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ON A TEST OF SQUARE-FREE MORPHISMS

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Abstract. A square-free word is one which does not contain two consecutive occurrences of the same factor; a square-free morphism is one which produces a square-free word whenever applied to a square-free word. For testing square-freeness of a morphism, the previous researches (Berstel, Ehrenfeucht-Rozenberg, Crochemore, Hsiao *et al.*, ...) dealt with the compactness claim: they proved that there is a bound (sometimes tight, depending on the morphism) such that a morphism is square-free if it is so on the words on the source alphabet of length up to this bound. In particular, when the morphism is ternary (the source alphabet of three letters) this bound is universally 5 and 5 is sharp on the target alphabet of 5 letters. In this paper we undertake a different approach: we do not search for any compactness bound or verify square-freeness excerpt for a few prerequisites; instead we define the *relator* on the source alphabet, the existence of which is relatively easy to verify by matching words for a common factor. As applications, we easily deduce all the previous bounds and we manifest the simplicity of performance by giving a short proof of a Lallement's result. More essentially, using it we show the optimum bound 5 for the ternary morphism on the alphabet of 4 letters and, by a more elaborate construction, on the alphabet of 3 letters. That gives the current problem the finishing touches.

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1. Introduction

We study square-free words, that is words containing no two identical adjacent occurrences of a factor. Although a binary word of length at least 4 contains a square, that there are infinitely many ternary square-free words is far from evident. It is easy to find many short square-free words. To find longer words we have to expand these shorter words in same ways. The first and foremost is the trial-and-error search. This kind of search can be used to show that the square-free words subject to certain restrictions are finite in length or in number. Another method is to use morphisms, repeatedly applied that is what Axel Thue did the seminal work of 1906 and since then various morphisms have been applied widely to construct square-free words. Actually, Thue's morphism, defined on a binary alphabet, and Thue's infinite word are not square-free, but overlap-free. It gives by an appropriate substitution a ternary square-free word. This fact is rediscovered several times later. In 1938, Morse and Hedlund did it by an identical method, due to which are the appellations Thue-Morse morphism, "Morse's trajectory," and Thue-Morse word. However, according to Raoul Bott, Thue's word was already mentioned in Morse's thesis of 1917 and was subsequently published in the paper on recurrent geodesics of 1921, where he

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proved that it is recurrent and nonperiodic. Moreover, another infinite ternary square-free word, dating from 1937, had recurred in Novikov's disproof of Burnside conjecture in 1959 and was there ascribed to a Russian researcher named Arshon [2]. S. Adian reproduced the Arshon's construction in [1]. But a striking tale did not commence simply like that; we know that Thue's sequence on two symbols was already described by Prouhet in 1851 (see [10], [4]) in connection with a problem of arithmetics known as Prouhet-Tarry-Escott problem.

Now we fix the terminology. We deal with a finite alphabet A of letters, the set A^* of words on A, the empty word ϵ, the set A^+ of nonempty words. For the words $u, v \in A^*$, uv denotes the concatenation (product) of them and for a word $w = a_1 a_2 \ldots a_n$ with the letters $a_1, a_2, \ldots, a_n$, n is the length of w, denoted $|w|$, and $a_n \ldots a_2 a_1$ is the reversal of w. By convention $|\epsilon| = 0$, the length of the empty word is null and the product of the empty sequence of words, the empty product, equals the empty word.

The word u is a factor of a word v if $v = xuy$ which says that u has an *occurrence* in the position $|x|$, or counting from the end of v, in the position $|x| - |v|$, and u ends in the position $|xu|$, or $|xu| - |v|$, of v. The word u is a prefix of v if it has a *prefix* occurrence in v, that is when $|x| = 0$, and a suffix if it has a *suffix* occurrence in v, that is, when $|y| = 0$. The factor u is a proper factor of v if $u \neq v$, or, $xy \neq \epsilon$. When v is a factor of w, transitively, u is a factor of w; each occurrence of u in v and each occurrence of v in w induce in a natural way an occurrence of u in w.

A *square* is a word of the form uu, where u is a nonempty word. A word is *square-free* if it does not contain any occurrence of a square; a morphism δ from the monoid Δ^* to A^* is *square-free* if $\delta(x)$ is square-free whenever x is square-free.

Now we recall the key notion of an overlap. Let u and v be two distinct nonempty words; u is said to *overlap* v if u has a nonempty suffix that is a prefix of v, that is,

$$u = rw, v = ws$$

for $r, s \in A^*, w \in A^+$ and w is an *overlap* of u and v. In general, we say that two words overlap if one of them overlaps the other. A word may overlaps itself in a proper way: u is said to be *bordered* if it has a proper nonempty suffix which is also a prefix of it,

$$u = rp = ps,$$

where $r, p, s \in A^+$ and p is a *border* of u. In the opposite case, u is *unbordered.* In a specific situation, an occurrence of u overlaps an occurrence of v in w if the first one has a suffix occurrence that coincides with a prefix occurrence of the latter, as occurrences in w; in other words, w contains an occurrence of rps.

