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ASYMPTOTIC BEHAVIOUR OF UNIVARIATE AND MULTIVARIATE ABSORPTION DISTRIBUTIONS

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Abstract. In this work we study the asymptotic behaviour of univariate, bivariate and multivariate absorption discrete q-distributions. Specifically, the pointwise convergence of the univariate absorption distribution to a deformed Gaussian distribution is established. Also, the pointwise convergence of the bivariate and multivariate absorption distributions to bivariate and multivariate deformed Gaussian ones respectively, are established Moreover, interesting applications of the asymptotic behaviour of univariate and multivariate absorption processes are presented.

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1. Brief introduction

Univariate discrete q-distributions had been studied in details the last decades by many authors. Among them we refer to Kemp and Newton [1], Kemp [2, 3], Charalambides [4, 16, 17], Kyriakoussis and Vamvakari [7–11].

Charalambides [16] presented discrete q-distributions based on stochastic models of sequences of independent Bernoulli trials with success probability varying geometrically, with rate q, either with the number of previous trials or with the number of previous successes or both with the number of previous trials and successes. Analytically, Charalambides [4, 16] considered independent q-Bernoulli trials with success probability varying geometrically, with rate q, with the number of previous trials and proved that the number X_n of successes in n such q-Bernoulli trials has the q-Binomial distributionof the first kind. Moreover, Charalambides [4, 16] considered independent q-Bernoulli trials with success probability varying geometrically, with rate q, with the number of previous successes and proved that the number X_n of failures in n such q-Bernoulli trials has the q-binomial distribution of the second kind.

Charalambides [16, 17] presented an absorption distribution as the distribution of the number of successes $n - X_n$, in the above q-binomial stochastic model of the second kind.

Charalambides [4, 16] derived Heine as direct approximation, as $n \to \infty$, of the q-binomial of the first kind, while Euler distribution as a direct approximation of the q-binomial of the second kind.

Keywords and phrases: q-Stirling's formula, q-factorials, pointwise convergence, univariate and multivariate discrete q-distributions, univariate absorption distribution, bivariate absorption distribution, multivariate absorption distribution, univariate absorption process, multivariate absorption process.

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Kyriakoussis and Vamvakari [10] introduced the Heine process $\{X_q(t), t > 0\}$, $0 < q < 1$, where the random variable $X_q(t)$, for every $t > 0$, represented the number of events during a time interval $(0, t]$.

Kyriakoussis and Vamvakari [8, 9, 11] have considered the asymptotic continuous behaviour of the above mentioned q-binomial of the first and second kind distributions, and the relative negative ones, as well as of the Heine and Euler distributions,

Recently, Vamvakari [12] introduced multivariate discrete q-distributions. Specifically, she derived the multivariate absorption distribution as a conditional distribution of a Heine process at a finite sequence of q-points in a time interval. Also, she deduced a multivariate q-hypergeometric distribution, as a conditional distribution of a multivariate absorption distribution.

Afterwards, Charalambides [13, 17] studied in detail q-multinomial, negative q- multinomial, multivariate q-Polya and inverse q-Polya distributions and also examined their limiting discrete distributions. Analytically, he considered a stochastic model of a sequence of independent Bernoulli trials with chain-composite successes (or failures), where the odds of success of a certain kind at a trial is assumed to vary geometrically, with rate q, with the number of previous trials and introduced the q-multinomial and negative q-multinomial distributions of the first kind as well as their discrete limit, multivariate Heine distribution. Also, he considered a stochastic model of a sequence of independent Bernoulli trials with chain-composite successes (or failures), where the probability of success of a certain kind at a trial varies geometrically, with rate q, with the number of previous successes and introduced the q-multinomial and negative q-multinomial distributions of the second kind kind as well as their discrete limit, multivariate Euler distribution.

The aim of this work is to initiate the study of continuous limiting behaviour of multivariate discrete q-distributions in the above q-multinomial stochastic model of the second kind, inspired by the limiting behaviour of the univariate ones. Specifically, we study the asymptotic behavior of the univariate, bivariate and multivariate absorption discrete q-distributions. The pointwise convergence of the univariate absorption distribution to a deformed Gaussian distribution is established. Also, the pointwise convergence of the bivariate and multivariate absorption distributions to bivariate and multivariate deformed Gaussian ones respectively, are established. Moreover, interesting applications of the asymptotic behaviour of univariate and multivariate absorption processes are presented.

2. Some preliminaries

Kyriakoussis and Vamvakari [8, 9, 11] proved limit theorems for the q-Binomial and the negative q-Binomial distributions of the first kind, as well as for the Heine distribution by using pointwise convergence in a "q-analogous sense" of the classical de Moivre–Laplace limit theorem. Specifically in [8], for the needs of their study they established a q-Stirling formula for $n \to \infty$ of the q-factorial of order n with $0 < q < 1$ constant,

$$[n]_q! = \frac{q^{-1/8}(2\pi(1-q))^{1/2}}{(q \log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}(1 + (q^{-n} - 1)q^{j-1})} \left(1 + O(n^{-1})\right), \tag{2.1}$$

where

$$[n]_q! = [1]_q [2]_q \cdots [n-1]_q [n]_q \text{ with } [n]_q = \frac{1-q^n}{1-q},\ n \geq 1.$$

Then, the pointwise convergence of the q-Binomial distribution of the first kind to a deformed continuous Stieltjes–Wigert distribution was established.

A similar asymptotic result has been provided in [9, 11] for the negative q-Binomial distribution of the first kind and the Heine distribution.

Note that, with $q := q(n)$ a sequence of n with $q(n) \to 1$ as $n \to \infty$, Vamvakari [14], studied the effect of this assumption on the $q(n)$-analogue of the Stirling-type and derived the following asymptotic formula

$$[n]_q! = \frac{q^{-1/8}(2\pi(1-q))^{1/2}}{(q \log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}(1 + (q^{-n} - 1)q^{j-1})} \left(1 + O\left(q^n(1-q)\right)\right), \tag{2.2}$$

where $q = q(n)$ with $q(n) \to 1$ as $n \to \infty$ and $q(n)^n = \Omega(1)$.

