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QUORUM COLORINGS OF MAXIMUM CARDINALITY IN LINEAR TIME FOR A SUBCLASS OF PERFECT TREES

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Abstract. A partition $\pi = \{V_1, V_2, ..., V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $1 \leq i \leq k$ is called a quorum coloring of G if for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N[v]$ of v have the same color as v. The maximum cardinality of a quorum coloring of G is called the quorum coloring number of G and is denoted by $\psi_q(G)$. A quorum coloring of G of order $\psi_q(G)$ is a ψ_q-coloring of G. In this paper, we design a linear-time algorithm for finding both a ψ_q-coloring and the quorum coloring number for every perfect tree whose the vertices of the same level have the same degree.

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1. Introduction

Quorum colorings in graphs [1] have several real-word applications such that, among others, data clustering problems consisting of partition a dataset into classes of data so that every data have more similarities with data of inside its class than with those that are outside (*cf.* [2–5]). To better understand formally this model related to graph coloring, we begin by recalling, in what follows, some standard definitions and notations of graph theory.

1.1. Definitions and notations

All the graphs of this paper are simple, that is, they are finite, undirected and have neither loops nor multiple edges.

Let $G = (V, E)$ be a graph. The *order* of G is denoted by n.The *induced subgraph* of G by a subset S of V is denoted by $G[S]$. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ in G is $|N_G(v)|$. A vertex of degree zero in G is an *isolated* vertex of G and a vertex of degree one in G is a *leaf* or a *pendent vertex* of G. The set of leaves of G is denoted by $L(G)$, or simply L when G is unambiguous. The maximum and minimum vertex degrees in G are respectively denoted by $\Delta(G)$ and $\delta(G)$. More generally for a vertex $v \in V$ and a subset $S \subseteq V$, the *open* and *closed neighborhoods* of v in S are respectively defined by the

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sets $N_S(v) = \{u \in S : uv \in E(G)\}$ and $N_S[v] = N_S(v) \cup \{v\}$, and the degree of v in S is $d_S(v) = |N_S(v)|$. In particular, one can see that for $S = V$ we have $N_S(v) = N_G(v)$, $N_G[v] = N_G[v]$ and $d_S(v) = d_G(v)$. A *tree* is a connected graph having no cycle. In this paper, we denote by *binary tree* every tree T with $\Delta(T) \leq 3$. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of the shortest $u-v$ path in G, and the *diameter* of G is the longest distance $diam(G) = \max\limits_{u,v \in V} d_G(u,v)$ between two vertices in G. A *matching* in a graph $G = (V,E)$ is a set $M \subseteq E$ whose edges are pairwise non adjacent. The *matching number* $\mu(G)$ equals the maximum cardinality of a matching in G.

A *rooted* tree $T = (V,E,r)$ is a tree $T = (V,E)$ with a distinguished vertex $r \in V \setminus L$ called the *root* of T. A vertex v is the *parent* of a vertex w in a rooted tree $T = (V,E,r)$ if $vw \in E$ and $d_T(w,r) = d_T(v,r) + 1$, in which case w is said to be a *child* of v. Two vertices having the same parent in a rooted tree $T = (V,E,r)$ are called *siblings*. A vertex x is a *descendant* of a vertex v in a rooted tree $T = (V,E,r)$ if $d_T(v,x) = d_T(x,r) - d_T(v,r)$ (in particular, we have $d_T(x,r) > d_T(v,r)$); in this case, v is said to be an *ancestor* of x. The *height* of a rooted tree $T = (V,E,r)$ (with respect to r) is defined by $h = \max\limits_{v \in V} d_T(r,v)$. , For every $i \in \{0,1,\ldots,h\}$, we denote by D_i the set $\{v \in V \mid d_T(r,v) = i\}$ of vertices of $T = (V,E,r)$ that are at distance i from r; this distance is called the *height* or the *level* of D_i's vertices. For convenience, we use the nation $D_i = \{v_{i,j}\}_{1 \leq j \leq \ell_i}$ for every $i \in \{0,1,\ldots,h\}$ throughout this paper (in particular, $r = v_{0,1}$).

A *perfect* tree is a rooted tree $T = (V,E,r)$ whose leaves are at the same distance h from the root r, where h is the height of T. A *perfect N_i-ary* tree *per level* is a perfect tree $T = (V,E,r)$ whose all the D_i's vertices have the same degree $N_i + 1$, that is, all the vertices that are at the same level i have the same number of children N_i. It can be seen from the above that every siblings of a perfect N_i-ary tree by level have the same degree. In particular, if $N_i = N$ for some integer $N \geq 2$ and every $i \in \{0,\ldots,h-1\}$ and $j \in \{1,\ldots,\ell_i\}$, then T is called a *perfect N-ary tree* (for $N = 2$, T is called *perfect binary* tree). Note from the definitions that every perfect N-ary tree is a perfect N_i-ary tree per level with $N_i = N$ for every $i \in \{0,1,\ldots,h-1\}$, but that obviously, the converse is not true. Note that for every perfect rooted tree $T = (V,E,r)$, we always have $|D_i| \leq |D_{i+1}|$, for every $0 \leq i \leq h$.

