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SEQUENCES FROM FIBONACCI TO CATALAN: A COMBINATORIAL INTERPRETATION *VIA* DYCK PATHS

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Abstract. We use Dyck paths having some restrictions in order to give a combinatorial interpretation for some famous number sequences. Starting from the Fibonacci numbers we show how the k-generalized Fibonacci numbers, the powers of 2, the Pell numbers, the k-generalized Pell numbers and the even-indexed Fibonacci numbers can be obtained by means of constraints on the number of consecutive valleys (at a given height) of the Dyck paths. By acting on the maximum height of the paths we get a succession of number sequences whose limit is the sequence of Catalan numbers. For these numbers we obtain a family of interesting relations including a *full history* recurrence relation. The whole study can be accomplished also by involving particular sets of strings *via* a simple encoding of Dyck paths.

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1. Introduction

The general term of the well known Fibonacci sequence is given by the recurrence

$$f_n = f_{n-1} + f_{n-2}.$$

It can be generalized as the sum of the k previous terms (so that the Fibonacci sequence is obtained with $k = 2$)

$$f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \ldots + f_{n-k}^{(k)},$$

giving the k-generalized Fibonacci sequence [1–3]. Considering the full history recurrence relation

$$f'_n = f'_{n-1} + f'_{n-2} + \ldots + f'_1 + f'_0$$

(*i.e.* a relation whose general term is a sum involving all the preceding terms), the sequence of the powers of 2 is obtained (starting with the suitable initial conditions), whose recurrence is very often written by

$$f'_n = 2f'_{n-1}.$$

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The general term of the Pell sequence is given by the recurrence

$$p_n = 2p_{n-1} + p_{n-2}.$$

It can be generalized by considering the sum of the k previous terms (so that the Pell sequence is obtained with $k = 2$ and the powers of 2 can be considered as the particular case with $k = 1$)

$$p_n^{(k)} = 2p_{n-1}^{(k)} + p_{n-2}^{(k)} + \ldots + p_{n-k}^{(k)},$$

giving the k-generalized Pell sequence [4]. Considering the (almost) full history recurrence relation

$$p'_n = 2p'_{n-1} + p'_{n-2} + \ldots + p'_1$$

the sequence of the even-indexed Fibonacci numbers $(1, 1, 2, 5, 13, \ldots)$ is obtained (starting with the suitable initial conditions), whose general term is indicated by

$$p'_n = 3p'_{n-1} - p'_{n-2}.$$

For clarity sake, in our work we consider the Fibonacci sequence defined by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, so that the even-indexed Fibonacci sequence starts with the terms $1, 2, 5, 13, \ldots$ which does not coincide with the one we are dealing with $(1, 1, 2, 5, 13, \ldots)$. Nevertheless, we use the same name since the two sequences differ only for the first two terms.

In the present work we describe a combinatorial interpretation of all these sequences by means of the Dyck paths subject to suitable constraints. More precisely, we consider the sets $\mathcal{D}_n^{(h,k)}$ of Dyck paths having semilength n and height at most h, and avoiding $k - 1$ valleys at height $h - 1$, and the sets $\mathcal{D}_n^{(h)}$ of Dyck paths having height at most h (without restriction on the numbers of consecutive valleys).

Starting from $h = 2$, we show how increasing values of $k \geq 2$ give rise to sets of Dyck paths (the $\mathcal{D}_n^{(2,k)}$'s) enumerated by the k-generalized Fibonacci numbers (in particular $\mathcal{D}_n^{(2,2)}$ is enumerated by the Fibonacci numbers), up to the set $\mathcal{D}_n^{(2)}$ enumerated by the powers of 2.

For $h = 3$, we show how increasing values of $k \geq 2$ give rise to sets of Dyck paths (the $\mathcal{D}_n^{(3,k)}$'s) enumerated by the k-generalized Pell numbers (in particular $\mathcal{D}_n^{(3,2)}$ is enumerated by the Pell numbers), up to the set $\mathcal{D}_n^{(3)}$ enumerated by the even-indexed Fibonacci numbers.

Furthermore, we extend our investigation to larger values of h, providing a general recurrence relation for the sequences enumerating the sets $\mathcal{D}_n^{(h)}$. Obviously, for each value of h, these sequences are getting closer and closer to the Catalan numbers, counting the sets of unrestricted Dyck paths. As a final result, from this additional step, we obtain some interesting relations involving the Catalan sequence itself.

A link between the Fibonacci and the Catalan numbers has been already studied, for example in [5] where pattern avoiding permutations have been used, with techniques similar to the ones developed in [6]. There, the authors found which sequences lay between Fibonacci's and Catalan's, thus defining a sort of "discrete continuity" between those famous numbers. Here, the aim is very similar and is accomplished by using Dyck paths.

2. Preliminaries

A Dyck path is a lattice path in the discrete plane $\mathbb{Z}^2$ from $(0, 0)$ to $(2n, 0)$ with up and down steps in $\{(1, 1), (1, -1)\}$, never crossing the x-axis. The number of up steps in any prefix of a Dyck path is greater or equal to the number of down steps and the total number of steps (the *length* of the path) is $2n$. We denote the set of Dyck paths having length $2n$ (or equivalently semilength n) by $\mathcal{D}_n$. A Dyck path can be codified by a string over the alphabet $\{U, D\}$, where U and D replace the up and down steps, respectively. The empty Dyck path is denoted by ε.

The height of a Dyck path P is the maximum ordinate reached by one of its steps. A *valley* of P is an occurrence of the substring DU while a *peak* is an occurrence of the substring UD. The height of a valley (peak) is the ordinate reached by D (U).