When the alphabet has more than two letters we may directly iterate a morphism to generate infinitely many square-free words. To yield square-free words the morphism is assumed to be square-free; to recognize a square-free morphism we have to develop certain necessary and sufficient conditions for that. We may mention first a test, due to Berstel and Crochemore, the crux of which is to examine the images of the morphism for the square-free source words up to some bound for square-freeness.

In the next section, we describe another test, more manageable and explicit, easier to verify. In the last section, we demonstrate the performance by giving a short proof of the square-freeness of the Lallement's morphism and we construct examples to show the sharpness of the bound 5 for ternary morphisms on four and on three letters.

2. Tests for square-free morphisms

We first state a characterization developed by Berstel [3] and made more precise by Crochemore [5]; see also [6]. Let Δ be another alphabet and h a morphism from Δ^* to A^* and denote by M the maximum and by m the minimum length of the words of $h(\Delta)$.

Proposition 2.1. *There exists a constant K such that h is square-free if and only if $h(w)$ is square-free for every square-free word $w \in \Delta^*$ of length at most K. If Δ has 3 letters, K is universally 5 and 5 is sharp.*

The bound K usually depends on h, Δ and A. To know if h is square-free, we take a square-free word w, $|w| \leq K$, compute $h(w)$ one by one and search for a square in it, if we find no one, h is square-free. We list the available values for K.

Berstel [3] first established $K = 2\lfloor \frac{M}{m} \rfloor + 2$;

Ehrenfeucht and Rozenberg [7] $K = \lfloor \frac{M}{m} \rfloor + 2$;

Crochemore [5] improved it to $K = \max\{3, \lceil \frac{M-3}{m} \rceil\}$ and mentioned that this bound is sharp and when $|\Delta| = 3$, $K = 5$ and 5 is sharp on A of 5 letters. However, in [6] Crochemore stated $K = \max\{3, \lceil \frac{M-3}{m} \rceil + 1\}$ and referred to [5] for sharpness;

Finally, Hsiao *et al.* [8] proved $K = \max\{|\Delta|, T+2\}$, where as a function of h,

$$T = \max\{t : h(\Delta) \cap A^* h(\Delta^t) A^* \neq \emptyset\}.$$

We shall derive all these estimates, as easy consequences, with sharpness, at the end of this section.

In this paper, our intention is different. To decide the square-freeness of h, we do not examine $h(w)$ if it contains a square. Instead, we try to clarify the tangible structure of those words that turn to non-square-free word under the morphism, as detailed as possible and use it to prove their nonexistence. We start with an almost evident statement, but the argument used in the proof is typical for this paper: matching words for a common factor. For the words u, v, denote by $\operatorname{lcp}(u, v)$ their longest common prefix and by $\operatorname{lcs}(u, v)$ their longest common suffix.

Proposition 2.2. *Suppose that x and y are square-free words. Then xy is not square-free if and only if there is a factorization $x = uv$ with $|v| > 0$ for which*

$$|\operatorname{lcp}(v, y)| + |\operatorname{lcs}(u, v)| \geq |v| \qquad (*)$$

or there is a factorization $y = uv$ with $|u| > 0$ for which

$$|\operatorname{lcp}(u, v)| + |\operatorname{lcs}(x, u)| \geq |u|. \qquad (**)$$

In these cases, xy contains a square of length $2|v|$ and $2|u|$, respectively.

Proof. It is enough to prove any one of the two possibilities, because we consider the reversal of xy for the other. Suppose, for instance, that $(*)$ holds, that is

$$|\operatorname{lcp}(v, y)| + |\operatorname{lcs}(u, v)| \geq |v|.$$

Then, v has a prefix v' common to y and a suffix v'' common to u such that

$$|v'| + |v''| = |v|,$$

hence

$$v = v'v''.$$

We have also

$$y = v'y''$$

for $y'' \in A^*$, and

$$u = u'v''$$

for $u' \in A^*$. Thus,

$$xy = (uv)(v'y'') = (u'v''v'v'')(v'y'') = u'(v''v'v''v')y''$$

which shows that xy contains the square $(v''v')^2$ with $|v''v'| = |v| > 0$.

Conversely, suppose that xy contains a square, rr. Since both x and y are square-free, rr does not occur inside x or y entirely, therefore

(a) the first occurrence of r occurs in x while the second one does not; or

(b) the second occurrence of r occurs in y but the first one does not.

Suppose that we have (b), the other case is handled by reversal. This means that

$$xy = x'rry''$$

for $x', y'' \in A^*$, where

$$|rry''| > |y| \geq |ry''|.$$

These inequalities imply that

$$y = v'ry''$$

for some $v' \in A^*$, and

$$rry'' = u''y$$

for some $u'' \in A^+$. It follows, on one hand, that

$$rry'' = u''y = u''(v'ry'')$$

and

$$r = u''v'.$$

On the other hand

$$xy = x'rry'' = x'(rry'') = x'(u''y)$$

hence

$$x = x'u''.$$

Now

$$y = v'ry'' = v'(u''v')y'' = (v'u'')(v'y'')$$

and putting $u = v'u'', v = v'y''$ we see that

$$y = uv$$

with v' a common prefix of u and v, u'' a common suffix of x and u and $|u| = |v'| + |u''| \geq |u''| > 0$ which shows that (**) holds and the proof is done.

