p+4}{p}(p+1)^{n-1}(2p+1)^n \\ &+ \frac{p+4}{2p}(2p+1)^n + \frac{3(2-p)}{p+2}(p+1)^{n-1} - \frac{1}{2}.\end{aligned}$$

Raise both sides of (3.2) to the second power and take the expectation. Hence, after all, we obtain

$$\begin{aligned}\mathbb{E}[Z_n^2] &= (2p+1)\mathbb{E}[Z_{n-1}^2] + p\Big(4\big(\mathbb{E}[X_{n-1}^2]\big)^2 + 6\big(\mathbb{E}[Z_{n-1}]\big)^2 + 12\big(\mathbb{E}[X_{n-1}]\big)^2 + 8\mathbb{E}[X_{n-1}^2] \\ &\quad + 16\mathbb{E}[X_{n-1}^2]\mathbb{E}[X_{n-1}] + 16\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}] + 4\mathbb{E}[Z_{n-1}]\big(\mathbb{E}[X_{n-1}]\big)^2 \\ &\quad + 8\mathbb{E}[Z_{n-1}X_{n-1}]\mathbb{E}[X_{n-1}] + 8\mathbb{E}[Z_{n-1}X_{n-1}] + 8\mathbb{E}[X_{n-1}] + 6\mathbb{E}[Z_{n-1}] + 1\Big).\end{aligned}$$

By substituting the expectations,

$$\begin{aligned}\mathbb{E}[Z_n^2] &= (2p+1)\mathbb{E}[Z_{n-1}^2] + \frac{8(p^4+9p^3+37p^2+25p+6)}{p(p+2)(p+1)^2}(p+1)^{4n-4} \\ &\quad - \frac{2(p+4)(p^2+11p+6)}{p(p+1)}(p+1)^{2n-2}(2p+1)^{n-1} - \frac{48p(1-p)}{(p+1)^2}(p+1)^{3n-3} \\ &\quad + \frac{3(p+4)^2}{2p}(2p+1)^{2n-2} + \frac{2p(p^3-16p^2+p+26)}{(p+2)(p+1)^2}(p+1)^{2n-2} - \frac{p}{2}.\end{aligned}$$

This has the form (2.3), with solution

$$\mathbb{E}[Z_n^2] = \frac{8(p^4+9p^3+37p^2+25p+6)}{p^2(p^3+4p^2+6p+2)(p+2)}(p+1)^{4n-2} + \mathcal{O}\big((2p+1)^n(p+1)^{2n}\big).$$

which completes the proof. □

4. Limit law for Zagreb index

In this section, we show the convergence in L_2 and characterize the limiting distribution of for $\hat{Z}_n := Z_n/(p+1)^{2n}$ by contraction method. By equation (3.2), we have

$$\begin{aligned}\frac{Z_n}{(p+1)^{2n}} &\stackrel{d}{=} (1-\mathbb{I})\cdot\frac{1}{(p+1)^2}\cdot\frac{Z'_{n-1}}{(p+1)^{2n-2}} + \mathbb{I}\cdot\frac{1}{(p+1)^{2n}}\cdot\big(2X_{n-1}^{(1)} + 2X_{n-1}^{(2)} + 1\big) \\ &\quad + \mathbb{I}\cdot\frac{1}{(p+1)^2}\cdot\Big(\frac{Z_{n-1}^{(1)}}{(p+1)^{2n-2}} + \frac{Z_{n-1}^{(2)}}{(p+1)^{2n-2}} + \frac{Z_{n-1}^{(3)}}{(p+1)^{2n-2}}\Big) \\ &\quad + \mathbb{I}\cdot\frac{2}{(p+1)^2}\cdot\Big(\frac{X_{n-1}^{(1)}}{(p+1)^{n-1}}\cdot\frac{X_{n-1}^{(2)}}{(p+1)^{n-1}}\Big).\end{aligned}$$

After simplification, the distributional equation for $\hat{Z}_n$ is

$$\begin{aligned}\hat{Z}_n &\stackrel{d}{=} \frac{1-\mathbb{I}}{(p+1)^2}\cdot\hat{Z}'_{n-1} + \frac{\mathbb{I}}{(p+1)^{2n}}\cdot\big(2X_{n-1}^{(1)} + 2X_{n-1}^{(2)} + 1\big) \\ &\quad + \frac{\mathbb{I}}{(p+1)^2}\cdot\big(\hat{Z}_{n-1}^{(1)} + \hat{Z}_{n-1}^{(2)} + \hat{Z}_{n-1}^{(3)}\big) + \frac{2\mathbb{I}}{(p+1)^2}\cdot\hat{X}_{n-1}^{(1)}\cdot\hat{X}_{n-1}^{(2)}.\end{aligned} \tag{4.1}$$

In the following theorem, all conditions of Theorem 3 of [17] are satisfied.

