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A SMALL MORPHISM FOR WHICH THE FIXED POINT HAS AN ABELIAN CRITICAL EXPONENT LESS THAN 2

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Abstract. It is known that there are infinite words over finite alphabets with Abelian critical exponent arbitrarily close to 1; however, the construction previously used involves huge alphabets. In this note we give a short cyclic morphism (length 13) over an 8-letter alphabet for which the fixed point has an Abelian critical exponent less than 1.8.

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1. Introduction

A central topic in combinatorics on words is the construction of infinite words avoiding repetitions. If k is an integer, then a **k-repetition** is a word

$$x^k = \underbrace{xxx\cdots x}_{k \text{ times}}$$

where x is a non-empty word. In the early 1900's, Thue [1, 2] constructed an infinite word over a 3-letter alphabet containing no 2-powers, and an infinite word over a 2-letter alphabet containing no 3-powers; these alphabet sizes are least possible.

Dejean [3] extended the notion of k-power to fractional exponents; for $1 < k < 2$, a k-power is a word $z^k = xyx$, where $z = xy$, and $|xyx|/|xy| = k$. For an integer $n > 1$, the **repetitive threshold** $\mathrm{RT}(n)$ is defined to be the infimum of r such that there exists an infinite sequence over an n-letter alphabet not containing any k-power with $k \geq r$. In 1972, Dejean [3] conjectured that

$$\mathrm{RT}(n) = \begin{cases} 7/4, & n = 3 \\ 7/5, & n = 4 \\ n/(n-1), & n \neq 3, 4 \end{cases}$$

This conjecture was resolved in a series of papers by Dejean, Pansiot, Moulin-Ollagnier, Carpi, and others [3–7] with the last cases being resolved simultaneously by the authors and by Michaël Rao [8, 9].

Keywords and phrases: Abelian repetition, Dejean's conjecture, critical exponent, repetition threshold, combinatorics on words.

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An important variation on repetitions in words is the study of *Abelian repetitions*. For an integer $k > 1$, an Abelian k-power is a word

$$x_1 x_2 \cdots x_k$$

where x_1 is non-empty, and the words x_i are anagrams of each other. In the late 1950's, Erdős [10] asked for the maximal length, as a function of n, of a word over an n-letter alphabet not containing an Abelian 2-power. He suggested [11] that perhaps there were infinite words not containing an Abelian 2-power over a finite alphabet, perhaps even over a 4-letter alphabet. In 1968, Evdokimov [12] constructed such a word over an alphabet of 26 letters. In 1970, Pleasants [13] reduced the alphabet size to 5. Finally, in 1992, Keränen [14] gave a construction over a 4-letter alphabet, which is the smallest alphabet possible. In related work, in 1979 Dekking [15] had shown that Abelian 3-powers can be avoided on a 3-letter alphabet, and that Abelian 4-powers can be avoided already on a 2-letter alphabet; these alphabets are smallest possible.

Inspired by Dejean's work with (non-Abelian) repetitions, Cassaigne and Currie [16] defined Abelian fractional powers: for $1 < k < 2$, an **Abelian k-power** is a word $x_1 y x_2$, where x_1 is an anagram of x_2, and $|x_1 y x_2|/|x_1 y| = k$. Several years later, this definition was extended (in three variant versions) to fractional powers greater than 2 by Samsonov and Shur [17].

Since 2003, one of the authors [18, 19] has pointed out the desirability of formulating and proving the correct Abelian version of Dejean's conjecture; that is, giving the value of the **Abelian threshold** $\mathrm{ART}(n)$, where 'k-power' in the definition of $\mathrm{RT}(n)$ is replaced by 'Abelian k-power'. Since Abelian 2-powers are unavoidable on 3 or fewer letters, this is particularly interesting for $n \geq 4$. In recent years, others [17, 20] have started to pay some attention to this problem. Based on numerical evidence the conjecture of Petrova and Shur [20] is that

$$\begin{aligned} 9/5 <& ART(4) \\ & ART(5) = 3/2 \\ 4/3 <& ART(6) < 3/2 \\ & ART(k) = (k-3)/(k-4) \text{ for } k \geq 7. \end{aligned}$$