It is to be observed that if xy is square-free, x does not overlap y, in particular, y is not a suffix of x and x is not a prefix of y.

Given a subset S of A^*, we usually assign a name to each element of S as follows. Let Δ be an alphabet having the same cardinality as S. A *coding map* δ for S is a bijection from Δ to S, which is extended in the usual manner to a *coding morphism* for S, also denoted δ, from Δ^* to A^*. We always use an arbitrary but fixed coding morphism δ for S.

We present now the background notion of a relator over a set of words.

Definition 2.3. For the set $S = \delta(\Delta)$ and the coding map δ
(i) The letter $x \in \Delta$, is a 1-*relator* if $\delta(x)$ is not square-free;
(ii) Let $k > 1$ be an integer. The square-free word

$$x_1x_2 \ldots x_{k-1}x_k,$$

on Δ, $x_1, \ldots, x_k \in \Delta$, of length k, is said to be a *k-relator* if either

$$\delta(x_k) = u's\delta(x_2 \ldots x_{k-1})u'u''$$

where s is a nonempty suffix of $\delta(x_1)$, $u', u'' \in A^*$, or

$$\delta(x_1) = u'u''\delta(x_2) \ldots \delta(x_{k-1})pu''$$

where p is a nonempty prefix of $\delta(x_k)$ and $u', u'' \in A^*$.
(iii) The square-free word

$$xdydz,$$

$x, y, z \in \Delta$, $x \neq y, y \neq z$, $d \in \Delta^*$, is said to be a *ps-relator* if

$$\delta(x) = p's, \delta(y) = ps, \delta(z) = ps'$$

$p, s \in A^+$, $p', s' \in A^*$; that is, p is a nonempty common prefix of $\delta(y)$ and $\delta(z)$ and s is a nonempty common suffix of $\delta(x)$ and $\delta(y)$.
(iv) The square-free word

$$xdyex,$$

$x, y \in \Delta, d, e \in \Delta^*$, is said to be an *o-relator* if $\delta(x)$ is a bordered word and

$$\delta(x) = w\delta(d)f'\bar{f}f''\delta(e)w$$

for a border w and

$$\delta(y) = f'f'', \bar{f} \in A^*.$$

(v) The square-free word

$$xdyez,$$

$x, y, z \in \Delta$, $d, e \in \Delta^*$, is said to be an *i-relator* if $\delta(y)$ is a bordered word and

$$\delta(y) = w\delta(d)h'g''\delta(e)w,$$

for a border w, where h' is a nonempty prefix of $\delta(z)$, g'' is a nonempty suffix of $\delta(x)$.

We inclusively call the k-, ps-, i-, and o-relator *relator*. By definition, $\delta(w)$ contains a square whenever w is (or contains) a relator. Note that, by definition, a relator r is square-free but $\delta(r)$ is non-square-free.

Remark 2.4. (i) When $k = 2$, $\delta(x_2)\ldots\delta(x_{k-1})$ vanishes, Proposition 2 says that x_1x_2, $x_1 \neq x_2$, is a 2-relator if and only if $\delta(x_1)\delta(x_2)$ is not square-free, provided x_1 and x_2 are not 1-relators.

(ii) xdydz is a ps-relator only if xyz is a ps-relator. Consequently, S has a ps-relator if and only if it has a ps-relator of length 3.

(iii) Suppose that $xdyex$ is an o-relator and d' is the first letter of d. Directly by definition $xd'x$ is a 3-relator: for w is a nonempty suffix of $\delta(x)$ and $|\delta(x)| > |\delta(d')|$, $d' \neq x$, $xd'x$ is square-free; similarly for the last letter of e. If $d = e = \epsilon$ then xyx itself is a 3-relator. Analogously for the i-relator.

(iv) If a word contains two distinct occurrences of the same factor f that overlap, that is, uvw, where $f = uv = vw$, $u, v, w \in A^+$, then it contains a square, namely, uu (and ww), of length shorter than $2|f|$.

Now we state our main results which means merely that the relator is inevitable for non-square-free morphism. Let S a subset of A^*, Δ a coding alphabet and $\delta : \Delta \to S$ an arbitrary but fixed coding map for S and we preliminarily assume that S has no 1-relator or 2-relator. We clarify, for an arbitrary w, when $\delta(w)$ contains a square.

Proposition 2.5. *Let $w \in \Delta^*$ be a square-free word and S have no 1- or 2-relator. Then $\delta(w)$ is square-free if and only if w does not contains any relator .*

Proof. We can dismiss the trivial cases $|w| = 1$ and $|w| = 2$ when the proposition holds by definition and by Remark 4 (i). Let now $|w| = k > 2$.

Suppose that $\delta(w)$ is square-free. Then, naturally, w does not contain those relators announced in the proposition, as their image contains a square. So, one direction is trivial.