Theorem 4.1. *The scaled Zagreb index $\hat{Z}_n := Z_n/(p+1)^{2n}$ in an exponential PORT of age n and index p, as $n \to \infty$, converges in distribution and in L_2 to some random variable Z with distribution, the unique solution* *of the following distributional equation*

$$Z \overset{d}{=} \frac{1-\mathbb{I}}{(p+1)^2}\, Z^{(0)} + \frac{\mathbb{I}}{(p+1)^2}\big(Z^{(1)} + Z^{(2)} + Z^{(3)}\big) + \frac{2\mathbb{I}}{(p+1)^2}\, X^{(1)}X^{(2)}, \tag{4.2}$$

where $Z^{(i)}$, $i = 0, 1, 2, 3$, *are independent copies of* Z *and independent of* $\mathbb{I}$, $X^{(1)}$ *and* $X^{(2)}$. *The random variables* $X^{(1)}$ *and* $X^{(2)}$ *are independent copies of* X *satisfied in the equation* (2.5). *Therefore,*

$$\mathbb{E}[Z] = \frac{2}{p}, \qquad \mathbb{E}[Z^2] = \frac{8(p^4 + 9p^3 + 37p^2 + 25p + 6)}{p^2(p^3 + 4p^2 + 6p + 2)(p+2)(p+1)^2}.$$

Proof. Let $\mathcal{M}_2$ be the space of all distributions with mean $\frac{2}{p}$ and finite absolute second moment. Consider the transformation $T_X : \mathcal{M}_2 \to \mathcal{M}_2$ defined by

$$T_X(\mu) = \frac{1-\mathbb{I}}{(p+1)^2}\, Z^{(0)} + \frac{\mathbb{I}}{(p+1)^2}\big(Z^{(1)} + Z^{(2)} + Z^{(3)}\big) + \frac{2\mathbb{I}}{(p+1)^2}\, X^{(1)}X^{(2)},$$

where $Z^{(i)}$, $i = 0, 1, 2, 3$ and $\mathbb{I}$ are independent and $Z^{(i)}$ have μ as distribution.

We first prove that the transformation T_X have a unique fixed point with respect to the metric $\mathcal{L}_2^2$ defined on $\mathcal{M}_2$. Let μ and ν be two measures of $\mathcal{M}_2$,

$$\begin{aligned} T_X(\mu) &= \frac{1-\mathbb{I}}{(p+1)^2}\, Z_\mu^{(0)} + \frac{\mathbb{I}}{(p+1)^2}\big(Z_\mu^{(1)} + Z_\mu^{(2)} + Z_\mu^{(3)}\big) + \frac{2\mathbb{I}}{(p+1)^2}\, X^{(1)}X^{(2)} \\ T_X(\nu) &= \frac{1-\mathbb{I}}{(p+1)^2}\, Z_\nu^{(0)} + \frac{\mathbb{I}}{(p+1)^2}\big(Z_\nu^{(1)} + Z_\nu^{(2)} + Z_\nu^{(3)}\big) + \frac{2\mathbb{I}}{(p+1)^2}\, X^{(1)}X^{(2)} \end{aligned}$$

$$\begin{aligned} \mathcal{L}_2^2\big(T_X(\mu),\, T_X(\nu)\big) &\le \mathbb{E}\left[\left|T_X(\mu) - T_X(\nu)\right|^2\right] \\ &= \frac{\mathbb{E}\left[(1-\mathbb{I})^2\right]}{(p+1)^4}\,\mathbb{E}\big[Z_\mu - Z_\nu\big]^2 + \frac{3\,\mathbb{E}[\mathbb{I}^2]}{(p+1)^4}\,\mathbb{E}\big[Z_\mu - Z_\nu\big]^2 \\ &= \frac{2p+1}{(p+1)^4}\,\mathbb{E}\big[Z_\mu - Z_\nu\big]^2. \end{aligned} \tag{4.3}$$

Take infimum on both sides of the equation (4.3), over all Z_μ and Z_ν, we have

$$\mathcal{L}_2^2\left(T_X(\mu),\, T_X(\nu)\right) \le \frac{2p+1}{(p+1)^4}\,\mathcal{L}_2^2\left(\mu,\, \nu\right).$$

Since $\frac{2p+1}{(p+1)^4} < 1$, then T_X is a contraction and it has a unique fixed point. Now, if $\hat{Z}_n$ converges in distribution to some random variable Z, the limit will be the unique fixed point of T_X. Define $\hat{Z}_n := \frac{Z_n}{(p+1)^{2n}}$. By (4.1) and (4.2),