We denote by $\mathcal{D}_n^{(h,k)}$ the set of Dyck paths having semilength n and height at most h, and avoiding $k-1$ consecutive valleys at height $h-1$. The set of Dyck paths having semilength n with height at most h (without restriction on the number of valleys) is denoted by $\mathcal{D}_n^{(h)}$. Moreover, $\mathcal{D}^{(h)} = \bigcup_{n \geq 0} \mathcal{D}_n^{(h)}$ and $\mathcal{D}^{(h,k)} = \bigcup_{n \geq 0} \mathcal{D}_n^{(h,k)}$.

The cardinalities of $\mathcal{D}_n^{(h,k)}$ and $\mathcal{D}_n^{(h)}$ are indicated by $D_n^{(h,k)}$ and $D_n^{(h)}$, respectively. Finally, the set $\mathcal{D}_n$ of unrestricted Dyck paths having semilength $n \geq 0$ is enumerated by the n-Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In the paper we refer to the sequence of the Fibonacci numbers defined as follows:

$$f_n = \begin{cases} 1 & \text{for } n = 0, 1 \\ f_{n-1} + f_{n-2} & \text{for } n \geq 2 \end{cases}, \tag{2.1}$$

while the k-generalized Fibonacci sequence is:

$$f_n^{(k)} = \begin{cases} 1 & \text{for } n = 0 \\ \displaystyle\sum_{i=1}^{k} f_{n-i}^{(k)} & \text{for } n \geq 1 \quad (f_\ell^{(k)} = 0 \text{ if } \ell < 0) \end{cases}. \tag{2.2}$$

The sequence of the powers of 2 and the Pell numbers have initial conditions slightly different with respect to their common definition. The reason lies in the fact that we are dealing with Dyck paths that for $n = 0$ and $n = 1$ force the sequences to start with $1, 1, \ldots$ (for the empty path having semilength 0 and the only Dyck path UD with semilength 1). We use the following statements:

$$b_n = \begin{cases} 1 & \text{for } n = 0, 1 \\ 2b_{n-1} & \text{for } n \geq 2 \end{cases} \tag{2.3}$$

for the powers of 2, and

$$p_n = \begin{cases} 1 & \text{for } n = 0, 1 \\ 2p_{n-1} + p_{n-2} & \text{for } n \geq 2 \end{cases}$$

for the Pell numbers.

We recall that the recurrence relation (2.3) is equivalent to the full history recurrence relation

$$f_n = \begin{cases} 1 & \text{for } n = 0 \\ \displaystyle\sum_{i=0}^{n-1} f_i & \text{for } n \geq 1 \end{cases}. \tag{2.4}$$

The k-generalized Pell numbers are often defined as [4, 7]

$$p_n^{(k)} = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ 2p_{n-1}^{(k)} + \sum_{i=1}^{k-1} p_{n-1-i}^{(k)} & \text{for } n \geq 2 \end{cases}.$$

In our work, we will use the sequences defined by:

$$p_n^{(k)} = \begin{cases} 1 & \text{for } n = 0, 1 \\ 2p_{n-1}^{(k)} + \sum_{i=1}^{k-1} p_{n-1-i}^{(k)} & \text{for } n \geq 2 \quad (p_\ell^{(k)} = 0 \text{ if } \ell \leq 0) \end{cases}. \tag{2.5}$$

The reasons for the small differences are always related to the use of Dyck paths and will become clear later on.

As mentioned in the Introduction, we refer to the even-indexed Fibonacci numbers as the ones defined by the following recurrence:

$$g_n = \begin{cases} 1 & \text{for } n = 0, 1 \\ 3g_{n-1} - g_{n-2} & \text{for } n \geq 1 \end{cases} \tag{2.6}$$

which can be also defined by means the following (almost) full history recurrence relation:

$$p'_n = \begin{cases} 1 & \text{for } n = 0, 1 \\ 2p'_{n-1} + \sum_{i=1}^{n-2} p'_{n-1-i} & \text{for } n \geq 2 \end{cases}. \tag{2.7}$$

3. Restrictions on the height and the number of valleys

A Dyck path $P \in \mathcal{D}_n^{(h,k)}$ avoids $k-1$ consecutive valleys at height $h-1$. Starting from this note, in this section we show how the sets $\mathcal{D}_n^{(2,k)}$ and $\mathcal{D}_n^{(3,k)}$ can be exhaustively generated by inserting suitable factors in Dyck paths of the same set having a shorter semilength. The construction allows an exact enumeration of the sets, involving the k-generalized Fibonacci and Pell numbers.

The results of this section aim to show, by means of Dyck paths, how it is possible to start from the Fibonacci sequence and obtain other famous sequences of numbers, in a continuous manner, as mentioned in the Introduction. More precisely in Section 3.1, moving from $\mathcal{D}_n^{(2,2)}$ enumerated by Fibonacci numbers, we increase the parameter k obtaining, for each value of k, the set $\mathcal{D}_n^{(2,k)}$ enumerated by k-generalized Fibonacci numbers. At the limit ($k \to \infty$) the set $\mathcal{D}_n^{(2)}$ is reached, which is enumerated by the powers of 2. Then (Sect. 3.2), the parameter h is increased ($h = 3$) and the sets $\mathcal{D}_n^{(3,k)}$ are considered. As before, the parameter k is increased showing that these sets are enumerated by k-generalized Pell numbers, until the set $\mathcal{D}_n^{(3)}$ is reached, which is enumerated by the even-indexed Fibonacci numbers.