A partition $\pi = \{V_1, V_2, ..., V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $i \in \{1,...,k\}$ is called a *quorum coloring* of G if for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N_G[v]$ have the same color as v, which means formally that for every $i \in \{1,\ldots,k\}$ and every $v \in V_i$, we have $|N_G[v] \cap V_i| \geq \frac{|N_G[v]|}{2}$. The color classes V_i are called *quorum classes* and for a fixed $i \in \{1,\ldots,k\}$, each vertex of V_i is called a *quorum vertex*. The maximum cardinality of a quorum coloring of G is called the *quorum coloring number* of G and is denoted by $\psi_q(G)$. A quorum coloring of G of cardinality $\psi_q(G)$ is called a *ψ_q-coloring* of G. For a quorum coloring π on a graph G and any vertex $v \in V(G)$, we denote the class of v with respect to π by $\pi(v)$.

1.2. Previous results

Quorum colorings is another name of partitions into defensive alliances in graphs (*cf.* [6]) that admit applications to data clustering (see [2, 3, 5]), and the quorum classes of a graph with respect to a given quorum coloring are nothing but defensive alliances. For further reading on the well-studied topic of defensive alliances in graphs, the reader can consult the references [2, 5, 7–14].

Quorum colorings in graphs were introduced in 2013 by Hedetniemi *et al.* [1] where the authors studied their basic properties as well as the quorum coloring number of some usual graphs among which the hypercubes, then concluded their article by raising and listing twelve open problems. In 2018, Sahbi and Chellali [15] answered three of these open problems and showed in particular that the decision problem associated with $\psi_q(G)$ is *NP*-complete for a general graph G. Furthermore, Sahbi pursued the study of the open questions raised by the authors [1] throughout [3, 16, 17] and in [4] with Belkina and Bennadji. In particular, he brought a partial answer to the following open question posed by Hedetniemi *et al.*

1. Can you design a linear-time algorithm for computing the value of $\psi_q(T)$ for any tree T?

In fact, the author [16] first established a lower bound on the quorum coloring number of nontrivial trees involving the matching number of the subtree induced by the nonpendent vertices, the order of the tree and the vertex degrees and showed that this bound is computable in linear time.

Theorem 1.1. *[16] Let $T = (V, E)$ be a nontrivial tree. Then,*

$$\psi_q(T) \geq \mu\left(T[V \setminus L]\right) + n - \sum_{v \in (V \setminus L)} \left\lfloor \frac{d_T(v)}{2} \right\rfloor .$$

This bound can be computed in linear time.

Then, he showed that the bound of Theorem 1.1 is attained by all binary trees.

Corollary 1.2. *[16] Let $T = (V, E)$ be a nontrivial binary tree. Then,*

$$\psi_q(T) = \mu(T[V \setminus L]) + |L|.$$

This value can be computed in linear time.

In this paper, we first recall some fundamental results on quorum colorings of graphs in Section 2, then we partially answer Question 1, in Section 3, in the affirmative by designing a linear-time algorithm both for finding a ψ_q-coloring and computing the quorum coloring number of any perfect N_i-ary tree per level.

2. Preliminary results

This section is devoted to the statement of some fundamental results on quorum colorings that have been previously established in the literature, some of which will be used in this work. We start by a linear relationship between the quorum coloring number of a disconnected graph and those of its components.

Proposition 2.1. *[6] Let G be a disconnected graph whose components are $G_1, G_2, \ldots, G_r$ $(r \geq 1)$. Then*

$$\psi_q(G) = \sum_{1 \leq i \leq r} \psi_q(G_i).$$

The next result provides sharp lower and upper bounds of the quorum coloring number of a disconnected graph of order at least three. However, one can easily see that this result remains true for any graph and any order.

Proposition 2.2. *[6] Let G be a disconnected graph of order $n \geq 3$. Then*

$$1 \leq \psi_q(G) \leq n.$$

The sharpness of the bounds of Proposition 2.2 can be seen thanks to the following two propositions.

Proposition 2.3. *[1] For the complete graph K_n of odd order, $\psi_q(K_n) = 1$, while for any complete graph K_n of even order, $\psi_q(K_n) = 2$.*

Proposition 2.4. *[1] Let G be a graph of order n. Then $\psi_q(G) = n$ if and only if $\Delta(G) \leq 1$, that is G consists of isolated vertices and disjoint copies of the complete graph K_2 of order 2.*

The next result was proved by Hedetniemi *et al.* It says that in a ψ_q-coloring, a quorum class always induces a connected subgraph.

Proposition 2.5. *[1] Let G be a graph, and let $\pi = \{V_1, V_2, \ldots, V_k\}$ be any ψ_q-coloring of G. Then, for every i, $1 \leq i \leq k$, the induced subgraph $G[V_i]$ is connected.*

In [16], Sahbi proved the following observation which gives four equivalent assertions to the quorum vertex property and the second of which will be used in Section 3.