Already in 1999, Cassaigne and Currie [16] showed that $\lim_{n\to\infty} ART(n) = 1$. However, to produce a word containing no Abelian k-power with $k \geq 1.8$ using their construction required an alphabet of size on the order of 5^{485}. In contrast to this, one of the authors [18] claimed, based on some computational evidence, that the fixed point of a certain small morphism on an 8-letter alphabet avoids such Abelian k-powers. In this paper we prove this claim. Our methods (and the morphism) in question should generalize to give an upper bound on $\mathrm{ART}(n)$.

In this article we focus on the definition of Abelian fractional exponent initiated by Cassaigne and Currie. As mentioned, there are a number of alternatives. Readers may wish to consult the excellent survey by Fici and Puzynina [21] on Abelian repetitions and related topics.

2. Preliminaries

We use letters such as u, v, and w to stand for **words**, that is, finite strings of letters. For the empty word, containing no letters, we write ϵ. We denote the set of all letters, or **alphabet**, by Σ. The length of word u is denoted by $|u|$, and if $a \in \Sigma$, then $|u|_a$ denotes the number of times letter a appears in word u. Thus if $\Sigma = \{a, b, n\}$ and $u = banana$, we have $|u| = 6$, $|u|_a = 3$, $|u|_n = 2$.

The set of all finite words over Σ is denoted by Σ^*, and can be regarded as the free semigroup generated by Σ, where the operation is concatenation, and is written multiplicatively. If u, $v \in \Sigma^*$ we say that u is an **anagram** of v, and write $u \sim v$ if for every letter $a \in \Sigma$ we have $|u|_a = |v|_a$. If $u, p, v, s \in \Sigma^*$ and $u = pvs$ we say that p, v, and s are respectively a **prefix**, a **factor**, and a **suffix** of u. The **index** of v in u is given by $|p|$. (Thus a prefix has index 0.)

A **morphism** f on Σ^* is a semigroup homomorphism, and is determined by giving $f(a)$ for each letter $a \in \Sigma$. A morphism is k**-uniform** if $|f(a)| = k$ for each $a \in \Sigma$. Iteration of morphisms is written using exponents; thus $f^2(u) = f(f(u))$, and so forth. We use the convention that $f^0(u) = u$.

If $a \in \Sigma$ and f is a morphism such that a is a prefix of $f(a)$, then for each non integer n we have $f^n(a)$ is a prefix of $f^{n+1}(a)$. In the case that $|f(a)| > 1$, we let the **infinite word** $f^\omega(a)$ be the sequence which has $f^n(a)$ as a prefix for each n. It is usual to refer to $f^\omega(a)$ as the fixed point of f starting with a, and to extend notions such as prefix, factor, index, *etc.*, to this infinite word in the obvious ways.

For the remainder of this paper, let $\Sigma = \{0, 1, 2, \dots, 7\}$. Let $\sigma : \Sigma^* \to \Sigma^*$ be the cyclic shift morphism where

$$\sigma(a) \equiv a + 1 \pmod 8 \text{ for } a \in \Sigma.$$

Consider the 13-uniform cyclic morphism $f : \Sigma^* \to \Sigma^*$ given by

$$f(0) = 0740103050260.$$

Thus for an integer n,

$$f(\sigma^n(0)) = \sigma^n(f(0)).$$

We see that

$$\begin{aligned} |f(w)| &= 13|w| \text{ for } w \in \Sigma^* \\ |f(w)|_k &= |w| + 5|w|_k \text{ for } w \in \Sigma^*, k \in \Sigma. \end{aligned}$$

3. Preimages and anagrams

Let $X_1, X_2 \in \Sigma^*$ such that $X_1 \sim X_2$, and such that for $i = 1, 2$, we have

$$\begin{aligned} X_i &= s_i f(x_i) p_i \\ f(a_i) &= p_i q_i, \text{ some } a_i \in \Sigma \cup \{\epsilon\}, |p_i| \leq 12, \\ f(b_i) &= r_i s_i, \text{ some } b_i \in \Sigma \cup \{\epsilon\}, |s_i| \leq 12. \end{aligned}$$

We can certainly parse words X_1 and X_2 this way if they are factors of $f^\omega(0)$ of length at least 12.