In this way we get similar results as the ones presented in [5], but by means a different combinatorial interpretation of those sequences.

3.1. Height at most 2

The set $\mathcal{D}^{(2,2)}$ contains all the Dyck paths with height at most 2 and without valleys at height 1. If $P \in \mathcal{D}_n^{(2,2)}$, with $n \geq 2$, then it starts with UD or $UUDD$ and is followed by a path of semilength $n-1$ or $n-2$, respectively. Therefore, for $n \geq 2$, the set $\mathcal{D}_n^{(2,2)}$ is obtained by the union of the concatenation of UD to all the paths in $\mathcal{D}_{n-1}^{(2,2)}$ and the concatenation of the string $UUDD$ to the paths of $\mathcal{D}_{n-2}^{(2,2)}$:

$$\mathcal{D}_n^{(2,2)} = UD \cdot \mathcal{D}_{n-1}^{(2,2)} \cup UUDD \cdot \mathcal{D}_{n-2}^{(2,2)}$$

The only path with semilength 0 is the empty path, while the only path with semilength 1 is UD. Regarding cardinality we can write:

$$D_n^{(2,2)} = \begin{cases} 1 & \text{if } n = 0, 1 \\ D_{n-1}^{(2,2)} + D_{n-2}^{(2,2)} & \text{if } n \geq 2 \end{cases}$$

coinciding with the definition of the Fibonacci numbers (2.1).

To a deeper analysis, the above construction is a particular case ($k = 2$) of the following one. A path $P \in \mathcal{D}_n^{(2,k)}$ starts with one of the factors $UD, UUDD, UUDUDD, \ldots, U(UD)^{k-1}D$ (where $(UD)^{k-1}$ is the string obtained by concatenating UD to itself $k-1$ times). Then, the path P is obtained by concatenating one of these factors to a path of suitable length of $\mathcal{D}^{(2,k)}$, and the set $\mathcal{D}_n^{(2,k)}$ can be generated by:

$$\mathcal{D}_n^{(2,k)} = \begin{cases} \varepsilon & \text{if } n = 0 \\ \displaystyle\bigcup_{j=0}^{k-1} U(UD)^j D \cdot \mathcal{D}_{n-1-j}^{(2,k)} & \text{if } n \geq 1 \end{cases}$$

so that its cardinality is given by the recurrence

$$D_n^{(2,k)} = \begin{cases} 1 & \text{if } n = 0 \\ \displaystyle\sum_{j=0}^{k-1} D_{n-1-j}^{(2,k)} & \text{if } n \geq 1 \qquad (D_{n-1-j}^{(2,k)} = 0 \text{ if } n-1-j < 0) \end{cases},$$

coinciding with the recurrence relation of the k-generalized Fibonacci numbers (2.2).

The paths of $\mathcal{D}^{(2)}$ have height at most 2 and do not have restrictions on the number of consecutive valleys. Therefore, moving from the construction of $\mathcal{D}_n^{(2,k)}$, a path $P \in \mathcal{D}_n^{(2)}$ is the concatenation of the factor $U(UD)^jD$ to a path $Q \in \mathcal{D}_{n-1-j}^{(2)}$, for $j = 0, 1, \ldots, n-1$. Note that the length of the beginning factor of P can be up to n (rather than k as in the construction of $D_n^{(2,k)}$). Clearly, in the case $j = n-1$ the path Q is the empty path.

The description of the construction of $\mathcal{D}_n^{(2)}$ is

$$\mathcal{D}_n^{(2)} = \begin{cases} \varepsilon & \text{if } n = 0 \\ \displaystyle\bigcup_{j=0}^{n-1} U(UD)^j D \cdot \mathcal{D}_{n-1-j}^{(2)} & \text{if } n \geq 1 \end{cases}$$

whence

$$D_n^{(2)} = \begin{cases} 1 & \text{if } n = 0 \\ \displaystyle\sum_{j=0}^{n-1} D_{n-1-j}^{(2)} & \text{if } n \geq 1 \end{cases},$$

coinciding with the full history recurrence relation (2.4) for the powers of 2.

An easy combinatorial interpretation of the equivalent relation (2.3) can be provided. A Dyck paths $P \in \mathcal{D}_n^{(2)}$ $(P \neq \varepsilon)$ starts with UD or UUD. Then, it can be obtained either by concatenating the prefix UD to a path $Q \in \mathcal{D}_{n-1}^{(2)}$ (possibly $Q = \varepsilon$) or by inserting UD after the first $\mathbf{U}$ step of a path $Q = \mathbf{U}Q' \in \mathcal{D}_{n-1}^{(2)}$ (with $Q \neq \varepsilon$). Summarizing:

$$\mathcal{D}_n^{(2)} = \begin{cases} \varepsilon & \text{if } n = 0 \\ UD & \text{if } n = 1 \\ UD \cdot Q \ \cup \mathbf{U}\ UD \cdot Q' & \text{if } n \geq 2, \text{ with } Q,\ \mathbf{U}Q' \in \mathcal{D}_{n-1}^{(2)} \end{cases}$$

leading to:

$$D_n^{(2)} = \begin{cases} 1 & \text{if } n = 0, 1 \\ 2D_{n-1}^{(2)} & \text{if } n \geq 2 \end{cases}.$$

3.2. Height at most 3

We consider the set $\mathcal{D}^{(3,k)}$ of Dyck paths with height at most 3 and avoiding $k-1$ consecutive valleys at height 2.