Observation 2.6. *[16] Let $G = (V, E)$ be a graph, $\pi = \{V_1, V_2, \ldots, V_k\}$ a quorum coloring of G and $i \in \{1, 2, \ldots, k\}$. Then the following assertions are equivalent:*

1. *V_i is a quorum class.*
2. *For every vertex $v \in V_i$, $|N_G[v] \cap V_i| \geq \left\lceil \frac{|N_G[v]|}{2} \right\rceil$.*
3. *For every vertex $v \in V_i$, $|N_G[v] \cap V_i| \geq |N_G[v] \cap (V \setminus V_i)|$.*
4. *For every vertex $v \in V_i$, $d_{V_i}(v) + 1 \geq d_{V \setminus V_i}(v)$.*
5. *For every vertex $v \in V_i$, $d_{V_i}(v) \geq \left\lfloor \dfrac{d_G(v)}{2} \right\rfloor$.*

As consequences of Observation 2.6, the author [16] deduced the three following corollaries.

Corollary 2.7. *[16] Let $G = (V, E)$, $\pi = \{V_1, V_2, \ldots, V_k\}$ a quorum coloring of G and $i \in \{1, 2, \ldots, k\}$. Then for every vertex $v \in V_i$, $|V_i| \geq \left\lfloor \dfrac{d_G(v)}{2} \right\rfloor + 1$.*

Corollary 2.8. *[16] Let $G = (V, E)$ be a graph, $\pi = \{V_1, V_2, \ldots, V_k\}$ a ψ_q-coloring of G, $i \in \{1, 2, \ldots, k\}$ a positive integer and $v \in V_i$ a vertex such that $d_G(v) = \max\limits_{u \in V_i} d_G(u)$. Then $|V_i| = 1$ if and only if $d_G(v) \leq 1$.*

Corollary 2.9. *[16] Let $G = (V, E)$, $\pi = \{V_1, V_2, \ldots, V_k\}$ a ψ_q-coloring of G and $i \in \{1, 2, \ldots, k\}$ a positive integer. Then $|V_i| \geq 2$ if and only if V_i contains a vertex of degree at least 2.*

Corollary 2.7 provides a lower bound of a quorum class in term of maximum degree of its vertices, while Corollary 2.8 states that the unique vertex of a singleton quorum class is necessarily pendent or isolated. Corollary 2.9, obtained by negating Corollary 2.8, shows that every vertex of degree at least two is contained in a quorum class of order at least two in any ψ_q-coloring, but this result can easily be extended to a quorum coloring that is not necessarily a ψ_q-coloring.

In the next section, we prove our result announced in Section 1.

3. Our partial answer to Question 1

In this section, we partially answer Question 1 stated in Section 1. To do this, we need the following definitions.

A quorum coloring π of a graph G is said to be *cost-effective* if every vertex of $V \setminus L$ satisfies the quorum vertex property with equality, that is, if for every vertex $v \in V$ we have $|N_G[v] \cap \pi(v)| = \left\lceil \frac{|N_G[v]|}{2} \right\rceil$. Given two quorum colorings π_1 and π_2 of G, we say that π_1 is *better* than π_2 if π_1 is cost-effective and $|\pi_1| \geq |\pi_2|$.

Before proving our main result, we first prove that we can modify any quorum coloring of an arbitrarily tree to obtain a better one in the sense of the above definition.

Theorem 3.1. *Let $T = (V, E, r)$ be a tree and π_0 a quorum coloring of T. Then, there exists a quorum coloring of T that is better than π_0.*

Proof. Let π_0 be a quorum coloring of T. We run the following algorithm that we denote by Algorithm 1.

- For $i = 0$ to $h - 1$ and $j = 1$ to ℓ_i, execute the following two steps.
  1. For $|N_T[v_{i,j}] \cap \pi_i(v_{i,j})| > \left\lceil \frac{|N_T[v_{i,j}]|}{2} \right\rceil$, chose arbitrarily a subset $S_{i,j}$ of $\pi_i(v_{i,j}) \cap D_{i+1}$ so that $|N_T[v_{i,j}] \cap \pi_i(v_{i,j})| - |S_{i,j}| = \left\lceil \frac{|N_T[v_{i,j}]|}{2} \right\rceil$, color the $S_{i,j}$'s vertices with $|S_{i,j}|$ new colors and for every vertex $v \in S_{i,j}$, assign the new v's color to all the descendants of v that were in $\pi_i(v)$; we denote the obtained coloring by π'_i. Then, go to step 2.