From $s_1 f(x_1) p_1 \sim s_2 f(x_2) p_2$ we have, in particular, that

$$\begin{aligned} |s_1 f(x_1) p_1| &= |s_2 f(x_2) p_2|, \text{ so} \\ |s_2 p_2| - |s_1 p_1| &= |f(x_1)| - |f(x_2)|. \end{aligned}$$

The definition of f and the conditions on the p_i and s_i imply that for each $k \in \Sigma$, $|s_i p_i|_k \leq 10$. For the remainder of this section we consider the words $X_i, x_i, p_i, q_i, r_i, s_i$ to be fixed. Define $\Delta = |x_1| - |x_2|$. Then

$$\begin{aligned} 13|\Delta| &= |13|x_1| - 13|x_2|| \\ &= ||f(x_1)| - |f(x_2)|| \\ &= ||s_2 p_2| - |s_1 p_1|| \\ &\leq 24. \end{aligned}$$

We conclude that $|\Delta| \leq 1$. In particular, $|x_1| \geq |x_2| - 1$.

Lemma 3.1. *Suppose that for some k we have $|x_1|_k \geq |x_2|_k + 1$. Then $a_2 = k$ or $b_2 = k$.*

Proof. We see that

$$\begin{aligned}|f(x_1)|_k &= |x_1| + 5|x_1|_k \\ &\geq (|x_2| - 1) + 5(|x_2|_k + 1) \\ &= |x_2| + 5|x_2|_k + 4 \\ &= |f(x_2)|_k + 4.\end{aligned}$$

Then

$$\begin{aligned}|s_2p_2|_k &= |s_2f(x_2)p_2|_k - |f(x_2)|_k \\ &= |s_1f(x_1)p_1|_k - |f(x_2)|_k \\ &\geq |s_1f(x_1)p_1|_k - (|f(x_1)|_k - 4) \\ &= |s_1p_1|_k + 4 \\ &\geq 4.\end{aligned}$$

It follows that either $|s_2|_k \geq 2$ or $|p_2|_k \geq 2$. We conclude that $a_2 = k$ or $b_2 = k$. □

It follows that there are at most 2 values k for which $|x_1|_k > |x_2|_k$, and at most two values j for which $|x_2|_j > |x_1|_j$.

Lemma 3.2. *Suppose that for some k we have $|x_1|_k \geq |x_2|_k + 2$. Then $b_1x_1a_1 \sim b_2x_2a_2$.*

Proof. Recall that $-1 \leq \Delta \leq 1$. We will show that $\Delta = 0$. First of all, we have

$$\begin{aligned}|f(x_1)|_k &= |x_1| + 5|x_1|_k \\ &\geq |x_1| + 5(|x_2|_k + 2) \\ &= |x_2| + \Delta + 5(|x_2|_k + 2) \\ &= |f(x_2)|_k + \Delta + 10.\end{aligned}$$

Then

$$\begin{aligned}10 &\geq |s_2p_2|_k \\ &= |s_1p_1|_k + |f(x_1)|_k - |f(x_2)|_k \\ &\geq |f(x_1)|_k - |f(x_2)|_k \\ &\geq \Delta + 10.\end{aligned}$$

and we deduce that $0 \geq \Delta$. Also,

$$\begin{aligned}|s_2p_2|_k &= |s_1p_1|_k + |f(x_1)|_k - |f(x_2)|_k \\ &\geq |f(x_1)|_k - |f(x_2)|_k \\ &= |x_1| + 5|x_1|_k - (|x_2| + 5|x_2|_k) \\ &= \Delta + 5(|x_1|_k - |x_2|_k) \qquad (3.1) \\ &\geq -1 + 5(2) \\ &= 9.\end{aligned}$$