Moreover, Kyriakoussis and Vamvakari [11] established the pointwise convergence of the q-binomial distribution of the second kind. In detail, they considered the q-Binomial distribution of the second kind with probability function given by

$$f_X(x) = \binom{n}{x}_q \theta^x \prod_{j=1}^{n-x}(1-\theta q^{j-1}), \quad x = 0,1,\ldots,n, \tag{2.3}$$

where $0<\theta<1$ and $0<q<1$ or $1<q<\infty$ with $\theta q^{n-1}<1$. Then, they transfer to the equal-distributed deformed random variable $[X]_q$ and by applying suitably the q-Stirling type (2.2), for $n\to\infty$, they obtained that the q-Binomial distribution of the second kind was approximated by a deformed standardized Gaussian distribution as follows:

$$f_X(x) \cong \frac{(\log q^{-1})^{1/2}}{\sigma_q(2\pi(1-q)^{1/2}}\, q^x \exp\left(-\frac{1}{2}\left(\left(\frac{1-q}{\log q^{-1}}\right)^{1/2}\cdot\frac{[x]_q-\mu_q}{\sigma_q}\right)^2\right), \;\; x\geq 0, \tag{2.4}$$

where $\theta=\theta_n=(1-q)^a$, $0<a<1$, $q=q(n)$ with $q(n)\to 1$, as $n\to\infty$, and $q(n)^n=\Omega(1)$, and μ_q and ${\sigma_q}^2$ the mean value and the variance of the random variable $[X]_q$, given respectively by

$$\mu_q = E\left([X]_q\right) = [n]_q\,\theta \quad \text{and} \quad (\sigma_q)^2 = V\left([X]_q\right) = [n]_q\theta\left(1-\theta\left(1-q^{n-1}\right)\right). \tag{2.5}$$

Similar asymptotic results, have been provided in [11] for the negative q-Binomial distribution of the second kind as well as for their discrete limit, the Euler distribution.

3. Main results

3.1. Continuous limiting behaviour of the univariate absorption distribution

Let X_n be the number of failures in a sequence of n independent Bernoulli trial, with probability of success at the jth geometric sequence of trials given by

$$p_j = 1-\theta q^{j-1}, j=1,2,\ldots,0<\theta<1, 0<q<1 \textit{ or } 1<q<\infty,$$

where, for $0<\theta<1$ and $1<q<\infty$, the number j of geometric sequences of trials is restricted by θq^{j-1}, ensuring that $0<p_j<1$.

The above success probability at the jth geometric sequence, in the case $1<q<\infty$, by replacing the parameter q by q^{-1}, with $0<q<1$ and substituting $\theta=q^r$, becomes

$$p_j = 1-q^{r-j+1}, \;\; j=1,2,\ldots,[r], \;\; 0<r<\infty, \;\; 0<q<1, \tag{3.1}$$

which is a geometrically decreasing sequence of a finite number of terms. Then the probability function of the number Y_n of successes in n independent Bernoulli trials is given by

$$f_{Y_n}(y) = P(Y_n=y) = \binom{n}{y}_q q^{(n-y)(r-y)}(1-q)^y[r]_{y,q}, \;\; y=0,1,\ldots,n, \tag{3.2}$$

for $0<r<\infty$, $0<q<1$, and $n\leq[r]$. This discrete q-distribution is known as *absorption distribution* (see Charalambides [16]).

In this section, we transfer from the random variable Y_n of the absorption distribution (3.2) to the equal-distributed deformed random variable $[Y_n]_q$, and using the q-analogue of Stirling's formula (2.2), we provide the convergence of the absorption distribution to a deformed standardized continuous Gaussian distribution. Initially, we need to compute the mean value and the variance of the random variable $[Y_n]_q$, say μ_q^A and $(\sigma_q^A)^2$, respectively. The mean and the variance of $[Y_n]_q$ are given respectively by

$$\begin{aligned}\mu_q^A &= E\left([Y_n]_q\right) = (1-q)[n]_q[r]_q \\ &\quad \text{and} \\ (\sigma_q^A)^2 &= V\left([Y_n]_q\right) = qE\left([Y_n]_{2,q}\right) + E\left([Y_n]_q\right) - \left(E\left([Y_n]_q\right)\right)^2 \\ &= q(1-q)^2[n]_{2,q}[r]_{2,q} - (1-q)^2[n]_q^2[r]_q^2 + (1-q)[n]_q[r]_q \\ &= q^{n+r-1}(1-q)[n]_q[r]_q.\end{aligned} \tag{3.3}$$

Next, we consider the q-standardized deformed random variable $Z = \frac{[Y_n]_q - \mu_q^A}{\sigma_q^A}$, where μ_q^A and σ_q^A are given in (3.3). Then, we apply pointwise convergence techniques to the probability function (3.2), by using suitably the q Stirling type (2.2) and we obtain the following theorem concerning the asymptotic behaviour of the univariate absorption distribution.

Theorem 3.1. *Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $r = O(n)$. Then, for $n \to \infty$, the absorption distribution given by the p.f. in (3.2) is approximated by a deformed standardized Gaussian distribution as follows:*

$$f_{Y_n}(y) \cong \frac{(\log q^{-1})^{1/2}}{\sigma_q^A(2\pi(1-q))^{1/2}}\, q^y \exp\left(-\frac{1}{2}\left(\left(\frac{1-q}{\log q^{-1}}\right)^{1/2} \cdot \frac{[y]_q - \mu_q^A}{\sigma_q^A}\right)^2\right), \quad y \geq 0, \tag{3.4}$$

where μ_q^A and $\left(\sigma_q^A\right)^2$ are the mean value and the variance of the random variable $[Y_n]_q$, respectively, given in (3.3).

Proof. Using the q-Stirling formula in (2.2), for $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $r = O(n)$, the absorption distribution (3.2) is approximated by

$$\begin{aligned}f_{Y_n}(y) &= \binom{n}{y}_q q^{(n-y)(r-y)}(1-q)^y[r]_{y,q} \\ &\cong \frac{(q\log q^{-1})^{1/2}}{q^{-1/8}(2\pi(1-q))^{1/2}} q^{(n-y)(r-y)} q^{-\binom{y}{2}} q^{y/2}[y]_{1/q}^{-(y+1/2)}[r]_{y,q} \\ &\quad \cdot \prod_{j=1}^{\infty}(1+q(q^{-y}-1)q^{j-1})\end{aligned}$$

or

$$\begin{aligned}f_{Y_n}(y) &\cong \frac{(\log q^{-1})^{1/2}}{q^{-1/8}(2\pi(1-q))^{1/2}} q^{(n-y)(r-y)} q^{y^2/2+y/2}[y]_q^{-(y+1/2)}[r]_{y,q} \\ &\quad \cdot \prod_{j=1}^{\infty}(1+q(q^{-y}-1)q^{j-1}).\end{aligned} \tag{3.5}$$