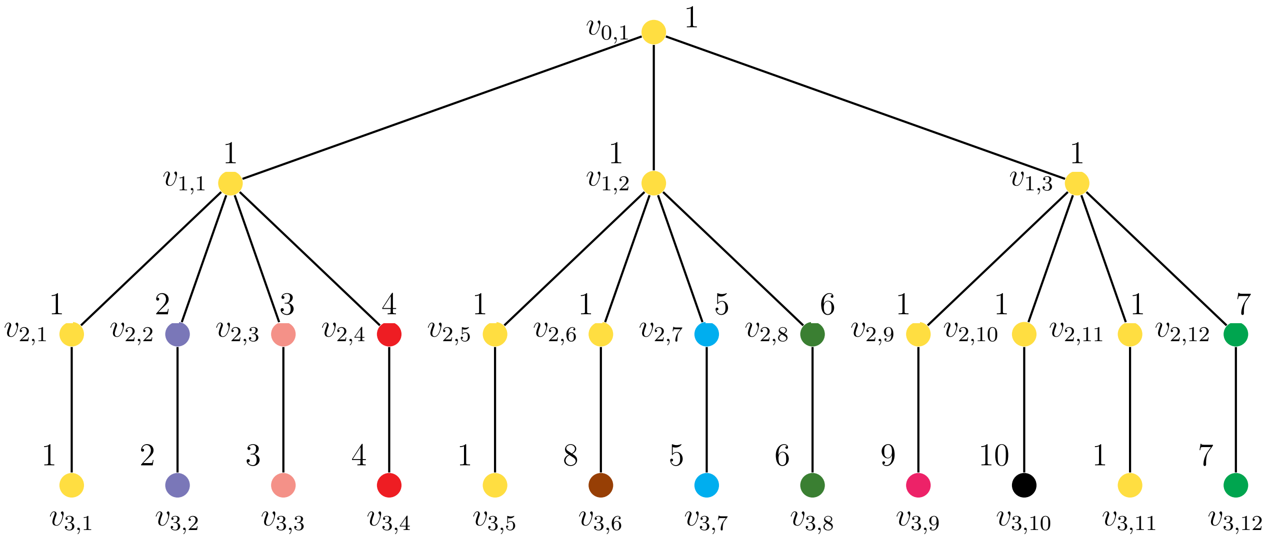


FIGURE 1. An arbitrary initial quorum coloring π_0 with $|\pi_0| = 10$.

2. For every vertex $v \in S_{i,j}$ such that $|N_T[v] \cap \pi'_i(v)| < \left\lceil \frac{|N_T[v]|}{2} \right\rceil$ (note in this case that we have necessarily $|N_T[v] \cap \pi'_i(v)| = \left\lceil \frac{|N_T[v]|}{2} \right\rceil - 1$ since by passing from π_i to π'_i, the unique neighbor of v that has no longer the v's color is $v_{i,j}$ implying that $|\pi_i(v)| - |\pi'_i(v)| = 1$), assign the v's color to an arbitrarily chosen vertex w from $(N_T(v) \cap D_{i+2}) \setminus \pi'_i(v)$ and to all the descendants of w that belonged to $\pi'_i(v)$; therefore, it can be seen without difficulty that the resulted coloring is a quorum coloring that we denote by π_{i+1}.

Algorithm 1 must terminate since T is of finite order. When it is finished, we clearly obtain a cost-effective quorum coloring π_h according to steps 1 and 2. Moreover, each time Algorithm 1 is run for some $i \in \{0, \dots, h-1\}$ and $j \in \{1, \dots, \ell_i\}$, the number of colors increases by at least one at the execution of step 1, while it decreases by at most one at the execution of step 2 and consequently, the number of colors does not decrease. Hence we deduce that π_h is better than π_0. □

Starting from a quorum coloring π_0 of any tree T, the proof of Theorem 3.1 shows that one can obtain a quorum coloring of T better than π_0 that is cost-effective. As consequence, if π_0 is a ψ_q-coloring then Algorithm 1 concludes with a cost effective ψ_q-coloring of T as output.

Corollary 3.2. *Every tree has a cost-effective ψ_q-coloring.*

Figures 1–7 illustrate how Algorithm 1 works on a perfect N_i-ary tree per level with $(N_i)_{0 \leq i \leq 2} = (3, 4, 1)$.

We are now ready to prove our central result by designing a linear-time algorithm both for finding a ψ_q-coloring and computing the quorum coloring number of any perfect N_i-ary tree per level.

Theorem 3.3. *For every perfect N_i-ary tree per level T of order n, one can both find a cost-effective ψ_q-coloring and compute $\psi_q(T)$ in $O(n)$.*

Proof. Let $T = (V, E, r)$ be a perfect N_i-ary tree per level of order n. We run the following algorithm denoted by Algorithm 2.