The shortest suffix of $f(k)$ containing 4 k's has length 8; the shortest suffix of $f(k)$ containing 5 k's has length 10; no proper suffix of $f(k)$ contains more than 5 k's; there is a single k in $f(\ell)$ for $\ell \in \Sigma - \{k\}$. The analogous statements hold for prefixes of $f(k)$. Thus from $|s_2p_2|_k \geq 9$ we conclude that we have $a_2 = b_2 = k$, and either $|p_2| \geq 10$ and $|s_2| \geq 8$, or $|p_2| \geq 8$ and $|s_2| \geq 10$; in both cases $|s_2p_2| \geq 18$.

We claim that $\Delta \geq 0$, *i.e.*, $|x_1| \geq |x_2|$. Otherwise, $|x_2| \geq |x_1| + 1$ so that

$$\begin{aligned} 24 + |f(x_1)| &\geq |s_1 f(x_1) p_1| \\ &= |s_2 f(x_2) p_2| \\ &\geq 13|x_2| + 18 \\ &\geq 13(|x_1| + 1) + 18 \\ &= 31 + |f(x_1)|, \end{aligned}$$

a contradiction. We earlier established that $\Delta \leq 0$, so we now conclude that $\Delta = 0$. From inequality (3.1), we see that $|s_2p_2|_k \geq 10$. This forces $|p_2|, |s_2| \geq 10$.

Since $\Delta = 0$, we have $|x_1| = |x_2|$. Since $|x_1|_k \geq |x_2|_k + 2$, either there are letters $j_1 \neq j_2$ such that $|x_2|_{j_i} > |x_1|_{j_i}$, or a single letter j such that $|x_2|_j \geq |x_1|_j + 2$. In the first case, by Lemma 3.1, one of a_1 and b_1 is j_1, and the other j_2; in the second case $a_1 = b_1 = j$. In both cases, $b_1x_1a_1 \sim b_2x_2a_2$. □

Lemma 3.3. *We can choose words $\hat{x}_i \in \{x_i, x_ia_i, b_ix_i, b_ix_ia_i\}$ for $i = 1, 2$ such that $\hat{x}_1 \sim \hat{x}_2$.*

Proof. If for each $k \in \Sigma$ we have $|x_1|_k = |x_2|_k$, then let $\hat{x}_i = x_i$, and the result is true. Assume then that $|x_1|_k \neq |x_2|_k$ for some value k. Again, if for some $k \in \Sigma$ we have $||x_1|_k - |x_2|_k| \geq 2$, then the result is true by Lemma 3.2. Suppose then that $||x_1|_k - |x_2|_k| \leq 1$ for each k. By Lemma 3.1, $|x_1|_k > |x_2|_k$ for at most two values of k, and $|x_2|_k > |x_1|_k$ for at most two values of k.

Without loss of generality, suppose that $|x_1| \geq |x_2|$, so that $\Delta \geq 0$. Then $\Delta = 0$ or $\Delta = 1$.

Case 1: We have $\Delta = 1$.

Since

$$\begin{aligned} \Delta &= |x_1| - |x_2| \\ &= \sum_{k\in\Sigma} |x_1|_k - \sum_{k\in\Sigma} |x_2|_k \\ &= \sum_{k\in\Sigma} (|x_1|_k - |x_2|_k) \end{aligned}$$

we must have $|x_1|_k > |x_2|_k$ for at least one value of k. By supposition, $||x_1|_k - |x_2|_k| \leq 1$, so that $|x_1|_k = 1 + |x_2|_k$ for this value. By Lemma 3.1, there can be at most two values of k such that $|x_1|_k = 1 + |x_2|_k$. We make cases on the number of such k.

Case 1a: There is a unique $k_1 \in \Sigma$ such that $|x_1|_{k_1} = 1 + |x_2|_{k_1}$.