Consider the random variable $[Y_n]_q = \frac{1-q^{Y_n}}{1-q}$ and the q-standardized random variable $Z = \frac{[Y_n]_q - \mu_q^A}{\sigma_q^A}$, with μ_q^A and $(\sigma_q^A)^2$ given by (3.3). Then the following relations are easily derived:

$$
\begin{aligned}
[y]_q &= \mu_q^A \left(\frac{\sigma_q^A}{\mu_q^A} z + 1 \right) \\
&= (1-q)[n]_q[r]_q \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right), \qquad (3.6) \\
q^y &= 1 - (1-q)\mu_q^A \left(\frac{\sigma_q^A}{\mu_q^A} z + 1 \right) \\
&= 1 - (1-q)^2[n]_q[r]_q \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right), \qquad (3.7) \\
y &= \frac{1}{\log q} \log \left(1 - (1-q)\mu_q^A \left(\frac{\sigma_q^A}{\mu_q^A} z + 1 \right) \right) \\
&= \frac{1}{\log q} \log \left(1 - (1-q)^2[n]_q[r]_q \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right) \right), \qquad (3.8) \\
q^{y^2/2} &= \exp \left(\frac{1}{2\log q} \log^2 \left(1 - (1-q)\mu_q^A \left(\frac{\sigma_q^A}{\mu_q^A} z + 1 \right) \right) \right) \\
&= \exp \left(\frac{1}{2\log q} \log^2 \left(1 - (1-q)^2[n]_q[r]_q \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right) \right) \right), \qquad (3.9)
\end{aligned}
$$

and

$$
\begin{aligned}
[y]_q^y &= (\mu_q^A)^y \exp \left(\frac{1}{\log q} \log \left(1 - (1-q)\mu_q^A \left(\frac{\sigma_q^A}{\mu_q^A} z + 1 \right) \right) \log \left(\frac{\sigma_q^A}{\mu_q^A} z + 1 \right) \right) \\
&= ((1-q)[n]_q[r]_q)^y \\
&\quad \cdot \exp \left(\frac{1}{\log q} \log \left(1 - (1-q)^2[n]_q[r]_q \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right) \right) \right. \\
&\qquad \left. \cdot \log \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right) \right). \qquad (3.10)
\end{aligned}
$$

Expanding the logarithms appearing in (3.10) in Taylor series and carrying out all the manipulations, we get

$$
\begin{aligned}
&\exp \left(\frac{1}{\log q} \log \left(1 - (1-q)^2[n]_q[r]_q \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right) \right) \right. \\
&\qquad \left. \cdot \log \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} z \right) \right) \\
&\quad = \exp \left(\frac{1-q}{\log q^{-1}} \frac{z^2}{2} \left(1 + \frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}} \right)^{1/2} + O(((1-q)[n]_q[r]_q)^{-1/2}) \right). \qquad (3.11)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\prod_{j=1}^{\infty}(1+q(q^{-y}-1)q^{j-1}) &= \prod_{j=1}^{\infty}(1+q(1-q)q^{-y}[y]_q q^{j-1}) \\
&= \prod_{j=1}^{\infty}\left(1+q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)q^{j-1}\right) \\
&= \exp\left(\sum_{j=1}^{\infty}\log\left(1+q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\right.\right. \\
&\qquad\qquad \left.\left.\cdot\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)q^{j-1}\right)\right), \qquad (3.12)
\end{aligned}
$$

with

$$
\begin{aligned}
&(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right) \\
&\quad = (1-q)^2[n]_q[r]_q\left(1-(1-q)^2[n]_q[r]_q\left(1+\frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}}z\right)\right)^{-1} \\
&\qquad\qquad \cdot\left(1+\frac{q^{(n+r-1)/2}}{((1-q)[n]_q[r]_q)^{1/2}}z\right), \qquad (3.13)
\end{aligned}
$$

which is derived by applying the Euler–Maclaurin summation formula (see Odlyzko [15], p. 1090) to the sum in (3.39) as follows:

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\log\left(1+q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)q^{j-1}\right) \\
&= \frac{1}{2\log q^{-1}}\log^2\left(1+q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\cdot\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right) \\
&+ \mathrm{Li}_2\left(\frac{q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)}{q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q}{\mu_q}z+1\right)+1}\right) \\
&+ \frac{1}{2}\log\left(1+q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right) \\
&+ \frac{\beta_2\log q}{2}\,\frac{q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)}{1+q(1-q)\mu_q^A\left(1-(1-q)\mu_q^A\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)\right)^{-1}\left(\frac{\sigma_q^A}{\mu_q^A}z+1\right)} \\
&+ O\left(\log q\right), \qquad (3.14)
\end{aligned}
$$

where Li_2 is the dilogarithm function and β_2 the second Bernoulli number.

Applying all the previous estimations (3.6)–(3.14) to the p.f. $f_{Y_n}(y)$ in (3.5), carrying out all the necessary manipulations and considering all the asymptotic expressions for $n \to \infty$, we derive our desired asymptotic result (3.4) and the proof is completed. □

Remark 3.2. Possible realizations of the sequence $q := q(n)$ considered in Theorem 3.1 above are

$$q(n) = 1 - \frac{\alpha}{n^\beta}, \quad \alpha \geq 1, \; \beta \geq 1$$

or

$$q(n) = 1 - \exp(-n).$$

An interesting application of the previous Theorem 3.1, concerning the asymptotic behaviour of an absorption process is presented in the following example.

Example 3.3. *A*symptotic Behaviour of the Absorption Process. Assume that batches of κ particles are sequentially propelled into a chamber of l consecutive lines cells, with capacity of each cell limited to one particle. Initially, a batch of κ particles occupies the κ leftmost cells. Then, a coin, with probability p of heads and $q = 1 - p$ of tails, is successively tossed. When tails occurs each of the k particles of the batch moves one cell to the right, while when heads occurs the batch of the κ particles is absorbed and the cells with the absorbed particles are removed. A batch of κ particles, which successfully reaches the κ rightmost cells, is said to have escaped and its particles are removed from the chamber without removing these cells. Subsequent batches of r particles are propelled into the chamber of the remaining cells. Then, the probability function of the number X_n of absorbed batches of κ particles, when n batches are propelled into the chamber of l cells, is given by (3.2), with q^κ instead of q (see Charalambides [4]). Therefore, the continuous limit of the probability function of X_n, for $q = q(n)$, with $q(n) \to 1$, as $n \to \infty$, $q(n)^{n\kappa} = \Omega(1)$ and $r = O(n)$, is the deformed standardized Gaussian distribution, given in (3.4), with $q = q^\kappa$.