Conversely, suppose that for

$$w = x_1 \ldots x_k \in \Delta^+,$$

$x_1, \ldots, x_k \in \Delta$, $\delta(w)$ contains a square rr and that rr is of the shortest length in $\delta(w)$. Since every word that contains rr contains a minimal factor (in factor order) that does so, we may write

$$rr = g''\delta(x_2)\ldots\delta(x_{k-1})h'$$

for a nonempty suffix g'' of $\delta(x_1)$,

$$\delta(x_1) = g'g'',$$

$g' \in A^*, g'' \in A^*$, and a nonempty prefix h' of $\delta(x_k)$,

$$\delta(x_k) = h'h'',$$

$h' \in A^+, h'' \in A^*$, otherwise we can shrink w to this effect. We shall prove that w is then a relator of one of the kind listed in the proposition.

If $|g''| \geq |r|$ then

$$g'' = ru''$$

for some $u'' \in A^*$, so

$$r = u''\delta(x_2)\dots\delta(x_{k-1})h',$$

and

$$\delta(x_1) = g'g'' = g'ru'' = g'u''\delta(x_2)\dots\delta(x_{k-1})h'u''$$

which means that

$$x_1 \dots x_k$$

is a k-relator for S, as h' is a prefix of $\delta(x_k)$ and w is square-free by assumption. Symmetrically, if $|h'| \geq |r|$, the same is true and we are done.

Thus, we can now assume that $|g''| < |r|$ and $|h'| < |r|$. Let m be the least index for which

$$|g''\delta(x_2)\dots\delta(x_{m-1})\delta(x_m)| \geq |r|,$$

or equivalently, $m-1$ the greatest index for which

$$|g''\delta(x_2)\dots\delta(x_{m-1})| < |r|.$$

Therefore,

$$r = g''\delta(x_2)\dots\delta(x_{m-1})f' \tag{2.1}$$

and

$$rf'' = g''\delta(x_2)\dots\delta(x_{m-1})\delta(x_m),$$

or equivalently,

$$r = f''\delta(x_{m+1})\dots\delta(x_{k-1})h' \tag{2.2}$$

$$\delta(x_m) = f'f''$$

for $f' \in A^+, f'' \in A^*$. Since $|g''\delta(x_2)\dots\delta(x_{k-1})| = |rr| - |h'| > |r|$, we obtain

$$|f''\delta(x_{m+1})\dots\delta(x_{k-1})| = |r| - |h'| > 0,$$

hence $1 < m \leq k-1$ and, always,

$$|\delta(x_2)\dots\delta(x_{m-1})f'| > 0.$$

We distinguish the following possibilities:

(i) $|f'| \leq |h'| \leq |\delta(x_2)\dots\delta(x_{m-1})f'|$. Then, by (2.1) and (2.2), f' is a suffix of h', which is a suffix of

$$r = g''\delta(x_2)\dots\delta(x_{m-1})f'.$$

We may write

$$h' = uf'$$

for some $u \in A^*$, so u is a suffix of $g''\delta(x_2)\dots\delta(x_{m-1})$.

If $u = \epsilon$ then

$$h' = f',$$

that is, f' is a common prefix of $\delta(x_m)$ and $\delta(x_k)$ and, moreover, in view of the expression for r in (2.1) and (2.2),

$$g''\delta(x_2)\dots\delta(x_{m-1}) = f''\delta(x_{m+1})\dots\delta(x_{k-1}).$$

Since two words of S, one of which is the suffix of the other, are equal, it follows

$$\delta(x_{k-1}) = \delta(x_{m-1}), \dots, \delta(x_{m+1}) = \delta(x_2)$$

and

$$g'' = f''.$$

Consequently,

$$x_2 \dots x_{m-1} = x_{m+1} \dots x_{k-1}$$

and, evidently, f'' is a common suffix of $\delta(x_1)$ and $\delta(x_m)$. All together, these facts show that

$$x_1(x_2 \dots x_{m-1})x_m(x_{m+1} \dots x_{k-1})x_k$$

is a ps-relator.

(ii) $|h'| < |f'|$. Then h' is a proper suffix of f',

$$f' = uh'$$

for $u \in A^+$.

If $|f''| \leq |g''\delta(x_2)\dots\delta(x_{m-1})|$, f'' is a prefix of $g''\delta(x_2)\dots\delta(x_{m-1})$ and we denote

$$f''v = g''\delta(x_2)\dots\delta(x_{m-1})$$

for some $v \in A^*$. On the other hand, by (2.1), $r = f''vf' = f''vuh'$, therefore, by (2.2),

$$\delta(x_{m+1})\dots\delta(x_{k-1}) = vu.$$

Note that $uh' = f'$ is a prefix of $\delta(x_m)$ and $h \neq \epsilon$, u is then a proper prefix of $\delta(x_m)$, hence $u \notin S^+$ (prefix condition of S). This equality shows that, for some i, $m+1 \leq i \leq k-1$, $\delta(x_i)$ overlaps the nonempty u, and

uh'. But this $\delta(x_i)$, as an "internal" occurrence in inside the second r, has the identical occurrence inside the first r, hence it overlaps

$$uh'f'' = f'f'' = \delta(x_m).$$

By Remark 2.4 (iv), it results in a square in

$$\delta(x_1)\dots\delta(x_{m-1})\delta(x_m)$$

of length shorter than $|\delta(x_i)| + |\delta(x_m)| \le |rr| - |g''| - |h'| < |rr|$, a contradiction.