A path $P \in \mathcal{D}_n^{(3,k)}$ can be obtained by:

- adding the prefix UD to each path of semilength $n-1$;
- inserting the factor $U(UD)^j D$, with $j = 0, 1, \ldots, k-1$, after the first up step $\mathbf{U}$ of a path $Q = \mathbf{U}Q'$ with suitable length.

We note that the path $\mathbf{U}Q'$ cannot be the empty path, since it must have a first up step, therefore $\mathbf{U}Q' \in \mathcal{D}_{n-1-j}^{(3,k)}$, with $n-1-j \geq 1$. If $n \geq k+1$ all the k insertions of $U(UD)^j D$ for $j = 0, 1, \ldots, k-1$ are allowed since all the paths inside the involved sets $D_{n-1}^{(3,k)}$ (when $j = 0$) , $D_{n-2}^{(3,k)}$ (when $j = 1$) , $\ldots$, $D_{n-k}^{(3,k)}$ (when $j = k-1$) have lengths greater than or equal to 1. In the case $n \leq k$, only $n-1$ insertions of the same factor are allowed. They are the ones involving the paths of the sets $D_{n-1}^{(3,k)}$ (when $j = 0$) , $D_{n-2}^{(3,k)}$ (when $j = 1$) , $\ldots$, $D_1^{(3,k)}$ (when $j = n-2$) having length greater than or equal to 1.

The construction of $\mathcal{D}_n^{(3,k)}$ is:

$$\mathcal{D}_n^{(3,k)} = \begin{cases} \varepsilon & \text{if } n=0 \\ UD & \text{if } n=1 \\ UD\cdot Q \ \cup \displaystyle\bigcup_{j=0}^{n-2} \mathbf{U}U(UD)^jD\cdot Q' & \text{if } 2\le n\le k \\ UD\cdot Q \ \cup \displaystyle\bigcup_{j=0}^{k-1} \mathbf{U}U(UD)^jD\cdot Q' & \text{if } n\ge k+1 \end{cases},$$

where $Q\in\mathcal{D}_{n-1}^{(3,k)}, \mathbf{U}Q'\in\mathcal{D}_{n-1-j}^{(3,k)}$.

We get a combinatorial interpretation of the sequence:

$$D_n^{(3,k)} = \begin{cases} 1 & \text{if } n=0,1 \\ D_{n-1}^{(3,k)} + \left(D_{n-1}^{(3,k)} + D_{n-2}^{(3,k)} + \dots D_2^{(3,k)} + D_1^{(3,k)}\right) & \text{if } 2\le n\le k \\ D_{n-1}^{(3,k)} + \displaystyle\sum_{j=0}^{k-1} D_{n-1-j}^{(3,k)} & \text{if } n\ge k+1 \end{cases}$$

equivalent to

$$D_n^{(3,k)} = \begin{cases} 1 & \text{if } n=0,1 \\ 2D_{n-1}^{(3,k)} + \displaystyle\sum_{j=1}^{k-1} D_{n-1-j}^{(3,k)} & \text{if } n\ge 2 \quad (D_{n-1-j}^{(3,k)}=0 \text{ if } n-1-j\le 0) \end{cases}$$

coinciding with the k-generalized Pell recurrence relation (2.5).

The paths of $\mathcal{D}^{(3)}$ have height at most 3 and do not have restriction on the number of consecutive valleys. Therefore, basing on the construction of $D_n^{(3,k)}$, the insertion of the factor $U(UD)^jD$ after the first up step $\mathbf{U}$ of $Q=\mathbf{U}Q'$ is allowed for $j=0,1,\dots,n-2$ (rather than up to $k-1$):

$$\mathcal{D}_n^{(3)} = \begin{cases} \varepsilon & \text{if } n=0 \\ UD & \text{if } n=1 \\ UD\cdot Q \ \cup \displaystyle\bigcup_{j=0}^{n-2} \mathbf{U}U(UD)^jD\cdot Q' & \text{if } n\ge 2, \end{cases}$$

where $Q\in\mathcal{D}_{n-1}^{(3)}, \mathbf{U}Q'\in\mathcal{D}_{n-1-j}^{(3)}$.

Regarding the cardinality we get:

$$D_n^{(3)} = \begin{cases} 1 & \text{if } n = 0, 1 \\ 2D_{n-1}^{(3)} + \sum_{j=1}^{n-2} D_{n-1-j}^{(3)} & \text{if } n \geq 2 \end{cases}$$

coinciding with (2.7) for the even-indexed Fibonacci numbers.

An easy combinatorial interpretation for the equivalent relation (2.6) defining the same numbers is described here. A path $P \in \mathcal{D}_n^{(3)}$ starts with UD, UUD, or $UUUD$. This leads to the following three possibilities for the construction of P:

$$P = UD \cdot Q \qquad \left(UD \text{ followed by } Q \in \mathcal{D}_{n-1}^{(3)}\right) \quad \text{or}$$

$$P = \mathbf{U}UD \cdot Q' \qquad \left(UD \text{ inserted after } \mathbf{U} \text{ in } UQ' \in \mathcal{D}_{n-1}^{(3)}\right) \quad \text{or}$$

$$P = \mathbf{UU}UD \cdot Q' \qquad \left(UD \text{ inserted after } \mathbf{UU} \text{ in } \mathbf{UU}Q' \in \mathcal{D}_{n-1}^{(3)}\right).$$