1. Assign a color to r and go to step 2.

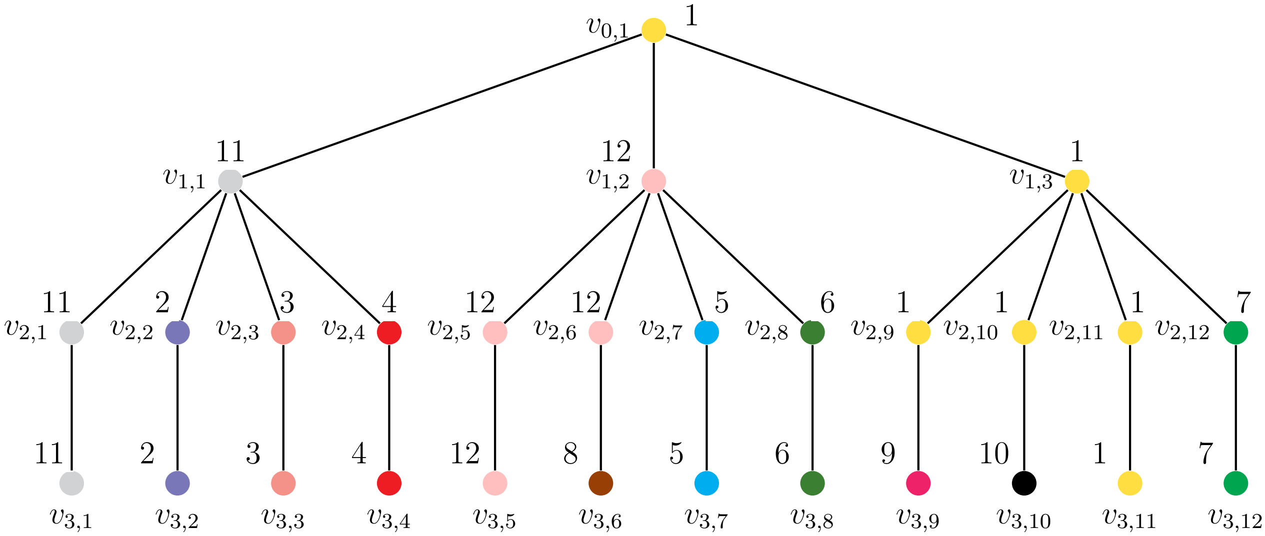
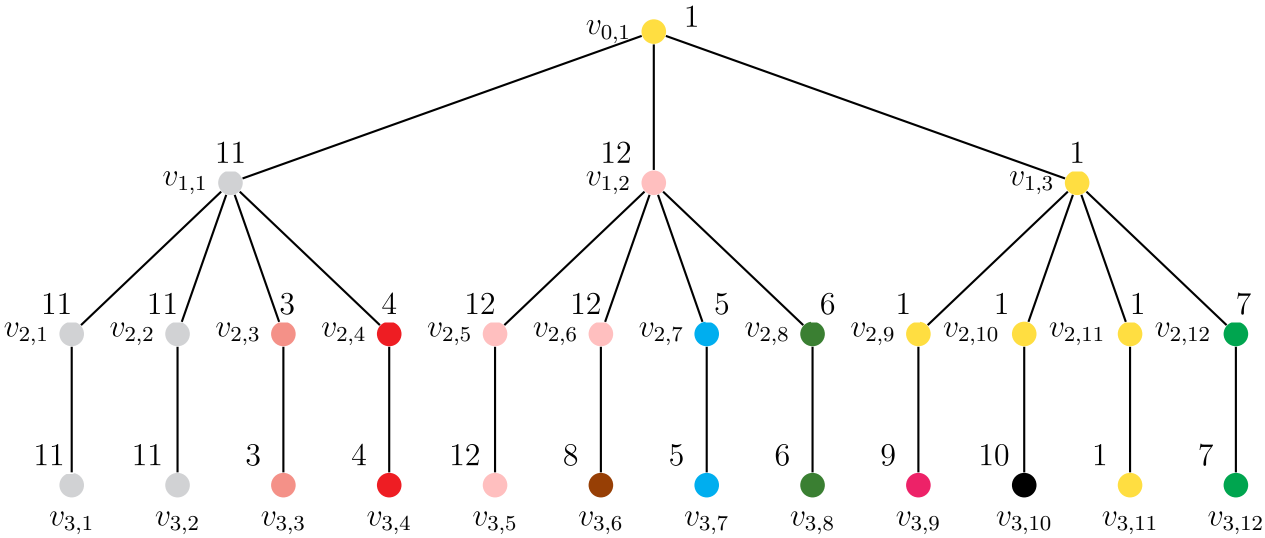


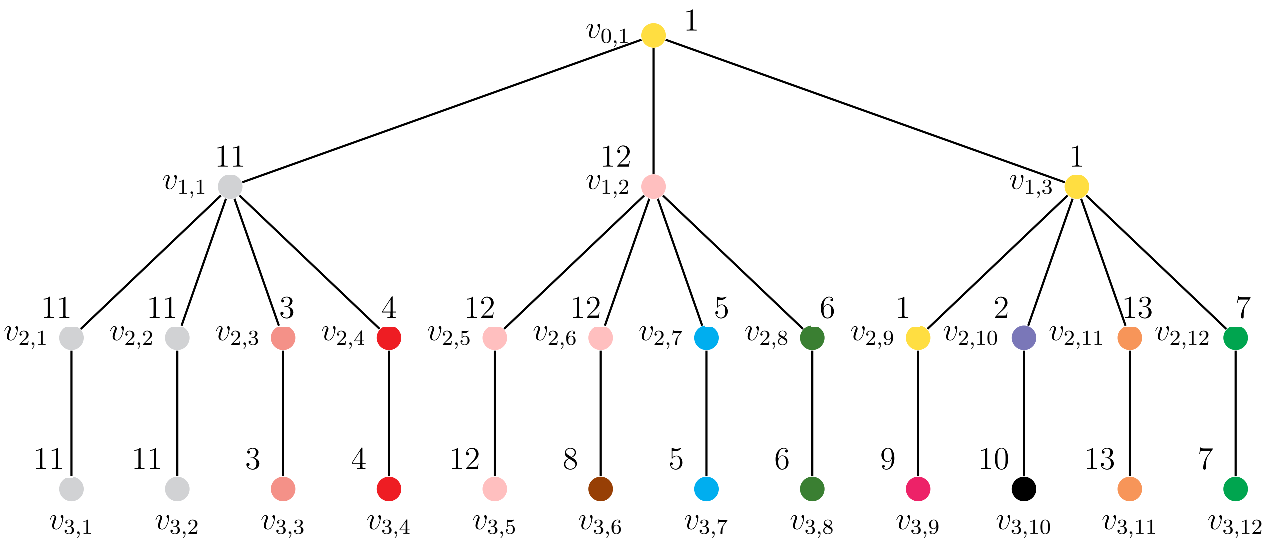
FIGURE 2. Iteration 1.1: obtaining of the coloring π'_0.