Thus, it should be

$$|f''| > |g''\delta(x_2)\dots\delta(x_{m-1})|$$

and $g''\delta(x_2)\dots\delta(x_{m-1}$ is a proper prefix of f'', so

$$f'' = g''\delta(x_2)\dots\delta(x_{m-1})w$$

for some nonempty word w. Substituting this expression for f'' in (2.2) and comparing with (2.1), we obtain

$$f' = w\delta(x_{m+1})\dots\delta(x_{k-1})h'$$

and

$$\delta(x_m) = f'f'' = w\delta(x_{m+1})\dots\delta(x_{k-1})h'g''\delta(x_2)\dots\delta(x_{m-1})w.$$

which shows that, by Definition 2.3 (v), where h' prefix of $\delta(x_k)$ and g'' suffix of $\delta(x_1)$, the square-free

$$x_1\dots x_m\dots x_k$$

is an i-relator. Finally,

(iii) $|h'| > |\delta(x_2)\dots\delta(x_{m-1})f'|$. This case is straightforward:

$$\delta(x_2)\dots\delta(x_{m-1})f'$$

is then a proper suffix of h'. We write again

$$h' = w\delta(x_2)\dots\delta(x_{m-1})f'$$

for a nonempty word w. As before, replacing h' with this expression in (2.2) and comparing with (2.1), we deduce

$$g'' = f''\delta(x_{m+1})\dots\delta(x_{k-1})w.$$

These two equalities show that g'' overlaps h' on w, hence $\delta(x_1)$ overlaps $\delta(x_k)$, so they are equal, $\delta(x_1) = \delta(x_k)$. We denote $x = x_1 = x_k$ and $\delta(x)$ is a bordered word with a border w, a prefix h' and a suffix g''. It is observed that

$$|g''| + |h'| \le |\delta(x)|,$$

otherwise h' overlaps g'' in $\delta(x)$, therefore

$$h'g'' = w\delta(x_2)\dots\delta(x_{m-1})f'f''\delta(x_{m+1})\dots\delta(x_{k-1})w$$

contains some square shorter than

$$|h'g''| = |rr| - |\delta(x_2)\dots\delta(x_{m-1})\delta(x_m)\delta(x_{m+1})\dots\delta(x_{k-1})| < |rr|.$$

as $k > 2$, contradicting the minimality of rr. It follows, finally, that

$$\delta(x) = h'\bar{f}g'' = w\delta(x_2)\dots\delta(x_{m-1})f'\bar{f}f''\delta(x_{m+1})\dots\delta(x_{k-1})w$$

for some word $\bar{f} \in A^*$, which means that

$$xx_2\dots x_m\dots x_{k-1}x = x_1\dots x_m\dots x_k$$

that is square-free, is an o-relator and the proof is complete.

Proposition 2.5 immediately has the following consequence.

Proposition 2.6. *Let $h : \Delta^* \to A^*$ be a morphism and winΔ^* square-free. Then $h(w)$ is square-free if and only if $h(\Delta)$ has no relator; in particular, h is a coding morphism for $h(\Delta)$.*

Remark 2.4 (ii) and (iii) permit us to materialize Proposition 2.6 in more explicit details as the following assertion.

Proposition 2.7. *h is square-free if and only if h is a coding morphism for $S = h(\Delta)$ and $h(\Delta)$ has no k-relator for all $k \geq 1$, or no ps-, i- or o-relator of length 3. Equivalently,*

(0) (no k-relator) S is square-free and for every $\delta(x) \in S$ and for arbitrary factorization $\delta(x) = \delta'\delta''$ over A^* let $s's_2\dots s_{k-2}$ be a common suffix of δ' and S^*, where s' a nonempty suffix of $h(x_1)$, $s_2 = h(x_2), \dots, s_{k-2} = h(x_{k-2})$, either

$$|\text{lcp}\,(\delta', \delta'')| + |\text{lcs}\,(\delta', S^*)| < |\delta'|$$

or $x_1x_2\dots x_{k-2}x$ is non-square-free on Δ whenever

$$\text{lcp}\,(\delta', \delta'')(s's_2\dots s_{k-2}) = \delta'$$

and, symmetrically, let $s_{k-2}\dots s_2p'$ be a common prefix of δ'' and S^*, where p' a nonempty prefix of $h(x_1)$, $s_2 = h(x_2), \dots, s_2 = h(x_{k-2})$, either

$$|\text{lcs}\,(\delta'', \delta')| + |\text{lcp}\,(\delta'', S^*)| < |\delta''|$$

or $xx_{k-2}\dots x_2x_1$ is non-square-free on Δ whenever

$$(s_{k-2}\dots s_2p')\text{lcs}\,(\delta'', \delta') = \delta''.$$

(i) (xyz is not a ps-relator) For every $x, y, z \in \Delta$, $x \neq y, y \neq z$,

$$|\text{lcp}\,(h(y), h(z))| + |\text{lcs}\,(h(x), h(y))| < |h(y)|;$$

(ii) (xyz is not an i-relator) For every $x, y, z \in \Delta$, $x \neq y, y \neq z$ such that $h(y)$ is a bordered word,

$$|\mathrm{lcp}\,(wh(z), h(y))| + |\mathrm{lcs}\,(h(y), h(x)w)| < |h(y)|$$

for every border w of $h(y)$;