Clearly, the path $\mathbf{U}Q'$ can not be the empty path ($\mathbf{U}Q' \neq \varepsilon$). Moreover, with $\mathbf{UU}Q' \in \mathcal{D}_{n-1}^{(3)}$ we denote the Dyck paths of $D_{n-1}^{(3)}$ starting with two up steps. For their cardinality we observe that they are as many as all the paths in $D_{n-1}^{(3)}$ minus the paths of the same lengths starting with UD. The last ones are obtained by the concatenation of UD to all the paths of $\mathcal{D}_{n-2}^{(3)}$. Therefore

$$\left|\{\mathbf{UU}Q' \in D_{n-1}^{(3)}\}\right| = D_{n-1}^{3} - D_{n-2}^{(3)}.$$

The above construction can be summarized in:

$$\mathcal{D}_n^{(3)} = \begin{cases} \varepsilon & \text{if } n = 0 \\ UD & \text{if } n = 1 \\ UD \cdot Q \ \cup \mathbf{U}UD \cdot Q' \ \cup \mathbf{UU}UD \cdot Q' & \text{if } n \geq 2, \end{cases},$$

with $Q, \mathbf{U}Q', \mathbf{UU}Q' \in \mathcal{D}_{n-1}^{(3)}$. Whence, the following recurrence for the cardinality, coinciding with (2.6) for the even-indexed Fibonacci numbers:

$$D_n^{(3)} = \begin{cases} 1 & \text{if } n = 0, 1 \\ 3D_{n-1}^{(3)} - \mathcal{D}_{n-2}^{(3)} & \text{if } n \geq 2 \end{cases}.$$

4. Restrictions only on the height

The aim of the present section is to find a recurrence relation for the sequence $\{D_n^{(h)}\}_{n\geq 0}$, depending on h, whose generating function of is indicated by $f_h(x)$, where x takes track of the semilength of the Dyck paths. We note that $f_h(x)$ is already known, thanks to study of R. Kemp [8]. Nevertheless, in our work, the generating function is obtained through the concept of production matrices [9], which is later than Kemp's work and allows to get $f_h(x)$ in a very simple way.

So, even if the results were already known, we believe we have found them by following a methodology that is probably more straightforward than Kemp's.

Clearly, for $h = 1$, there is exactly one Dyck path of height at most 1 for each n (more precisely the path $(UD)^n$, for $n \geq 0$). Then, we have:

$$\{D_n^{(1)}\}_{n\geq 0} = \{1, 1, 1, \ldots\} \quad \text{and} \quad f_1(x) = \frac{1}{1-x} \tag{4.1}$$

The set $\mathcal{D}^{(h)} = \bigcup_{n\geq 0} \mathcal{D}_n^{(h)}$ of Dyck paths of height at most $h \geq 1$ can be exhaustively generated by means the ECO operator (see [10]) as in [11]. Here, we only recall the succession Ω_h rule arising from the construction:

$$\Omega_h : \begin{cases} \begin{array}{ccl} (1) & & \\ (1) & \rightsquigarrow & (2) \\ (2) & \rightsquigarrow & (2)(3) \\ \vdots & \vdots & \vdots \\ (h-1) & \rightsquigarrow & (2)(3)\cdots(h-1)(h) \\ (h) & \rightsquigarrow & (2)(3)\cdots(h)(h) \end{array} \end{cases} \tag{4.2}$$

According to the theory developed in [9], the production matrix P_h associated to Ω_h is:

$$P_h = \begin{pmatrix} 0 & u^t \\ 0 & P_{h-1} + eu^t \end{pmatrix} \tag{4.3}$$

where u^t is the row vector $(1\ 0\ 0\ldots)$ and e is the column vector $(1\ 1\ 1\ \ldots)^t$, of appropriate size and for what the generating function $f_h(x)$ of the sequence corresponding to Ω_h is concerned, we have [9]:

$$f_h(x) = \frac{1}{1 - xf_{h-1}(x)}. \tag{4.4}$$

Since $f_1(x)$ is rational, thanks to (4.4) it is possible to deduce that also $f_h(x)$ is rational, too. Therefore, we write:

$$f_h(x) = \frac{p_h(x)}{q_h(x)}, \tag{4.5}$$

where $p_h(x)$ and $q_h(x)$ are polynomials with suitable degrees. From (4.4) and (4.5) we obtain:

$$\begin{aligned} p_h(x) &= q_{h-1}(x) \\ q_h(x) &= q_{h-1}(x) - xq_{h-2}(x). \end{aligned} \tag{4.6}$$

It is not difficult to show by induction that (we omit the easy proof):

Proposition 4.1. *The degree of the polynomial $q_h(x)$ is $\lceil \frac{h}{2} \rceil$.*

Thanks to the above proposition we can assume:

$$q_h(x) = a_{h,0} - a_{h,1}x - a_{h,2}x^2 - \ldots - a_{h,j}x^j \quad \text{with} \quad j = \left\lceil \frac{h}{2} \right\rceil.$$

Clearly, it is $a_{h,j} = 0$ if $j > \lceil \frac{h}{2} \rceil$.