By Lemma 3.1, we have $a_2 = k_1$ or $b_2 = k_1$. Let $\hat{x}_1 = x_1$ and

$$\hat{x}_2 = \begin{cases} b_2x_2, & \text{if } b_2 = k_1; \\ x_2a_2, & \text{otherwise. (In this case, } a_2 = k_1.) \end{cases}$$

We have

$$\begin{aligned} 1 &= \Delta \\ &= |x_1| - |x_2| \\ &= \sum_{k\in\Sigma} (|x_1|_k - |x_2|_k) \\ &= 1 + \sum_{k\in\Sigma-\{k_1\}} (|x_1|_k - |x_2|_k) \end{aligned}$$

Then $\sum_{k\in\Sigma-\{k_1\}}(|x_1|_k - |x_2|_k) = 0$. However, each term of this sum is non-positive, since $|x_1|_k - |x_2|_k < 1$ for $k \neq k_1$. It follows that no term of this sum can be negative, so that $|x_1|_k = |x_2|_k$ whenever $k \neq k_1$. It follows that

$$\hat{x}_1 = x_1 \sim kx_2 \sim \hat{x}_2, \text{ as desired.}$$

Case 1b: There are two values $k_1 \neq k_2 \in \Sigma$ such that for $i = 1, 2$, $|x_1|_{k_i} = 1 + |x_2|_{k_i}$.
By Lemma 3.1 we have $\{a_2, b_2\} = \{k_1, k_2\}$. Let $\hat{x_2} = b_2x_2a_2$. From

$$\begin{aligned} 1 &= \Delta \\ &= |x_1| - |x_2| \\ &= \sum_{k\in\Sigma}(|x_1|_k - |x_2|_k) \\ &= 2 + \sum_{k\in\Sigma-\{k_1,k_2\}}(|x_1|_k - |x_2|_k) \end{aligned}$$

and the fact that $|x_1|_k - |x_2|_k \leq 0$ for $k \neq k_1, k_2$, we conclude that there is exactly one value k_3 such that $|x_2|_{k_3} > |x_1|_{k_3}$. Then $|x_2|_{k_3} = 1 + |x_1|_{k_3}$, and by Lemma 3.1 we conclude that $k_3 \in \{a_1, b_1\}$. Let

$$\hat{x}_1 = \begin{cases} b_1x_1, & \text{if } b_1 = k_3; \\ x_1a_1, & \text{otherwise. (In this case, } a_1 = k_3.) \end{cases}$$

Then $\hat{x_1} \sim \hat{x}_2$, as desired.
Case 2: We have $\Delta = 0$.
By Lemma 3.1, the number of k such that $|x_1|_k > |x_2|_k$ is 1, or 2. (We are assuming there is at least one k where these differ.) Since

$$\begin{aligned} 0 &= \Delta \\ &= \sum_{k\in\Sigma}(|x_1|_k - |x_2|_k), \end{aligned}$$

there will be the same number of k such that $|x_2|_k > |x_1|_k$.
Case 2a: There is a unique $k_1 \in \Sigma$ such that $|x_1|_{k_1} = 1 + |x_2|_{k_1}$ and a unique $k_2 \in \Sigma$ such that $|x_2|_{k_2} = 1 + |x_1|_{k_2}$
By Lemma 3.1, we have $a_2 = k_1$ or $b_2 = k_1$ and $a_1 = k_2$ or $b_1 = k_2$. Let

$$\hat{x}_2 = \begin{cases} b_2x_2, & \text{if } b_2 = k_1; \\ x_2a_2, & \text{otherwise. (In this case, } a_2 = k_1.) \end{cases}$$

Let

$$\hat{x}_1 = \begin{cases} b_1x_1, & \text{if } b_1 = k_2; \\ x_1a_1, & \text{otherwise. (In this case, } a_1 = k_2.) \end{cases}$$

Case 2b: There are distinct $k_1, k_2, k_3, k_4 \in \Sigma$ such that

$$\begin{aligned} |x_1|_{k_1} &= 1 + |x_2|_{k_1} \\ |x_1|_{k_2} &= 1 + |x_2|_{k_2} \\ |x_2|_{k_3} &= 1 + |x_1|_{k_3} \\ |x_2|_{k_4} &= 1 + |x_1|_{k_4} \end{aligned}$$