(iii) (xyx is not an o-relator) For every $x, y \in \Delta$ such that $h(x)$ is a bordered word,

$$|\mathrm{lcp}\,(wh(y), h(x))| + |\mathrm{lcs}\,(h(x), h(y)w)| < |w| + |h(y)| + |w| = |h(y)| + 2|w|$$

for every border w of $h(x)$.

Proof. The fact that h is a coding morphism is obvious. Since the image of a relator contains a square, the first claim follows by Proposition 2.6 and by Remark 2.4 (ii) and (iii). For the latter claim, one direction is obvious: provided S is square-free, (0) is just the transcription by definition of the fact that the words $x_1x_2 \dots x_{k-2}x$ and $xx_{k-2} \dots x_2x_1$ are not k-relators and (i),(ii) and (iii) are nothing else but that S has no ps-, i- or o-relator of length 3.

Conversely, suppose that S has no k-relator for all $k \geq 1$, no ps-, i- or o-relator of length 3. First, let $\delta(x) \in S$ and $h(x) = h'h'', h', h'' \in A^+$. Denote

$$u = \mathrm{lcp}\,(h', h''), s = \mathrm{lcs}\,(h', S^*).$$

Since $h(x)$ is square-free, $|u| < |h'|$. If $|u| + |s| \geq |h'|$ then $|s| > 0$, hence $s = s'h(x_2) \dots h(x_{k-2})$, for s' nonempty suffix of $h(x_1)$, $x_1, x_2, \dots x_{k-2} \in \Delta$, $k \geq 2$, which implies that $x_1x_2 \dots x_{k-2}x$ is non-square-free as S has no k-relator. Similarly, for the symmetrical possibility $|u| + |s| < |h'|$ that is (0).

Now, for (i), let $x, y, z \in \Delta, x \neq y, y \neq z$ and denote

$$p' = \mathrm{lcp}\,(h(y), h(z)), s'' = \mathrm{lcs}\,(h(x), h(y)).$$

Since $x \neq y$ and $y \neq z$ we have $|s''| < |h(y)|$ and $|p'| < |h(y)|$. The word xyz is not a ps-relator if and only if $p' = \epsilon$, or $s'' = \epsilon$ or $|\delta(y)| > |p'| + |s''|$. All three possibilities yield

$$|p'| + |s''| < |h(y)|,$$

which is (i).

Next, let $x, y, z \in \Delta, x \neq y, y \neq z$ and suppose that xyz is not an i-relator but $h(y)$ is bordered with a border w. We have

$$h(y) = wuw,$$

with $u \in A^+$, as $h(y)$ is square-free. Moreover, as there is no 3-relator, no word of $h(\Delta)$ is a prefix or suffix of u. Thus, now, for an i-relator $d = e = \epsilon$ only. So, we denote by

$$h' = \mathrm{lcp}\,(h(z), u), g'' = \mathrm{lcs}\,(u, h(x)),$$

$h' \neq u$ and $g'' \neq u$, as there are no 2-relators. That xyz is not an i-relator is equivalent to that $g'' = \epsilon$, or $h' = \epsilon$ or

$$|h'| + |g''| < |u|.$$

As above, we have this inequality anyway, which is equivalent to (ii).

Finally, suppose that $x, y \in \Delta, x \neq y$ and xyx is not an o-relator but $h(x)$ is a bordered word. By the same reason as above,

$$h(x) = wuw, u \in A^+,$$

with a border w. Consider

$$f' = \mathrm{lcp}\,(h(y), u), f'' = \mathrm{lcs}\,(u, h(y)).$$

Since yxy is not an i-relator, by (ii),

$$|f'| + |f''| < |u|$$

and, since xyx is not an o-relator, we obtain

$$|f'| + |f''| < |h(y)|$$

which is equivalent to (iii) and that completes the proof.

Thus, to know the square-free status of h, we search for a relator. We do not need any compactness bound and just to verify square-freeness of the would-be relators. Instead, we proceed by matching words — non-overlapping distinct words when there are no 1 or 2-relators, — for a common prefix or suffix until a first relator is found; if none is found h is square-free. So, in the worst case, we should consider all words of S. However, the reader may feel unsecured for whom we give a trivial proof of Proposition 2.1. For a morphism h, we let $r(h)$ be the shortest relator of h, if it exists, and $r(h) = 0$ if h is square-free. It is straight to see by Proposition 2.5 that given a class C of morphisms, the value $K = \max\{r(h) : h \in C\}$ is a compactness bound for C. Moreover, K is sharp, that is, the lowest. Denote for each h

$$L = \lfloor \frac{M-4}{m} \rfloor$$

and recall that

$$T = \max\{t : h(\Delta) \cap A^* h(\Delta^t) A^* \neq \emptyset\}.$$

We formulate Proposition 2.1 as follows.