$$|\pi_1| = 11$$

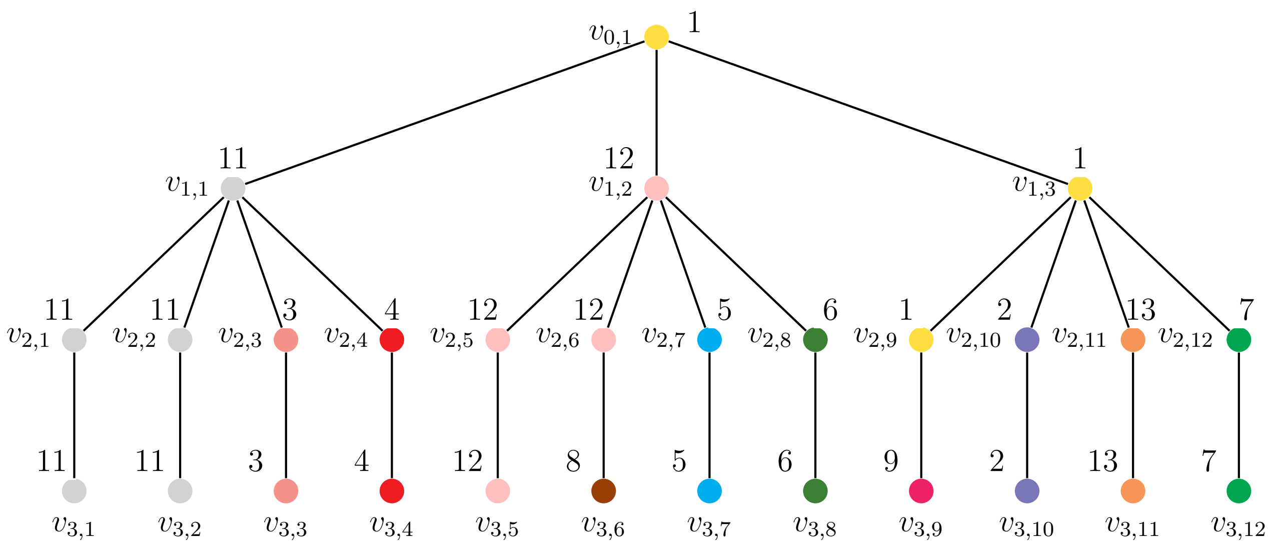
FIGURE 3. Iteration 1.2: obtaining of the quorum coloring π_1.

2. Assign arbitrarily the r's color to exactly $\left\lfloor \dfrac{d_T(r)}{2} \right\rfloor$ of its children, color its $\left\lceil \dfrac{d_T(r)}{2} \right\rceil$ remaining children with $\left\lceil \dfrac{d_T(r)}{2} \right\rceil$ new colors and set $\alpha_1^1 = \left\lceil \dfrac{N_0}{2} \right\rceil + 1$. Then, go to step 3.
3. For $i = 1$ to $h-1$ and $j = 1$ to ℓ_i, consider the following mutually exclusive two cases.



π'_1

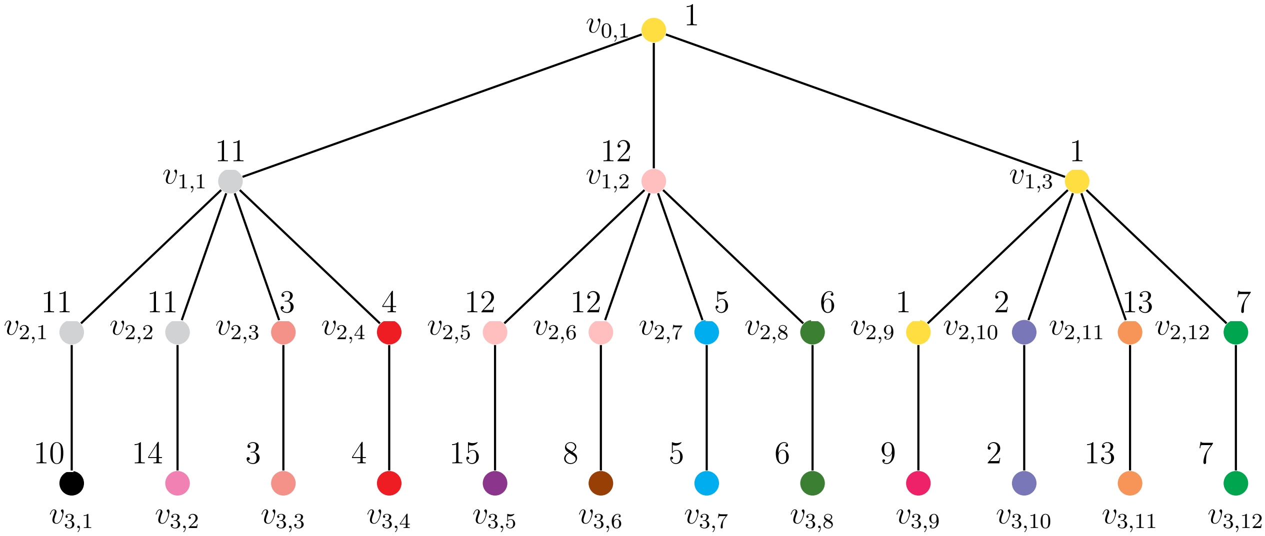
FIGURE 4. Iteration 2.1: obtaining of the coloring π'_1.