3.2. Continuous limiting behaviour of the bivariate and multivariate absorption distributions

let $Y_{n,j}$ be the number of successes of a jth kind in a sequence of n independent Bernoulli trials with chain composite failures, where the conditional probability of success of the jth kind at any trial, given that $i-1$ successes of the $j$the kind occur in the previous trials, is given by

$$p_{j,i} = 1 - \theta_j q^{i-1}, j = 1, 2, \ldots, i = 1, 2, \ldots, 0 < \theta_j < 1, 0 < q < 1 \textit{ or } 1 < q < \infty,$$

where, for $0 < \theta_j < 1$ and $1 < q < \infty$, the number j of geometric sequences of trials is restricted by $\theta_j q^{i-1}$, ensuring that $0 < p_{j,i} < 1$.

The above success probability at the jth geometric sequence, in the case $1 < q < \infty$, by replacing the parameter q by q^{-1}, with $0 < q < 1$ and substituting $\theta_j = q^{m_j}$, becomes

$$p_{j,i} = 1 - q^{m_j - j + 1}, \;\; 0 < m_j < \infty, \;\; j = 1, 2, \ldots, k, \;\;\; i = 1, 2, \ldots, [m_j], \;\; 0 < q < 1, \tag{3.15}$$

which is a geometrically decreasing sequence of a finite number of terms. Then the joint probability function of the random vector $\mathcal{Y} = (Y_{n,1}, Y_{n,2}, \ldots, Y_{n,k})$ is given by

$$\begin{aligned} f_{\mathcal{Y}}(y_1, y_2, \ldots, y_k) &= P(Y_{n,1} = y_1, Y_{n,2} = y_2, \ldots, Y_{n,k} = y_k) \\ &= \binom{n}{y_1, y_2, \ldots, y_k}_q q^{\sum_{j=1}^k (n - s_j)(m_j - y_j)} \prod_{j=1}^k (1-q)^{y_j} [m_j]_{y_j, q}, \end{aligned} \tag{3.16}$$

for $y_j = 0, 1, 2, \ldots, n$, with $\sum_{j=1}^{k} y_j \le n$, $s_j = \sum_{i=1}^{j} y_i$, $\ 0 < m_j < \infty$, $\ 0 < q < 1$, and $n \le [m_j]$, $j = 1, 2, \ldots, k$. This discrete q-distribution is known as a *multivariate absorption distribution* (see Charalambides [17]).

Note that, initially Vamvakari [12], has derived the multivariate absorption distribution as a conditional distribution of a Heine process at a finite sequence of q-points in a time interval.

Next we will study the asymptotic behaviour of the *bivariate absorption* distribution.

Let the discrete bivariate random variable $(Y_{n,1}, Y_{n,2})$ with joint probability function

$$\begin{aligned} f_{Y_{n,1},Y_{n,2}}(y_1, y_2) = &\binom{n}{y_1, y_2}_q (1-q)^{y_1+y_2} q^{(m-y_1-y_2)(n-y_1-y_2)} q^{y_1(m_2-y_2)} \\ &\cdot [m_1]_{y_1,q} [m_2]_{y_2,q}, \end{aligned} \tag{3.17}$$

where $y_j = 1, 2, \ldots, n$, $j = 1, 2$, with $y_1 + y_2 \le n$, and $m = m_1 + m_2$, m_1, m_2 nonnegative integers. The distribution of the bivariate random variable $(Y_{n,1}, Y_{n,2})$ is known as a *bivariate absorption distribution.*

The marginal probability function of the random variable $Y_{n,2}$, is distributed according to the univariate absorption distribution with probability function

$$f_{Y_{n,2}}(y_2) = \binom{n}{y_2}_q (1-q)^{y_2} q^{(m_2-y_2)(n-y_2)} [m_2]_{y_2,q}, \quad y_2 = 0, 1, 2, \ldots, n, \tag{3.18}$$

for $0 < q < 1$ and $n \le m_2$.

The mean and the variance of the deformed variable $[Y_{n,2}]_q$ are given by

$$\begin{aligned} \mu_{[Y_{n,2}]_q} &= E\left([Y_{n,2}]_q\right) = (1-q)[n]_q [m_2]_q \\ &\text{and} \\ (\sigma_{[Y_{n,2}]_q})^2 &= V\left([Y_{n,2}]_q\right) \\ &= (1-q)^2 [n]_{2,q} [m_2]_{2,q} - (1-q)^2 [n]_q^2 [m_2]_q^2 + (1-q)[n]_q [m_2]_q \\ &= q^{n+m_2-1}(1-q)[n]_q [m_2]_q, \end{aligned} \tag{3.19}$$

respectively.

The conditional random variable $Y_{n,1}|Y_{n,2}$, is distributed according to the univariate absorption distribution with probability function

$$\begin{aligned} f_{Y_{n,1}|Y_{n,2}}(y_1|y_2) = &\binom{n-y_2}{y_1}_q (1-q)^{y_1} q^{(m_1-y_1)(n-y_1-y_2)} [m_1]_{y_1,q}, \\ &y_1 = 0, 1, \ldots, n-y_2, \end{aligned} \tag{3.20}$$

for $0 < q < 1$ and $n - y_2 \le m_1$.

The conditional mean and the conditional variance of the deformed variable $[Y_{n,1}]_q$ given $Y_{n,2} = y_2$ are given by

$$\begin{aligned} \mu_{[Y_{n,1}]_q|Y_{n,2}} &= E\left([Y_{n,1}]_q|y_2\right) = (1-q)[n-y_2]_q [m_1]_q \\ &\text{and} \\ (\sigma_{[Y_{n,1}]_q|Y_{n,2}})^2 &= V\left([Y_{n,1}]_q|y_2\right) \\ &= q(1-q)^2 [n-y_2]_{2,q} [m_1]_{2,q} - (1-q)^2 [n-y_2]_q^2 [m_1]_q^2 \\ &\quad +(1-q)[n-y_2]_q [m_1]_q \\ &= q^{n-y_2+m_1-1}(1-q)[n-y_2]_q [m_1]_q, \end{aligned} \tag{3.21}$$

respectively.

Note 3.4. The conditional q-mean, $\mu_{[X_2]_{1/q}|X_1}$, can be interpreted as a q-regression curve.