$$\begin{aligned} \mathcal{L}_2^2\big(Z,\, \hat{Z}_n\big) \le{}& \frac{2p+1}{(p+1)^4}\big\|\hat{Z}_{n-1}^{(1)} - Z^{(1)}\big\|_2^2 + \frac{p}{(p+1)^{4n}}\big\|2X_{n-1}^{(1)} + 2X_{n-1}^{(2)} + 1\big\|_2^2 \\ &+ \frac{4p}{(p+1)^4}\big\|(\hat{X}_{n-1}^{(1)} - X^{(1)})(\hat{X}_{n-1}^{(2)} - X^{(2)})\big\|_2^2 + \frac{8p}{(p+1)^4}\big\|\hat{X}_{n-1}^{(1)} - X^{(1)}\big\|_2^2\|X^{(2)}\|_2^2 \\ &+ \frac{6p}{(p+1)^4}\big(\mathbb{E}[\hat{Z}_{n-1}^{(1)} - Z^{(1)}]\big)^2 + \frac{16p}{(p+1)^4}\mathbb{E}\big[X^{(1)}(\hat{X}_{n-1}^{(1)} - X^{(1)})\big]\mathbb{E}\big[(\hat{X}_{n-1}^{(2)} - X^{(2)})^2\big] \end{aligned}$$

$$
\begin{aligned}
&+\frac{8p}{(p+1)^4}\big(\mathbb{E}\big[X^{(1)}(\hat{X}^{(1)}_{n-1}-X^{(1)})\big]\big)^2+\frac{16p}{(p+1)^4}\mathbb{E}[\hat{Z}^{(3)}_{n-1}-Z^{(3)}]\mathbb{E}[\hat{X}^{(2)}_{n-1}-X^{(2)}]\mathbb{E}[X^{(1)}]\\
&+\frac{8p}{(p+1)^4}\mathbb{E}\big[(\hat{Z}^{(1)}_{n-1}-Z^{(1)})(\hat{X}^{(1)}_{n-1}-X^{(1)})\big]\mathbb{E}[\hat{X}^{(2)}_{n-1}-X^{(2)}]\\
&+\frac{4p}{(p+1)^4}\mathbb{E}[\hat{Z}^{(3)}_{n-1}-Z^{(3)}]\big(\mathbb{E}[\hat{X}^{(1)}_{n-1}-X^{(1)}]\big)^2+\frac{16p}{(p+1)^{2n+2}}\mathbb{E}[\hat{X}^{(1)}_{n-1}]\mathbb{E}[\hat{Z}^{(2)}_{n-1}-Z^{(2)}]\\
&+\frac{8p}{(p+1)^4}\mathbb{E}\big[(\hat{Z}^{(1)}_{n-1}-Z^{(1)})(\hat{X}^{(1)}_{n-1}-X^{(1)})\big]\mathbb{E}[X^{(2)}]+\frac{6p}{(p+1)^{2n+2}}\mathbb{E}[\hat{Z}^{(1)}_{n-1}-Z^{(1)}]\\
&+\frac{8p}{(p+1)^{n+3}}\mathbb{E}\big[\hat{X}^{(1)}_{n-1}(\hat{Z}^{(1)}_{n-1}-Z^{(1)})\big]+\frac{16p}{(p+1)^{n+3}}\mathbb{E}\big[\hat{X}^{(1)}_{n-1}(\hat{X}^{(1)}_{n-1}-X^{(1)})\big]\mathbb{E}[X^{(2)}]\\
&+\frac{16p}{(p+1)^{n+3}}\mathbb{E}\big[\hat{X}^{(1)}_{n-1}(\hat{X}^{(1)}_{n-1}-X^{(1)})\big]\mathbb{E}[\hat{X}^{(2)}_{n-1}-X^{(2)}]+\frac{4p}{(p+1)^{2n+2}}\big(\mathbb{E}[\hat{X}^{(1)}_{n-1}-X^{(1)}]\big)^2\\
&+\frac{8p}{(p+1)^{2n+2}}\mathbb{E}[\hat{X}^{(1)}_{n-1}-X^{(1)}]\mathbb{E}[X^{(2)}]+\frac{16p}{(p+1)^{n+3}}\mathbb{E}[\hat{X}^{(2)}_{n-1}-X^{(2)}]\mathbb{E}[\hat{X}^{(1)}_{n-1}]\mathbb{E}[X^{(1)}].
\end{aligned}
$$

Using Theorems 2.1,2.2 and 3.1, this implies

$$
\lim_{n\to\infty}\mathcal{L}_2^2\big(Z,\,\hat{Z}_n\big)\le\frac{2p+1}{(p+1)^4}\lim_{n\to\infty}\big\|\hat{Z}^{(1)}_{n-1}-Z^{(1)}\big\|_2^2=\frac{2p+1}{(p+1)^4}\lim_{n\to\infty}\big\|\hat{Z}_{n-1}-Z\big\|_2^2.
$$

Then we deduce

$$
\lim_{n\to\infty}\mathcal{L}_2^2\big(Z,\,\hat{Z}_n\big)\le\frac{2p+1}{(p+1)^4}\lim_{n\to\infty}\mathcal{L}_2^2\big(Z,\,\hat{Z}_{n-1}\big)<\lim_{n\to\infty}\mathcal{L}_2^2\big(Z,\,\hat{Z}_n\big).
$$

Hence we have $\lim_{n\to\infty}\mathcal{L}_2^2\big(Z,\,\hat{Z}_n\big)=0$. □

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