Theorem 2.7. *For the class of all morphisms on Δ, a sharp compactness bound is*

$$K = \max\{3, \min\{T, L\} + 2\}$$

and for the ternary morphisms, K is universally 5.

Incidentally, $\lfloor \frac{M-4}{m} \rfloor + 2 = \lceil \frac{M-3}{m} \rceil + 1$ for all $M \geq m$, the bound reported in [6].

Proof. Given h, if S has a 1-, 2-, or 3-relator, $r(h) \leq 3$. Otherwise, S has no ps-, i- or o-relator at all, only the k-relators, $k > 3$, matter, so we are going to estimate their length. Consider, for instance, the relator

$$x_1 x_2 \ldots x_{k-1} x_k,$$

with $x_1, \ldots, x_k \in \Delta$ and

$$h(x_k) = u' s h(x_2) \ldots h(x_{k-1}) u' u''$$

where s is a nonempty suffix of $h(x_1)$, $u', u'' \in A^*$.

If $|u'| = 0$ then $h(x_1) = h(x_k)$ and s is a suffix of its. As $k-1 > 2$, $x_1x_2x_k$ square-free, but $h(x_1x_2x_k)$ is not, containing sx_2sx_2, is a 3-relator, despite assumption. Hence $|u'| > 0$

Further, if $|u''| = 0$ then by the same reason $x_kx_{k-1}x_k$, square-free, is 3-relator, a contradiction again. Hence $|u''| > 0$.

Consequently,

$$(k-2)m \leq |h(x_2)\dots h(x_{k-1})| = |h(x_k)| - |u'| - |s| - |u'| - |u''| \leq M - 4,$$

or

$$k \leq \frac{M-4}{m} + 2 = L + 2.$$

At the same time, simply by definition

$$k - 2 \leq T.$$

Together that means $r(h) \leq \min\{T, L\} + 2$. Thus, the estimate K above follows:

The following example shows that the bound $L + 2 = \max\{3, \lfloor \frac{M-4}{m} \rfloor + 2\}$ is sharp.

Example 2.8. Consider the morphism h defined as

$$h(x_k) = aa_1a_2\dots a_{k-1}ab, h(x_1) = a_1, \dots, h(x_{k-1}) = a_{k-1}$$

where $\Delta = \{x_1 \dots, x_k\}, A = \{a_1, \dots, a_k, a, b\}$. We see that $M = k+2, m = 1$, $\max\{3, \lfloor \frac{M-4}{m} \rfloor + 2\} = k$ for $k \geq 3$ and $x_1 \dots x_{k-1}x_k$ is a relator, of length k.

Let w a minimal word for $h(h)$ to contains a square, RR. There is no such w if R does not contain the letter a or b; if R contains a but not b then $w = x_1 \dots x_{k-1}x_k$ length k; R cannot contain both a and b because two distinct words of S have distinct last letters (and the first ones) w is then a square. Thus, h has no relator shorter than k. Note that $L = k-2 < T = k-1$ for this h.

The next example shows that the function $\max\{3, T+2\}$, which is another compactness bound by Proposition 2.7, is also sharp.

Example 2.9. The morphism h defined by

$$h(x_k) = aa_1a_2\dots a_{k-1}ab, h(x_1) = ca_1, h(x_2) = a_2 \dots, h(x_{k-1}) = a_{k-1}.$$

where $\Delta = \{x_1, \dots, x_k\}, A = \{c, a_1, \dots a_k, a, b\}$. We have $K = T = k-2$. Following the same argument as above we see that $x_1 \dots x_{k-1}x_k$, of length $k = L+2 = T+2$, is a only relator of h.

We defer proving the latter claim until the next section. Such a universal bound exists only for the ternary set; see [6].

3. Applications

We apply Proposition 2.6 to give a short proof of a result of Lallement [9], which will be used in a forthcoming construction.

Let $\Delta = A = \{a, b, c\}$ and ϕ defined by

$$\phi(a) = abcbacbcabcba, \phi(b) = bcacbacabcacb, \phi(c) = cabacbabcabac.$$

Note that $\phi(a), \phi(b)$ and $\phi(c)$ all are bordered palindrome words with two borders of length 1 and 5, and they do not overlap each other. Moreover,

$$p(\phi(a)) = \phi(b), p(\phi(b)) = \phi(c), p(\phi(c)) = \phi(a)$$

under the permutation $p(a) = b, p(b) = c, p(c) = a$ of A. That helps reducing the amount of computation.

Proposition 3.1 ([9]). *The morphism ϕ is square-free.*

Proof. Certainly, ϕ is a coding morphism for $S = \{\phi(a), \phi(b), \phi(c)\}$. We show that S has no relators.