By Lemma 3.1, we have $\{a_2, b_2\} = \{k_1, k_2\}$ and $\{a_1, b_1\} = \{k_3, k_4\}$. Let $\hat{x}_1 = b_1 x_1 a_1$, $\hat{x}_2 = b_2 x_2 a_2$. Then $\hat{x}_1 \sim \hat{x}_2$. □

4. Preimages of Abelian powers

If $k \in \Sigma$, then the index of $f(k)$ in $f^\omega(0)$ is always a multiple of 13. Suppose $f^n(0)$ contains a prefix PX_1YX_2 with $X_1 \sim X_2$, $|X_1| > 24$. For $i = 1, 2$, write

$$\begin{aligned} X_i &= s_i f(x_i) p_i \\ f(a_i) &= p_i q_i, \text{ some } a_i \in \Sigma \cup \{\epsilon\}, |p_i| \leq 12, \\ f(b_i) &= r_i s_i, \text{ some } b_i \in \Sigma \cup \{\epsilon\}, |s_i| \leq 12. \end{aligned}$$

Since each of the $|X_i| > 24$, we conclude that each x_i is non-empty. It follows that the indices of $f(x_1)$ and $f(x_2)$ are multiples of 13, which forces $|p_1 Y s_2|$ to be a multiple of 13, and $|p_1 s_2|$ to be a multiple of 13. We therefore write $PX_1YX_2 = f(Qb_1x_1a_1yb_2x_2)q_2$ where $Qb_1x_1a_1yb_2x_2a_2$ is a prefix of $f^{n-1}(0)$, $P = f(Q)r_1$, $Y = q_1 f(y) r_2$, and $y \in \Sigma^*$. Note that the $a_i, b_i, p_i, q_i, r_i, s_i, y$ may all be empty.

Lemma 4.1. *Suppose $n > 1$ and $f^\omega(0)$ contains an Abelian power X_1YX_2 with $X_1 \sim X_2$, $|X_1| \geq 24n$. Then $f^\omega(0)$ contains an Abelian power $\hat{x}_1\hat{y}\hat{x}_2$ with $\hat{x}_1 \sim \hat{x}_2$,*

$$|X_1 Y| \geq \frac{13n}{n+1} |\hat{x}_1 \hat{y}| \text{ and}$$

$$\frac{|X_1 Y X_2|}{|X_1 Y|} \leq \frac{|\hat{x}_1 \hat{y} \hat{x}_2|}{|\hat{x}_1 \hat{y}|} + \frac{72}{|X_1 Y|}.$$

Proof. Choose $\hat{x}_1$ and $\hat{x}_2$ as in Lemma 3.3. Then $\hat{x_1} \sim \hat{x_2}$. Let

$$\hat{y} = \begin{cases} y, & \text{if } \hat{x}_1 \in \{b_1x_1a_1, x_1a_1\}, \hat{x}_2 \in \{b_2x_2a_2, b_2x_2\}; \\ a_1 y, & \text{if } \hat{x}_1 \in \{b_1x_1, x_1\}, \hat{x}_2 \in \{b_2x_2a_2, b_2x_2\}; \\ y b_2, & \text{if } \hat{x}_1 \in \{b_1x_1a_1, x_1a_1\}, \hat{x}_2 \in \{x_2a_2, x_2\}; \\ a_1 y b_2, & \text{if } \hat{x}_1 \in \{b_1x_1, x_1\}, \hat{x}_2 \in \{x_2a_2, x_2\}. \end{cases}$$

Then $f^{n-1}(0)$ has prefix $Qb_1x_1a_1yb_2x_2a_2$ as in the previous discussion, which contains the factor $\hat{x}_1\hat{y}\hat{x}_2$ by our choice of $\hat{x}_1, \hat{y}$, and $\hat{x}_2$.