Next, we consider the deformed random variables $[Y_{n,2}]_q$ and $[Y_{n,1}]_q$ as well as the q-standardized random variables $Z = \frac{[Y_{n,2}]_q - \mu_{[Y_{n,2}]_q}}{\sigma_{[Y_{n,2}]_q}}$ and $W = \frac{[Y_{n,1}]_q - \mu_{[Y_{n,1}]_q|Y_{n,2}}}{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}$ with $\mu_{[Y_{n,2}]_q}$,$\sigma_{[Y_{n,2}]_q}$ and $\mu_{[Y_{n,1}]_q|Y_{n,2}}$, $\sigma_{[Y_{n,1}]_q|Y_{n,2}}$, given by (3.19) and (3.21), respectively. Then, we apply pointwise convergence techniques to the joint probability function (3.18), by using suitably the q Stirling type (2.2), and we obtain the following theorem concerning the asymptotic behaviour of the bivariate absorption distribution.

Theorem 3.5. *Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $m_i = O(n), i = 1, 2$. Then, for $n \to \infty$, the bivariate absorption distribution given by the p.f. in (3.17), is approximated by a deformed bivariate standardized Gaussian distribution as follows:*

$$
\begin{aligned}
f_{Y_{n,1},Y_{n,2}}(y_1, y_2) \cong\ & \frac{\log q^{-1}}{2\pi(1-q)\sigma_{[Y_{n,2}]_q}\sigma_{[Y_{n,1}]_q|Y_{n,2}}} q^{y_1+y_2} \\
& \cdot \exp\left(-\frac{1-q}{2\log q^{-1}}\left(\left(\frac{[y_2]_q - \mu_{[Y_{n,2}]_q}}{\sigma_{[Y_{n,2}]_q}}\right)^2 + \left(\frac{[y_1]_q - \mu_{[Y_{n,1}]_q|Y_{n,2}}}{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}\right)^2\right)\right), \\
& y_1, y_2 \geq 0,
\end{aligned}
\tag{3.22}
$$

where $\mu_{[Y_{n,2}]_q}$ and $\sigma_{[Y_{n,2}]_q}$, given in (3.19), are the mean value and the variance of the random variable $[Y_n]_q$ and $\mu_{[Y_{n,1}]_q|Y_{n,2}}$, $\sigma_{[Y_{n,1}]_q|Y_{n,2}}$, given in (3.21), are the conditional mean value and the conditional variance of the random variable $[Y_{n,1}]_q$ given $Y_{n,2} = y_2$.

Proof. Using the q-Stirling formula in (2.2), for $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$, and $m_i = O(n), i = 1, 2,$ the bivariate absorption distribution (3.18) is approximated by

$$
\begin{aligned}
f_{Y_{n,1},Y_{n,2}}(y_1, y_2) &= f_{Y_{n,2}}(y_2) f_{Y_{n,1}|Y_{n,2}}(y_1|y_2) \\
&= \binom{n}{y_2}_q (1-q)^{y_2} q^{(m_2-y_2)(n-y_2)} (m_2)_{y_2,q} \\
&\quad \cdot \binom{n-y_2}{y_1}_q (1-q)^{y_1} q^{(m_1-y_1)(n-y_1-y_2)} (m_1)_{y_1,q} \\
&\cong \frac{(q\log q^{-1})^{1/2}}{q^{-1/8}(2\pi(1-q))^{1/2}} q^{(n-y_2)(m_2-y_2)} q^{-\binom{y_2}{2}} q^{y_2/2} [y_2]_{1/q}^{-(y_2+1/2)} \\
&\quad \cdot [m_2]_{y_2,q} \prod_{j=1}^{\infty} (1 + q(q^{-y_2} - 1)q^{j-1}) \\
&\quad \cdot \frac{(q\log q^{-1})^{1/2}}{q^{-1/8}(2\pi(1-q))^{1/2}} q^{(n-y_2-y_1)(m_1-y_1)} q^{-\binom{y_1}{2}} q^{y_1/2} [y_1]_{1/q}^{-(y_1+1/2)} \\
&\quad \cdot [m_1]_{y_1,q} \prod_{j=1}^{\infty} (1 + q(q^{-y_1} - 1)q^{j-1})
\end{aligned}
$$

or

$$
\begin{aligned}
f_{Y_{n,1},Y_{n,2}}(y_1, y_2) \cong\ & \frac{q^{1/4}\log q^{-1}}{2\pi(1-q)} q^{(n-y_2)(m_2-y_2)} q^{(n-y_2-y_1)(m_1-y_1)} \\
& \cdot q^{{y_2}^2/2+y_2/2} q^{{y_1}^2/2+y_1/2} [y_1]_q^{-(y_1+1/2)} [y_2]_q^{-(y_2+1/2)} \\
& \cdot [m_2]_{y_1,q} [m_1]_{y_1,q} \\
& \cdot \prod_{j=1}^{\infty} (1 + q(q^{-y_2} - 1)q^{j-1}) \prod_{j=1}^{\infty} (1 + q(q^{-y_1} - 1)q^{j-1}).
\end{aligned}
\tag{3.23}
$$

Consider the random variables $[Y_{n,2}]_q$ and $[Y_{n,1}]_q$ as well as the q-standardized random variables $Z = \frac{[Y_{n,2}]_q - \mu_{[Y_{n,2}]_q}}{\sigma_{[Y_{n,2}]_q}}$ and $W = \frac{[Y_{n,1}]_q - \mu_{[Y_{n,1}]_q|Y_{n,2}}}{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}$ with $\mu_{[Y_{n,2}]_q}$,$\sigma_{[Y_{n,2}]_q}$ and $\mu_{[Y_{n,1}]_q|Y_{n,2}}$, $\sigma_{[Y_{n,1}]_q|Y_{n,2}}$, given by (3.19) and (3.21), respectively. Then the following relations are easily derived:

$$\begin{aligned}
[y_2]_q &= \mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\\
&= (1-q)[n]_q[m_2]_q\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right), \qquad (3.24)\\
[y_1]_q &= \mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\\
&= (1-q)[n-y_2]_q[m_1]_q\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right), \qquad (3.25)\\
q^{y_2} &= 1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\\
&= 1-(1-q)^2[n]_q[m_2]_q\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right), \qquad (3.26)\\
q^{y_1} &= 1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\\
&= 1-(1-q)^2[n-y_2]_q[m_1]_q\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right), \qquad (3.27)\\
y_2 &= \frac{1}{\log q}\log\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)\\
&= \frac{1}{\log q}\log\left(1-(1-q)^2[n]_q[m_2]_q\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right)\right), \qquad (3.28)\\
y_1 &= \frac{1}{\log q}\log\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)\\
&= \frac{1}{\log q}\log\left(1-(1-q)^2[n-y_2]_q[m_1]_q\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right)\right), \qquad (3.29)\\
q^{y_2^2/2} &= \exp\left(\frac{1}{2\log q}\log^2\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)\right)\\
&= \exp\left(\frac{1}{2\log q}\log^2\left(1-(1-q)^2[n]_q[m_2]_q\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right)\right)\right), \qquad (3.30)\\
q^{y_1^2/2} &= \exp\left(\frac{1}{2\log q}\log^2\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)\right)\\
&= \exp\left(\frac{1}{2\log q}\log^2\left(1-(1-q)^2[n-y_2]_q[m_1]_q\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right)\right)\right), \qquad (3.31)
\end{aligned}$$

and

$$\begin{aligned}
[y_2]_q^{y_2} &= (\mu_{[Y_{n,2}]_q})^{y_2}\\
&\quad\cdot\exp\left(\frac{1}{\log q}\log\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)\log\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)\\
&= ((1-q)[n]_q[m_2]_q)^{y_2}\\
&\quad\cdot\exp\left(\frac{1}{\log q}\log\left(1-(1-q)^2[n]_q[m_2]_q\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right)\right)\right.\\
&\qquad\left.\cdot\log\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right)\right), \qquad (3.32)
\end{aligned}$$

$$
\begin{aligned}
[y_1]_q^{y_1} &= (\mu_{[Y_{n,1}]_q|Y_{n,2}})^{y_1} \\
&\quad \cdot \exp\left(\frac{1}{\log q}\log\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)\cdot\log\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right) \\
&= ((1-q)[n-y_2]_q[m_1]_q)^{y_1} \\
&\quad \cdot \exp\left(\frac{1}{\log q}\log\left(1-(1-q)^2[n-y_2]_q[m_1]_q\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right)\right)\right. \\
&\qquad\qquad \left.\cdot\log\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right)\right). \qquad (3.33)
\end{aligned}
$$

Expanding the logarithms appearing in (3.32) and (3.33) in Taylor series and carrying out all the manipulations, we get respectively

$$
\begin{aligned}
&\exp\left(\frac{1}{\log q}\log\left(1-(1-q)^2[n]_q[m_2]_q\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right)\right)\log\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right)\right) \\
&\qquad = \exp\left(\frac{1-q}{\log q^{-1}}\frac{z^2}{2}\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}\right)^{1/2}+O(((1-q)[n]_q[m_2]_q)^{-1/2})\right) \qquad (3.34)
\end{aligned}
$$

and

$$
\begin{aligned}
&\exp\left(\frac{1}{\log q}\log\left(1-(1-q)^2[n-y_2]_q[m_1]_q\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right)\right)\right. \\
&\qquad\qquad \left.\cdot\log\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right)\right) \\
&\qquad = \exp\left(\frac{1-q}{\log q^{-1}}\frac{w^2}{2}\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}\right)^{1/2}\right. \\
&\qquad\qquad \left.+O(((1-q)[n-y_2]_q[m_1]_q)^{-1/2})\right). \qquad (3.35)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\prod_{j=1}^{\infty}(1+q(q^{-y_2}-1)q^{j-1}) = \prod_{j=1}^{\infty}(1+q(1-q)q^{-y_2}[y_2]_q q^{j-1}) \\
&\quad = \exp\left(\sum_{j=1}^{\infty}\log\left(1+q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\right.\right. \\
&\qquad\qquad \left.\left.\cdot\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)q^{j-1}\right)\right), \qquad (3.36)
\end{aligned}
$$

with

$$
\begin{aligned}
&(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right) \\
&\quad = (1-q)^2[n]_q[m_2]_q\left(1-(1-q)^2[n]_q[m_2]_q\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right)\right)^{-1} \\
&\qquad\qquad \cdot\left(1+\frac{q^{(n+m_2-1)/2}}{((1-q)[n]_q[m_2]_q)^{1/2}}z\right), \qquad (3.37)
\end{aligned}
$$

which is derived by applying the Euler–Maclaurin summation formula (see Odlyzko [15], p. 1090) to the sum in (3.36) as follows:

$$\begin{aligned}
&\sum_{j=1}^{\infty}\log\left(1+q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)q^{j-1}\right)\\
&\quad=\frac{1}{2\log q^{-1}}\log^2\left(1+q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\right.\\
&\qquad\qquad\left.\cdot\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)\\
&\quad+\mathrm{Li}_2\left(\frac{q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)}{q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)+1}\right)\\
&\quad+\frac{1}{2}\log\left(1+q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)\\
&\quad+\frac{\beta_2\log q}{2}\,\frac{q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)}{q(1-q)\mu_{[Y_{n,2}]_q}\left(1-(1-q)\mu_{[Y_{n,2}]_q}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,2}]_q}}{\mu_{[Y_{n,2}]_q}}z+1\right)+1}\\
&\quad+O\left(\log q\right),
\end{aligned}\tag{3.38}$$

where Li_2 is the dilogarithm function and β_2 the second Bernoulli number.

Furthermore, we have

$$\begin{aligned}
&\prod_{j=1}^{\infty}(1+q(q^{-y_1}-1)q^{j-1})=\prod_{j=1}^{\infty}(1+q(1-q)q^{-y_1}[y_1]_q q^{j-1})\\
&\quad=\prod_{j=1}^{\infty}\left(1+q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\right.\\
&\qquad\qquad\left.\cdot\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)q^{j-1}\right)\\
&\quad=\exp\left(\sum_{j=1}^{\infty}\log\left(1+q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\cdot\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\right.\right.\\
&\qquad\qquad\left.\left.\cdot\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)q^{j-1}\right)\right),
\end{aligned}\tag{3.39}$$

with

$$\begin{aligned}
&(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\\
&\quad=(1-q)^2[n-y_2]_q[m_1]_q\left(1-(1-q)^2[n-y_2]_q[m_1]_q\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right)\right)^{-1}\\
&\qquad\qquad\cdot\left(1+\frac{q^{(n-y_2+m_1-1)/2}}{((1-q)[n-y_2]_q[m_1]_q)^{1/2}}w\right),
\end{aligned}\tag{3.40}$$

which is derived by applying the Euler–Maclaurin summation formula (see Odlyzko [15], p. 1090) to the sum in (3.39) as follows:

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\log\left(1+q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\right.\\
&\qquad\qquad\qquad\qquad\left.\cdot\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)q^{j-1}\right)\\
&=\frac{1}{2\log q^{-1}}\log^2\left(1+q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\right.\\
&\qquad\qquad\qquad\qquad\left.\cdot\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)\\
&+\mathrm{Li}_2\left(\frac{q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)}{q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)+1}\right)\\
&+\frac{1}{2}\log\left(1+q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)\\
&+\frac{\beta_2\log q}{2}\,\frac{q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)}{1+q(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(1-(1-q)\mu_{[Y_{n,1}]_q|Y_{n,2}}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)\right)^{-1}\left(\frac{\sigma_{[Y_{n,1}]_q|Y_{n,2}}}{\mu_{[Y_{n,1}]_q|Y_{n,2}}}w+1\right)}\\
&+O\left(\log q\right).
\end{aligned}
\tag{3.41}
$$

Applying all the previous estimations (3.24)–(3.41) to the p.f. $f_{Y_{n,1},Y_{n,2}}(y_1,y_2)$ in (3.23), carrying out all the necessary manipulations and considering all the asymptotic expressions for $n\to\infty$, we derive our desired asymptotic result (3.22) and the proof is completed. □

Next we expand our study on the asymptotic behaviour of the multivariate absorption distribution with joint p.f. (3.16).

The marginal probability function of the random variable $Y_{n,k}$, is distributed according to the univariate absorption distribution with probability function

$$
f_{Y_{n,k}}(y_k)=\binom{n}{y_k}_q(1-q)^{y_k}q^{(m_k-y_k)(n-y_k)}[m_k]_{y_k,q},\quad y_k=0,1,2,\ldots,n,
\tag{3.42}
$$

for $0<q<1$ and $n\le m_k$.

The mean and the variance of the deformed variable $[Y_{n,k}]_q$ are given by

$$
\begin{aligned}
\mu_{[Y_{n,k}]_q}&=E\left([Y_{n,k}]_q\right)=(1-q)[n]_q[m_k]_q\\
&\quad\text{and}\\
(\sigma_{[Y_{n,k}]_q})^2&=V\left([Y_{n,k}]_q\right)\\
&=q(1-q)^2[n]_{k,q}[m_k]_{2,q}-(1-q)^2[n]_q^2[m_k]_q^2+(1-q)[n]_q[m_k]_q,
\end{aligned}
\tag{3.43}
$$

respectively.

The conditional random variable $Y_{n,k-1}|Y_{n,k}$, is distributed according to the univariate absorption distribution with probability function

$$\begin{aligned} f_{Y_{n,k-1}|Y_{n,k}}(y_{k-1}|y_k) &= \binom{n-y_k}{y_{k-1}}_q (1-q)^{y_{k-1}} q^{(m_{k-1}-y_{k-1})(n-y_{k-1}-y_k)}[m_{k-1}]_{y_{k-1},q}, \\ & y_{k-1}=0,1,2,\ldots,n-y_k, \end{aligned} \tag{3.44}$$

for $0<q<1$ and $n-y_k \le m_{k-1}$, $k\ge 2$.

The conditional mean and the conditional variance of the deformed variable $[Y_{n,k-1}]_q$ given $Y_{n,k}=y_k$ are given by

$$\begin{aligned} \mu_{[Y_{n,k-1}]_q|Y_{n,k}} &= E\left([Y_{n,k-1}]_q|y_k\right) = (1-q)[n-y_k]_q[m_{k-1}]_q \\ & \text{and} \\ (\sigma_{[Y_{n,k-1}]_q|Y_{n,k}})^2 &= V\left([Y_{n,k-1}]_q|y_k\right) \\ &= q(1-q)^2[n-y_k]_{2,q}[m_{k-1}]_{2,q} - (1-q)^2[n-y_k]_q^2[m_{k-1}]_q^2 \\ &\quad +(1-q)[n-y_k]_q[m_{k-1}]_q, \end{aligned} \tag{3.45}$$

respectively.

The conditional random variables $Y_{n,j}|\left(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k}\right)$, $1\le j\le k-1$, $k\ge 2$ are distributed according to univariate absorption distributions with probability functions

$$\begin{aligned} & f_{Y_{n,j}|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})}(y_j|y_{j+1},y_{j+2},\ldots,y_k) \\ &= \binom{n-\sum_{i=j+1}^k y_i}{y_j}_q (1-q)^{y_j} q^{(m_j-y_j)(n-\sum_{i=j}^k y_i)}[m_j]_{y_j,q}, \\ & y_j = 0,1,2,\ldots,n-\sum_{i=j+1}^k y_i,\ j=1,\ldots,k-1,\ k\ge 2. \end{aligned}$$

The conditional mean and conditional variance of the deformed variables $[Y_{n,j}]_q$ given $Y_{n,j+1}=y_{j+1},\ldots,y_{n,k}=y_k$, $j=1,\ldots,k-1$, $k\ge 2$, are given respectively by

$$\begin{aligned} \mu_{[Y_{n,j}]_q|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})} &= E\left([Y_{n,j}]_q|\left(y_{j+1},y_{j+2},\ldots,Y_k\right)\right) \\ &= (1-q)\left[n-\sum_{i=j+1}^k y_i\right]_q [m_j]_q \\ \text{and} & \\ \sigma^2_{[Y_{n,j}]_q|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})} &= V\left([Y_{n,j}]_q|\left(y_{j+1},y_{j+2},\ldots,Y_k\right)\right) \\ &= q(1-q)^2\left[n-\sum_{i=j+1}^k y_i\right]_{2,q}[m_j]_{2,q} - (1-q)^2\left[n-\sum_{i=j+1}^k y_i\right]_q^2[m_j]_q^2 \\ &\quad +(1-q)\left[n-\sum_{i=j+1}^k y_i\right]_q[m_j]_q. \end{aligned} \tag{3.46}$$

Note 3.6. The conditional q-means, $\mu_{[Y_{n,j}]_q|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})}$, $1 \le j \le k-2$, $k \ge 3$, can be interpreted as q-regression hyperplanes.