Recalling $f_1(x) = \frac{1}{1-x}$, it is $a_{1,0} = 1$ and from (4.6) we have $a_{h,0} = a_{h-1,0}$, so that $a_{h,0} = 1$ for each $h \geq 1$. Then,

$$q_h(x) = 1 - a_{h,1}x - a_{h,2}x^2 - \ldots - a_{h,j}x^j \quad \text{with} \quad j = \left\lceil \frac{h}{2} \right\rceil. \tag{4.7}$$

Using the expression for $q_h(x)$ in (4.6) and the trivial equality $\lceil \frac{h-1}{2} \rceil = \lfloor \frac{h}{2} \rfloor$, we obtain:

$$\begin{aligned} q_h(x) = 1 &- a_{h-1,1}x - a_{h-1,2}x^2 - \ldots - a_{h-1,\lfloor \frac{h}{2} \rfloor}x^{\lfloor \frac{h}{2} \rfloor} \\ &- x\left(1 - a_{h-2,1}x - a_{h-2,2}x^2 - \ldots - a_{h-2,\lfloor \frac{h-1}{2} \rfloor}x^{\lfloor \frac{h-1}{2} \rfloor}\right). \end{aligned} \tag{4.8}$$

From the identity theorem for polynomials, comparing formulas (4.7) and (4.8) for $q_h(x)$, it is not difficult to show the following expression for the coefficients $a_{h,j}$ which holds for each $h \geq 2$ (while for $h = 1$, it is $a_{1,1} = 1$):

$$a_{h,j} = \begin{cases} a_{h-1,1} + 1 & \text{for } j = 1 \\ a_{h-1,j} + a_{h-2,j-1} & \text{for } j = 2, 3, \ldots, \lceil \frac{h}{2} \rceil. \end{cases} \tag{4.9}$$

We have an explicit formula for the coefficients $a_{h,j}$, thanks to the following proposition. At first, being $a_{1,1} = 1$ and $a_{h,j} = a_{h-1,1} + 1$ for (4.9), we note that $a_{h,1} = h$.

Proposition 4.2. *For $h \geq 2$ we have*

$$a_{h,j} = \binom{h-j+1}{j}(-1)^{j+1} \tag{4.10}$$

for $j = 1, 2, \ldots, \lceil \frac{h}{2} \rceil$.

Proof. We proceed by induction on h. For $h = 2$ the statement is true since $a_{2,1} = 2$ is the same value given by (4.10).

Then, using $a_{h,j} = a_{h-1,j} + a_{h-2,j-1}$ for $j = 2, 3, \ldots, \lceil \frac{h}{2} \rceil$ from (4.9) and the inductive hypothesis, we have:

$$\begin{aligned} a_{h,j} = \ & a_{h-1,j} + a_{h-2,j-1} \\ = \ & \binom{h-1-j+1}{j}(-1)^{j+1} + \binom{h-2-j+1+1}{j-1}(-1)^{j-1+1} \\ = \ & \binom{h-j}{j}(-1)^{j+1} - \binom{h-j}{j-1}(-1)^{j} \\ = \ & \begin{cases} -\binom{h-j}{j} - \binom{h-j}{j-1} & \text{if } j \text{ is } \textit{even} \\ \binom{h-j}{j} + \binom{h-j}{j-1} & \text{if } j \text{ is } \textit{odd} \end{cases} = \begin{cases} -\binom{h-j+1}{j} & \text{if } j \text{ is } \textit{even} \\ \binom{h-j+1}{j} & \text{if } j \text{ is } \textit{odd} \end{cases} \\ = \ & \binom{h-j+1}{j}(-1)^{j+1}. \end{aligned}$$

For the case $j = 1$ we already noted that $a_{h,1} = h$. Since expression (4.10) returns h also for $j = 1$, then the proposition is proved.

□

The expression for $q_h(x)$ becomes:

$$q_h(x) = 1 - \sum_{j=1}^{\lceil \frac{h}{2} \rceil} \binom{h-j+1}{j} (-1)^{j+1} \tag{4.11}$$

or, equivalently,

$$q_h(x) = 1 - \sum_{j \geq 1} \binom{h-j+1}{j} (-1)^{j+1}$$

since $\binom{h-j+1}{j} = 0$ if $j > \lceil \frac{h}{2} \rceil$.

For our aim (recurrence relation for $D_n^{(h)}$) we focus our attention on the generating function

$$f_h(x) = \sum_{n \geq 0} D_n^{(h)} x^n. \tag{4.12}$$

For (4.5) and (4.6) it is

$$f_h(x) = \frac{q_{h-1}(x)}{q_h(x)} \tag{4.13}$$

and adapting expression (4.11) for $q_{h-1}(x)$ we have:

$$f_h(x) = \frac{1 - \displaystyle\sum_{j=1}^{\lfloor \frac{h}{2} \rfloor} \binom{h-j}{j} (-1)^{j+1} x^j}{1 - \displaystyle\sum_{j=1}^{\lceil \frac{h}{2} \rceil} \binom{h-j+1}{j} (-1)^{j+1} x^j}. \tag{4.14}$$

Comparing (4.12) and (4.14) we get

$$\left(1 - \sum_{j=1}^{\lceil \frac{h}{2} \rceil} \binom{h-j+1}{j} (-1)^{j+1} x^j\right) \left(\sum_{n \geq 0} D_n^{(h)} x^n\right) = 1 - \sum_{j=1}^{\lfloor \frac{h}{2} \rfloor} \binom{h-j}{j} (-1)^{j+1} x^j$$

and sorting the first part according to the increasing powers of x we obtain:

$$\sum_{n \geq 0} \left(D_n^{(h)} - \sum_{j=1}^{\lceil \frac{h}{2} \rceil} D_{n-j}^{(h)} \binom{h-j+1}{j} (-1)^{j+1} \right) x^n = 1 - \sum_{j=1}^{\lfloor \frac{h}{2} \rfloor} \binom{h-j}{j} (-1)^{j+1} x^j$$

where $D_\ell^{(h)} = 0$ whenever $\ell = 0$.