We note that

$$\begin{aligned} |\hat{x}_1 \hat{y}| &\leq |b_1 x_1 a_1 y b_2| \\ &= \frac{|f(b_1 x_1 a_1 y b_2)|}{13} \\ &= \frac{|X_1 Y| + |r_1 s_2|}{13} \end{aligned}$$

$$\begin{aligned} &\leq \frac{|X_1Y| + 24}{13} \\ &\leq \frac{|X_1Y|\frac{n+1}{n}}{13} \end{aligned}$$

Thus

$$|X_1Y| \geq \frac{13n}{n+1}|\hat{x}_1\hat{y}|.$$

Also,

$$\begin{aligned} |\hat{x}_1\hat{y}\hat{x}_2| &\geq |x_1a_1yb_2x_2| \\ &= \frac{|f(x_1a_1yb_2x_2)|}{13} \\ &= \frac{|X_1YX_2| - |s_1p_2|}{13} \\ &\geq \frac{|X_1YX_2| - 24}{13} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{|X_1YX_2|}{|X_1Y|} - \frac{|\hat{x}_1\hat{y}\hat{x}_2|}{|\hat{x}_1\hat{y}|} &\leq \frac{|X_1YX_2|}{|X_1Y|} - \frac{\frac{|X_1YX_2|-24}{13}}{\frac{|X_1Y|+24}{13}} \\ &= \frac{|X_1YX_2|}{|X_1Y|} - \frac{|X_1YX_2| - 24}{|X_1Y| + 24} \\ &= \frac{|X_1YX_2|(\cancel{|X_1Y|} + 24) - |X_1Y|(\cancel{|X_1YX_2|} - 24)}{|X_1Y|(|X_1Y| + 24)} \\ &\leq \frac{2\cancel{|X_1Y|}24 + \cancel{|X_1Y|}24}{\cancel{|X_1Y|}(|X_1Y| + 24)} \\ &= \frac{72}{|X_1Y| + 24} \\ &\leq \frac{72}{|X_1Y|}, \text{ as desired.} \end{aligned}$$

□

Corollary 4.2. *Suppose $n > 1$ and $c \in \mathbb{R}$ are such that whenever x_1yx_2 is an Abelian power in $f^\omega(0)$ with $x_1 \sim x_2$ and $|x_1| \leq 24n$, then $\frac{|x_1yx_2|}{|x_1y|} \leq c$. Then if X_1YX_2 is an Abelian power in $f^\omega(0)$ we have*

$$\frac{|X_1YX_2|}{|X_1Y|} \leq c + \frac{k}{1-r},$$

where $k = \frac{3}{n}$ and $r = \frac{n+1}{13n}$

Proof. Repeatedly using Lemma 4.1, create a sequence of Abelian powers

$$\left(X_1^{(i)}Y^{(i)}X_2^{(i)}\right)_{i=0}^{m}$$

such that

$$X_1^{(0)} = X_1, Y^{(0)} = Y, X_2^{(0)} = X_2,$$

$$|X_1^{(m)}| \leq 24n$$

$$|X_1^{(i)}| > 24n, \text{ for } i = 1, 2, \ldots, m-1$$

$$X_1^{(i+1)} \sim X_2^{(i+1)} \text{ for } i = 0, 1, \ldots, m-1$$

$$|X_1^{(i)} Y^{(i)}| \geq \frac{13n}{n+1} |X_1^{(i+1)} Y^{(i+1)}| \tag{4.1}$$

$$\frac{|X_1^{(i)} Y^{(i)} X_2^{(i)}|}{|X_1^{(i)} Y^{(i)}|} \leq \frac{|X_1^{(i+1)} Y^{(i+1)} X_2^{(i+1)}|}{|X_1^{(i+1)} Y^{(i+1)}|} + \frac{72}{|X_1^{(i)} Y^{(i)}|}, \text{ for } i = 0, 1, \ldots, m-1.$$