Consider the deformed random variables $[Y_{n,j}]_q$, $j = 1, 2, \ldots, k$, $k \ge 1$, and the q-standardized random variables

$$Z_k = \frac{[Y_{n,k}]_q - \mu_{[Y_{n,k}]_q}}{\sigma_{[Y_{n,k}]_q}}, \;\; Z_j = \frac{[Y_{n,j}]_q - \mu_{[Y_{n,j}]_q|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})}}{\sigma_{[Y_{n,j}]_q|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})}},$$

$j = 1, \ldots, k-1$, $k \ge 2$, with $\mu_{[Y_{n,k}]_q}$,$\sigma_{[Y_{n,k}]_q}$ and $\mu_{[Y_{n,j}]_q|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})}$, $\sigma_{[Y_{n,j}]_q|(Y_{n,j+1},Y_{n,j+2},\ldots,Y_{n,k})}$ given by (3.43) and (3.46), respectively. Then, by applying pointwise convergence techniques to the joint probability function (3.16) similar to the previous Theorem 3.5, we derive analogously the following theorem concerning the asymptotic behaviour of the multivariate absorption distribution.

Theorem 3.7. *Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $m_j = O(n), j = 1, 2, \ldots, k$. Then, for $n \to \infty$, the multivariate absorption distribution given by the p.f. in (3.16), is approximated by a deformed multivariate standardized Gaussian distribution as follows:*

$$f_{\mathcal{Y}}(y_1, y_2, \ldots, y_k) \cong \left(\frac{\log q^{-1}}{2\pi(q^{-1}-1)}\right)^{k/2} \frac{q^{\sum_{i=1}^{k} y_i}}{\sigma_{[Y_{n,k}]_q} \prod_{j=1}^{k-1} \sigma_{[Y_{n,j}]_q|\left(Y_{n,j+1},\ldots,Y_{n,k}\right)}}$$

$$\cdot \exp\left(-\frac{1-q}{2\log q^{-1}}\left(\left(\frac{[y_k]_q - \mu_{[Y_{n,k}]_q}}{\sigma_{[Y_{n,k}]_q}}\right)^2 + \sum_{j=1}^{k-1}\left(\frac{[y_j]_q - \mu_{[Y_{n,j}]_q|\left(Y_{n,j+1},\ldots,Y_{n,k}\right)}}{\sigma_{[Y_{n,j}]_q|\left(Y_{n,j+1},\ldots,Y_{n,k}\right)}}\right)^2\right)\right),$$
$$y_j \ge 0, j = 1, 2, \ldots, k, k \ge 2 \tag{3.47}$$

where $\mu_{[Y_{n,k}]_q}$ and $\sigma_{[Y_{n,k}]_q}$ given in (3.43), are the mean value and the variance of the random variable $[Y_{n,k}]_q$, while
$\mu_{[Y_{n,j}]_q|(Y_{n,j+1},\ldots,Y_{n,k})}$ and $\sigma_{[Y_{n,j}]_q|(Y_{n,j+1},\ldots,Y_{n,k})}$ given in (3.46), are the conditional mean values and the conditional variances of the random variables $[Y_{n,j}]_q$ given $Y_{n,j+1} = y_{j+1}, \ldots, Y_{n,k} = y_k$, $j = 1, \ldots, k-1$, $k \ge 2$.

An interesting application of the previous Theorem 3.7, regarding the asymptotic behaviour of a multivariate extension of the absorption process is presented in the following example.

Example 3.8. *A*symptotic Behaviour of a Multivariate Absorption Process. Consider a set of k consecutive chambers, c_j, $j = 1, 2, \ldots, k$, containing l_j, $j = 1, 2, \ldots, k$ consecutive lined cells, with the capacity of each cell limited to one particle. Assume that batches of s particles are sequentially propelled into the chambers starting from the first chamber as follows. At some stage, a batch of s particles are sequentially propelled at the s leftmost cells of the chamber c_j with l_j lined cells. Then, a coin, with probability p of heads and $q = 1 - p$ of tails, is succesively tossed. When tails occurs each of the s particles of the batch moves one cell to the right, fail of jth kind , f_j, while when heads occurs the batch of s particles is absorbed and the cells with the absorbed particles are removed, success of jth kind, s_j. A batch of s particles, which successfully reaches the s rightmost cells, is said to have escaped and its particles are removed from the chamber c_j without removing the remaning cells. Subsequent batches of s particles are propelled into into the chamber c_{j+1}, with l_{j+1} lined cells with possible results $\{s_{j+1}, f_{j+1}\}$ for $j = 1, 2, \ldots, k-1$. Also, a batch of s particles which successfully reaches the s rightmost cells of the last chamber c_k is escaped and its particles are subsequently propelled into the first chamber c_1 of the remaining cells. Then, the conditional probability of an absorption of jth kind of a batch of m particles, given that $i-1$ absorptions of the jth kind occur, is given by

$$p_{j,i} = 1 - q^{l_j - is+1}, \, 0 < q < 1, \, i = 1, 2, \ldots, l_j, \, j = 1, 2, \ldots, k.$$

Substituting $l_j = (m_j + 1)s - 1,\ m_j > 0$, it follows that

$$p_{j,i} = 1 - q^{s(m_j - i + 1)},\ i = 1, 2, \ldots, [m_j],\ j = 1, 2, \ldots, k$$

Therefore, the joint probability function of the numbers $Y_{n,j}$ of absorbed batches of m particles of the jth kind, when n batches are propelled into the chamber c_j of l_j cells, $j = 1, 2, \ldots, k$, is given (3.16), with q^s instead of q. This process defines a *multivariate absorption stochastic process*.

According to Theorem 3.7 , the continuous limit of the joint probability function of the random variables $X_{n,j}, j = 1, 2, \ldots, k$, for $q = q(n)$, with $q(n) \to 1$, as $n \to \infty$, $q(n)^{ns} = \Omega(1)$ and $m_j = O(n)$, $j = 1, 2, \ldots, k$, is the deformed multivariate standardized Gaussian distribution, given in (3.47).

Note that for $k = 1$ the above process becomes the univariate absorption process presented in Example 1.

4. Concluding remarks

The aim of this work was to initiate the study of continuous limiting behaviour of multivariate discrete q-distributions, inspired by the limiting behaviour of the univariate ones. Specifically, we studied the asymptotic behavior of the univariate, bivariate and multivariate absorption discrete q-distributions. The pointwise convergence of the univariate absorption distribution to a deformed Gaussian distribution was established. Also, the pointwise convergence of the bivariate and multivariate absorption distributions to bivariate and multivariate deformed Gaussian ones respectively, were established. Moreover, interesting applications of the asymptotic behaviour of univariate and multivariate absorption processes were presented.

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