From the identity theorem for polynomials we can deduce, after some easy consideration on the binomial coefficients, the desired recurrence relation:

$$D_n^{(h)} = \begin{cases} 1 & \text{for } n = 0 \\ \displaystyle\sum_{j=1}^{\lceil \frac{h}{2} \rceil} D_{n-j}^{(h)} \binom{h-j+1}{j} (-1)^{j+1} - \binom{h-n}{n} (-1)^{n+1} & \text{for } n \geq 1. \end{cases} \tag{4.15}$$

We observe that for suitably large value of n with respect to h (more precisely for $n > h/2$) the binomial coefficient $\binom{h-n}{n}$ disappears, and the relation allows to find $D_n^{(h)}$ involving only $\lceil \frac{h}{2} \rceil$ preceding terms.

Similar results on the generating functions $f_h(x)$ have been obtained also in [12], via different tools with respect to what we did. In this work, the expression of the coefficients of the polynomials allowed to get the exact expression of $D_n^{(h)}$.

A very interesting note arises when, once h is fixed, we ask for the number $D_n^{(h)}$ of Dyck paths having semilength $n \leq h$. Clearly, in this case, it is $D_n^{(h)} = C_n$ since all the Dyck paths of a certain semilegth $n \leq h$ have height at most equal to n. Thanks to the above argument it is possible to derive interesting relations involving Catalan numbers. Indeed, for the above remark, posing $h = n + \alpha$, we can write:

$$D_n^{(n+\alpha)} = C_n \quad \forall \alpha \geq 0$$

where α is integer. Then, for (4.15), we get:

$$C_n = \sum_{j=1}^{\lceil \frac{n+\alpha}{2} \rceil} C_{n-j} \binom{n+\alpha-j+1}{j} (-1)^{j+1} - \binom{\alpha}{n} (-1)^{n+1} ,$$

which is equivalent to

$$C_n = \sum_{j=1}^{n} C_{n-j} \binom{n+\alpha-j+1}{j} (-1)^{j+1} - \binom{\alpha}{n} (-1)^{n+1}$$

(since the first binomial coefficient is zero if $j \geq \lceil \frac{n+\alpha}{2} \rceil + 1$) or, including the term C_n within the sum, to:

$$\sum_{j=0}^{n} C_{n-j} \binom{n+\alpha-j+1}{j} (-1)^{j+1} = \binom{\alpha}{n} (-1)^{n+1} , \qquad \forall \alpha \geq 0. \tag{4.16}$$

which, to the best of our knowledge, is new.

As particular cases of (4.16) we explicit the expression for $\alpha = 0$ and $\alpha = n$. The first case corresponds to the value $h = n$ in (4.15) and leads to

$$C_n = \sum_{j=1}^{\lceil \frac{n}{2} \rceil} C_{n-j} \binom{n-j+1}{j} (-1)^{j+1} , \tag{4.17}$$

which links the n-Catalan numbers with the $\lceil \frac{n}{2} \rceil$ preceding ones.

The case $\alpha = n$ corresponds to $h = 2n$ and gives

$$C_n = \sum_{j=1}^{n} C_{n-j} \binom{2n-j+1}{j} (-1)^{j+1} - (-1)^{n+1} \ , \tag{4.18}$$

which is a *full history* recurrence relation for the Catalan numbers.

We observe that the closed form for $D_n^{(h)}$ proposed in [8] would probably have led to similar considerations to ours regarding the relationships involving Catalan numbers expressed by (4.16). Nevertheless, in our work, only simple combinatorial considerations have been used to obtain them.

5. Strings

Each Dyck path can be codified by a suitable sequence of non-negative integers where each number takes tracks of the height reached by each down step. Clearly, if $u = u_1, u_2, \dots, u_n$ is a sequence corresponding to a Dyck path, whenever $u_i < u_{i-1}$ then $u_i = u_{i-1} - 1$, for $i = 2, 3, \dots, n$. Being the last step of a Dyck path a down step, we can omit the last 0 in the corresponding sequence, so that its length is one less than the semilength of the Dyck path. Moreover, the sequences end with 0 or 1. We observe that if u_n=0, the corresponding Dyck path ends with UD. For example, the path $P = UUUDDDUDUDD$ becomes the sequence $2, 1, 1, 1$, while $P = UUUDDDUDDUD$ is codified in $2, 1, 1, 0$.

We note that these sequences are a restriction of the *reverse* of the Catalan words (defined as $a_1 = 0$ and $0 \le a_{i+1} \le a_i + 1$), proposed in Exercise 80 in [13].

In the following we describe the bijections between the sets $D_n^{(2,k)}$, $D_n^{(2)}$, $D_n^{(3,k)}$, and $D_n^{(3)}$ with the corresponding sets of sequences omitting the easy proofs.

By means of the above encoding the paths in $D_n^{(2,k)}$, for $n \ge 1$, are in bijection (we omit the easy proof) with the set $F_{n-1}^{(k)}$ of binary strings of length $n-1$ avoiding the pattern 1^k (i.e. the pattern constituted by k consecutive 1's), which are known to be enumerated by k-generalized Fibonacci numbers [2, 14–16].

Moreover, it is not difficult to see that the set $\mathcal{D}_n^{(2)}$, for $n \ge 1$, is in bijection with the set B_{n-1} of unrestricted binary strings which can be generated by concatenating 0 and 1 to each string of B_{n-2}. These concatenations corresponds to add the prefix UD (concatenation of 0) or to insert the factor UD after the first up step of the Dyck paths of $\mathcal{D}_{n-1}^{(2)}$ in order to obtain $\mathcal{D}_n^{(2)}$.