$|\pi_2| = 12$

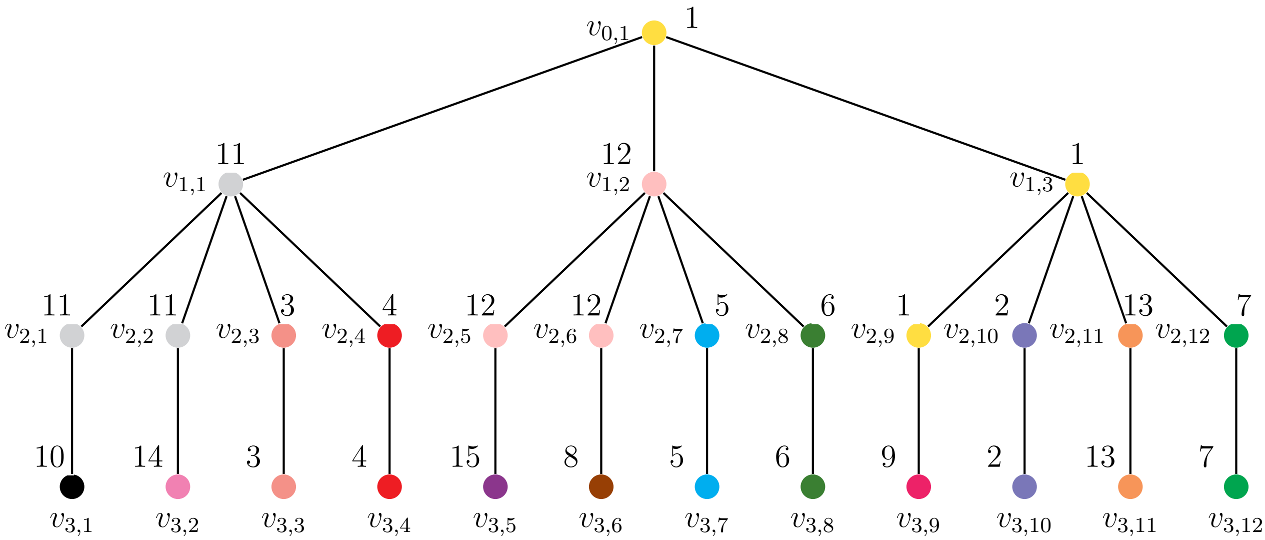
FIGURE 5. Iteration 2.2: obtaining of the quorum coloring π_2.

(i) If $v_{i,j}$ has the same color as its parent, then assign arbitrarily the color of $v_{i,j}$ to exactly $\left\lceil \frac{N_i}{2} \right\rceil - 1$ of its children, color its $\left\lfloor \frac{N_i}{2} \right\rfloor + 1$ remaining others with $\left\lfloor \frac{N_i}{2} \right\rfloor + 1$ new colors and set $\alpha_{i+1}^{j} = \left\lfloor \frac{N_i}{2} \right\rfloor + 1$.



π'_2

FIGURE 6. Iteration 3.1: obtaining of the coloring π'_2.



$|\pi_3| = 15 > |\pi_0| = 10$

FIGURE 7. Iteration 3.2: obtaining of the quorum coloring π_3.

(ii) Otherwise, if $v_{i,j}$ has not the same color as its parent, then assign arbitrarily the $v_{i,j}$'s color to exactly $\left\lceil \frac{N_i}{2} \right\rceil$ of its children, color its $\left\lfloor \frac{N_i}{2} \right\rfloor$ remaining children with $\left\lfloor \frac{N_i}{2} \right\rfloor$ new colors and set $\alpha^j_{i+1} = \left\lfloor \frac{N_i}{2} \right\rfloor$.