Property (4.1) coming from Lemma 4.1 ensures that m is finite. We see that for $0 \leq i \leq m-1$

$$\frac{1}{|X_1^{(i)} Y^{(i)}|} \leq \frac{r^i}{|X_1^{(0)} Y^{(0)}|}.$$

Since $k = \frac{3}{n} \geq \frac{72}{|X_1^{(0)} Y^{(0)}|}$, this means that

$$\frac{|X_1^{(0)} Y^{(0)} X_2^{(0)}|}{|X_1^{(0)} Y^{(0)}|} \leq \frac{|X_1^{(m)} Y^{(m)} X_2^{(m)}|}{|X_1^{(m)} Y^{(m)}|} + k \sum_{i=1}^{m} r^i \leq c + \frac{k}{1-r}.$$

□

5. Search depth

Remark 5.1. Say that word u is *equivalent* to word v if $u = \sigma^i(v)$ for some integer i. Any length 2 factor v of $f^\omega(0)$ is equivalent to a factor of 07401030502, the length 11 prefix of $f^\omega(0)$ Since f is cyclic, if u is equivalent to v, then every factor of $f(u)$ is equivalent to a factor of $f(v)$.

Let

$$g(x) = \left\lfloor \frac{x+24}{13} \right\rfloor.$$

Lemma 5.2. *Suppose that u is factor of $f^\omega(0)$, and $g^t(|u|) \leq 2$. Then u is equivalent to a factor of the prefix of $f^\omega(0)$ of length* $13^t(11)$.

Proof. Create a sequence of words $u = u_0, u_1, \ldots, u_t$ such that u_i is a factor of $f(u_{i+1})$ for each i, and u_{i+1} is a factor of $f^\omega(0)$ with $|f(u_{i+1})| \leq |u_i| + 24$. Then

$$u_0 \text{ is a factor of } f^t(u_t) \text{ and}$$

$$|u_t| \leq g^t(|u_0|) \leq 2.$$

It follows that u_t is equivalent to a factor of 07401030502, the prefix of $f^\omega(0)$ of length 11. Therefore, $u = u_0$ is equivalent to a factor of $f^t(07401030502)$, which has length $13^t(11)$. □

Lemma 5.3. *Suppose that $f^\omega(0)$ contains a factor X_1YX_2 such that*

$$\begin{aligned} X_1 &\sim X_2 \\ \frac{|X_1YX_2|}{|X_1Y|} &> 1.713 \\ |X_1| &\leq 1000. \end{aligned}$$

Then $f^5(0)$ contains such a factor.

Proof. For such a word, we have

$$\begin{aligned} |X_1YX_2| &> 1.713|X_1Y| \\ &= 1.713(|X_1YX_2| - |X_2|) \\ &= 1.713(|X_1YX_2| - |X_1|) \end{aligned}$$

so that

$$1.713|X_1| > 0.713|X_1YX_2|$$

and

$$\begin{aligned} |X_1YX_2| &< \frac{1.713|X_1|}{0.713} \\ &\leq \frac{1713}{0.713} \\ &< 2403. \end{aligned}$$

We find that $g^5(2403) = 2$. The result follows from Lemma 5.2. □

6. Search result and bound

We searched the word $f^5(0)$. All its factors X_1YX_2 with $X_1 \sim X_2$ and $|X_1| \leq 1000$ obey

$$\frac{|X_1YX_2|}{|X_1Y|} \leq \frac{841}{491} \approx 1.71283.$$

We conclude from Lemma 5.3 that $f^\omega(0)$ contains no factor X_1YX_2 with

$$\begin{aligned} X_1 &\sim X_2 \\ \frac{|X_1YX_2|}{|X_1Y|} &> 1.713 \\ |X_1| &\leq 1000. \end{aligned}$$

Theorem 6.1. *Word $f^\omega(0)$ contains no factor X_1YX_2 with $X_1 \sim X_2$ and*

$$\frac{|X_1YX_2|}{|X_1Y|} > \frac{876775}{489527} \approx 1.79107.$$

Proof. This follows from Corollary 4.2, letting $n = 1000/24$ and $c = \frac{841}{491}$. □

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