Always referring to the cited encoding, the set $D_n^{(3,k)}$ is in bijection, for $n \ge 1$, with the set $P_{n-1}^{(k)}$ of strings on $\Sigma = \{0, 1, 2\}$ ending with 0 or 1 with length $n-1$, and avoiding the patterns 20 (trivial) and 2^k1 (since paths cannot have $k-1$ consecutive valleys at height 2). The strings of $P_n^{(k)}$ can be obtained by collecting the concatenations of 0 to each string of $P_{n-1}^{(k)}$ together with all the concatenations of 2^j1 to each string of $P_{n-j-1}^{(k)}$, for $j = 0, 1, \dots, k-1$. These constructions corresponds to the generation of $\mathcal{D}_n^{(3,k)}$ by means of the addition of the prefix UD to each path in $\mathcal{D}_{n-1}^{(3,k)}$ (concatenation of 0), and the insertion of the factor $U(UD)^jD$ to each path $\mathbf{U}Q' \in D_{n-1-j}^{(3,k)}$ after its first $\mathbf{U}$ step (concatenation of 2^j1), for $j = 0, 1, 2, \dots, k-1$.

The set $D_n^{(3)}$ is in bijection with the set P_{n-1} of strings of length $n-1$ on $\{0, 1, 2\}$ avoiding 20 and ending with 0 or 1, *via* the mentioned encoding. The set P_n can be generated by concatenating 0 and 1 to each string of P_{n-1} and 2 to each string of P_{n-1} whose first symbol is not 0. This correspond to the generation of $D_{n+1}^{(3)}$ by adding the prefix UD to each paths in $D_n^{(3)}$ (concatenation of 0), by inserting UD after the first U step of each path in $D_n^{(3)}$ (concatenation of 1), and by inserting UD after the first two U steps of those paths in $D_n^{(3)}$ which do not start with UD (concatenation of 21).

In Table 1 we summarize the obtained results connecting the Dyck paths and the corresponding regular languages for each number sequence we have dealt with. We note that the strings of the languages described in

Table 1. Paths and languages

Sequence	Dyck paths	Language
Fibonacci numbers	height ≤ 2 and no valleys at height 1	$\{0, 10\}^* \cdot \{\varepsilon, 1\}$
k-generalized Fibonacci numbers	height ≤ 2 and no $k-1$ consecutive valleys at height 1	$\{0, 10, \ldots, 1^{k-1}0\}^* \cdot \{\varepsilon, 1, 11, \ldots, 1^{k-1}\}$
2^{n-1}	height ≤ 2	$\{0, 1\}^*$
Pell numbers	height ≤ 3 and no valleys at height 2	$\{0, 1, 21\}^*$
k-generalized Pell numbers	height ≤ 3 and no $k-1$ consecutive valleys at height 2	$\{0, 1, 21, \ldots, 2^{k-1}1\}^*$
even-indexed Fibonacci numbers	height ≤ 3	$\{0, 2^*1\}^*$

the first three rows of Table 1 coincide with the inversion arrays of the pattern avoiding permutations studied in [15].

6. Conclusions and further developments

After considering the case of $\mathcal{D}^{(2,k)}$ and $\mathcal{D}^{(3,k)}$ (Sect. 3), we developed a general argument leading to the recurrence relation for $D_n^{(h)}$, for each $h \geq 2$, whose number sequence approaches the Catalan numbers, as h increases (Sect. 4). One could carry out the journey towards Catalan numbers also by analysing the sets $\mathcal{D}^{(h,k)}$ for $h = 4, 5, \ldots$ and $k = 2, 3, \ldots$ for each h.

For this purpose, we observe that the paths in $\mathcal{D}_n^{(4,2)}$ can be obtained by adding the prefix UD to any path $v \in \mathcal{D}_{n-1}^{(4,2)}$, by inserting the factor UD after the first U step of v, by inserting the factor UD after the first two U steps of those paths $w \in \mathcal{D}_{n-1}^{(4,2)}$ which do not start with UD, and by inserting the factor $UUDD$ after the first two U steps of those paths $w' \in \mathcal{D}_{n-2}^{(4,2)}$ which do not start with UD. Note that there are as many paths w as the paths in $\mathcal{D}_{n-1}^{(4,2)} \setminus \mathcal{D}_{n-2}^{(4,2)}$, and there are as many paths w' as the paths in $\mathcal{D}_{n-2}^{(4,2)} \setminus \mathcal{D}_{n-3}^{(4,2)}$, so that $\mathcal{D}_n^{(4,2)}$ is enumerated by $a_n = 3a_{n-1} - a_{n-3}$ (sequence A052963 in [17]).

The mentioned construction for $\mathcal{D}_n^{(4,2)}$ can be easily generalized to $\mathcal{D}_n^{(4,k)}$ for $k > 2$, and it is not difficult to show that the obtained enumerating sequences are $a_n = 3a_{n-1} - a_{n-k-1}$, which do not appear in [17] (except for $k = 2$).

Although it should be possible to find the set construction case by case for every value of h and k, it would be very complicated as soon as these values increase, even slightly. Nevertheless, it should be possible to find the combinatorial interpretation of many number sequences that probably do not yet have it.

It should be interesting to analyse the possibility to define a Gray code on the set of strings we obtained, except the ones of the sets $F_n^{(k)}$ already studied [18, 19] and the ones enumerated by the powers of 2, these being the unrestricted binary strings for which the famous Binary Reflected Gray Code already exists. For the other strings such a code could be defined using the techniques developed in [20].

Recently, a generalization of k-Fibonacci numbers for k being any positive rational number has been proposed [21]. Could we find some interesting restrictions on Dyck paths enumerated by these numbers?

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