4. Set $\alpha = \sum_{i=0}^{h-1} \sum_{j=1}^{\ell_i} \alpha_{i+1}^j$.

Algorithm 2 must terminate since T is finite. By Assertion 2 of Observation 2.6, this algorithm concludes with a cost-effective quorum coloring π of T as output thanks to steps 2 and 3. Moreover, since the D_i's vertices have the same degree and therefore play the same role in T (indeed, their descendants that are at the same distance all have the same degree) then, even if it means permuting some colors, it follows that π is the unique cost-effective quorum-coloring of T. Therefore, this implies by Corollary 3.2 that π is a ψ_q- coloring of T. Besides, on the one hand the total number of comparison tests between the color of a vertex and that of its parent equals to the order n of T. On the other hand, the total number of color assignments equals to n too. In addition, the assignments of the α_{i+1}^j's requires n operations. Finally, the sum calculated in step 4 is performed in $n-1$ operations. Consequently, the total number of elementary operations necessary to both find π and calculate $\psi_q(T) = \alpha$ is at most equal to $n+n+n+n-1 < 4n$, whence the theorem. □

The next four corollaries follow immediately from Theorem 3.3 and from the fact that every perfect N-ary tree is also a perfect N_i-ary tree per level with $N_i = N$ for every $i \in \{0, 1, \ldots, h-1\}$.

Corollary 3.4. *For every perfect N_i-ary tree per level T of order n, one can find a cost-effective ψ_q-coloring in $O(n)$.*

Corollary 3.5. *For every perfect N_i-ary tree per level T of order n, one can compute $\psi_q(T)$ in $O(n)$.*

Corollary 3.6. *For every perfect N-ary tree T of order n, one can find a cost-effective ψ_q-coloring in $O(n)$.*

Corollary 3.7. *For every perfect N-ary tree T of order n, one can compute $\psi_q(T)$ in $O(n)$.*

It can be seen without difficulty that the quorum coloring π_3 of Figure 7 can be obtained by running Algorithm 2, which illustrates the unicity of a ψ_q-coloring in a perfect N_i-ary tree per level. In particular, one can check that we have:

$$\alpha_1^1 = \left\lceil \frac{N_0}{2} \right\rceil + 1 = \left\lceil \frac{3}{2} \right\rceil + 1 = 3,\ \alpha_2^1 = \left\lfloor \frac{N_1}{2} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor = 2,\ \alpha_2^2 = \left\lfloor \frac{N_1}{2} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor = 2,$$

$$\alpha_2^3 = \left\lfloor \frac{N_1}{2} \right\rfloor + 1 = \left\lfloor \frac{4}{2} \right\rfloor + 1 = 3,\ \alpha_3^1 = \left\lfloor \frac{N_2}{2} \right\rfloor + 1 = \left\lfloor \frac{1}{2} \right\rfloor + 1 = 1,\ \alpha_3^2 = \left\lfloor \frac{N_2}{2} \right\rfloor + 1 = \left\lfloor \frac{1}{2} \right\rfloor + 1 = 1,$$

$$\alpha_3^3 = \left\lfloor \frac{N_2}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0,\ \alpha_3^4 = \left\lfloor \frac{N_2}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0,\ \alpha_3^7 = \left\lfloor \frac{N_2}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0,$$

$$\alpha_3^8 = \left\lfloor \frac{N_2}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0,\ \alpha_3^5 = \left\lfloor \frac{N_2}{2} \right\rfloor + 1 = \left\lfloor \frac{1}{2} \right\rfloor + 1 = 1,\ \alpha_3^6 = \left\lfloor \frac{N_2}{2} \right\rfloor + 1 = \left\lfloor \frac{1}{2} \right\rfloor + 1 = 1,$$

$$\alpha_3^9 = \left\lfloor \frac{N_2}{2} \right\rfloor + 1 = \left\lfloor \frac{1}{2} \right\rfloor + 1 = 1,\ \alpha_3^{10} = \left\lfloor \frac{N_2}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0,\ \alpha_3^{11} = \left\lfloor \frac{N_2}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0,$$

$$\alpha_3^{12} = \left\lfloor \frac{N_2}{2} \right\rfloor = \left\lfloor \frac{1}{2} \right\rfloor = 0 \text{ and that } \alpha = \sum_{i=0}^{h-1} \sum_{j=1}^{\ell_i} \alpha_{i+1}^j = 15.$$

4. Concluding remarks

In this article, we have designed a linear-time algorithm finding a quorum coloring of maximum cardinality together with the quorum coloring number of any perfect tree satisfying the property that the vertices at the same level have the same degree. Since every perfect N-ary tree is a perfect N_i-ary tree per level, then the same algorithm is applicable for perfect N-ary trees, which answers Question 1 in the affirmative. This being established, there are two questions of interest to go further in our investigation. First, one can think of generalizing somewhat the work made in this paper to perfect trees whose siblings have the same degree; we will call them *locally perfect* $N_{i,j}$*-ary trees*. The children of a given vertex would then not necessarily play the same role since the descendants of any two siblings of same level would not necessarily have the same degree. The second question concerns the determination of the exact value of the quorum coloring number of perfect N-ary trees. In particular, one can verify without difficulty using the exact value of Corollary 1.2 that for any perfect binary tree T of order n and height h, we have

$$\psi_q(T) = \frac{2^{h+2}}{2^2-1}\left[1-\left(\frac{1}{4}\right)^{\left\lfloor \frac{h}{2}\right\rfloor+1}\right].$$

The latter formula could be the initialization of an induction proof.

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No competing interests to declare.

Data availability statement

No new data/codes were created or analyzed in this study.

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