It is routine to verify that S has no 1- or 2-relator, "by hand". But we demonstrate that by an application of Proposition 2.2. We shows, for instance, that

$$\phi(a)\phi(b)$$

is square-free; the other instances equally are true, due to the particular property of S. Let, for example,

$$\phi(b) = uv,$$

$u \in A^+, v \in A^*$. We may consider only when v begins with b, otherwise, the inequality ($**$) does not hold as there is no overlapping of two different words in S. Observe that the prefix (and suffix) of length 4 of any word of S has no more occurrence in the other words and only one more in position 8 in itself. Consequently,

$$|\text{lcs}\,(\phi(a), u)| \leq 3, |\text{lcp}\,(u, v)| \leq 3.$$

for all $u, |u| \neq 8$, or else, when $|u| = 8$

$$u = bcacbaca, v = bcacb$$

for which $\text{lcs}\,(\phi(a), u) = \epsilon$, $\text{lcp}\,(u, v) = bcacb$ and

$$|\text{lcs}\,(\phi(a), u)| + |\text{lcp}\,(u, v)| = 0 + 5 < 8 = |u|$$

that means that if

$$|u| > 6$$

the inequality ($**$) does not hold.

If

$$|u| \leq 6,$$

that is when $u = bcac, v = bacabcacb$, we obtain

$$|\text{lcs}\,(\phi(a), u)| = |\epsilon| = 0, |\text{lcp}\,(u, v)| = |b| = 1$$

and the inequality ($**$) does not hold either.

Since the words of S are no factor of each other, S has no k-relator for $k \geq 3$, hence S has no k-relator for all $k \geq 1$.

Next, S has no ps-relator, as two distinct words of its have no common prefix or suffix.

Lastly, denote

$$\phi(b) = wuw,$$

where w is a border of $\phi(a)$, $u \in A^+$. If

$$w = bcacb, u = aca,$$

we obtain

$$|\mathrm{lcp}\,(u, \phi(a))| = |\mathrm{lcs}\,(\phi(a), u)| = |a| = 1$$

and

$$|\mathrm{lcp}\,(\phi(c), u)| = |\mathrm{lcs}\,(u, \phi(c))| = |\epsilon| = 0.$$

Therefore, for $x, z \in \{a, c\}$,

$$|\mathrm{lcp}\,(u, \phi(x))| + |\mathrm{lcs}\,(\phi(z), u)| \leq 1 + 1 = 2 < 5 = |u|,$$

or

$$|\mathrm{lcp}\,(\phi(b), w\phi(x))| + |\mathrm{lcs}\,(w\phi(z), \phi(b))| \leq 5 + 1 + 1 + 5 = 12 < 13 = |\phi(b)|.$$

Similarly, for the remaining case

$$w = b, u = cacbacabcac,$$

we have

$$|\mathrm{lcp}\,(u, \phi(a))| = |\mathrm{lcs}\,(\phi(a), u)| = |\epsilon| = 0$$

and

$$|\mathrm{lcp}\,(\phi(c), u)| = |\mathrm{lcs}\,(u, \phi(c))| = |ca| = 2.$$

Therefore, for $x, z \in \{a, c\}$,

$$|\mathrm{lcp}\,(u, \phi(x))| + |\mathrm{lcs}\,(\phi(z), u)| \leq 2 + 2 = 4 < 11 = |u|,$$

or

$$|\mathrm{lcp}\,(\phi(b), w\phi(x))| + |\mathrm{lcs}\,(w\phi(z), \phi(b))| \leq 1 + 4 + 1 = 6 < 13.$$

All together, these inequalities show that xbz, $x \neq b, z \neq b$, are not i-relators and bxb, $x \neq b$, are not o-relators. That the other square-free words xyz are not relators is just a question of application of p. The proof is completed.

Now we show that the bound 5 in Proposition 2.7, and in Proposition 2.1, is tight, that is, we cannot lower the value 5. Namely, we show that there is a set S of three words which has no relator of length shorter than 5 but has one of length 5. It had been done in [5] with an example on 5 letters. Below we give a simple example on 4 letters.

Example 3.2. Let $\Delta = \{x, y, z\}, A = \{a, b, c, d\}$ and

$$\delta(x) = b, \delta(y) = c, \delta(z) = dadbcbdacad.$$

Notice that d is the only border of $\delta(z)$. We show that S has only one relator, which is a 5-relator, $zxyxz$.

It takes no effort to verify that S has no 1- or 2-relator.

Since there is no common prefix or suffix between two distinct words of S, S has no ps-relator, and no common prefix or suffix among the words b, c and $adbcbdacba$, S has no i- or o-relator of length 3. It remains to determine the other k-relators.

Let, for instance,

$$x_1 \dots x_k \in \Delta^+$$

be a k-relator of length $k > 2$, for which

$$\delta(x_k) = u's\delta(x_2)\dots\delta(x_{k-1})u'u'',$$

s is a nonempty suffix of $\delta(x_1)$, $u', u'' \in A^*$. Evidently, $x_k = z$, $x_2, \dots, x_{k-1} = x$ or y.

If $u' = \epsilon$ then $\delta(x_1)$ overlaps $\delta(x_k)$ on s, hence, they are equal bordered words, $\delta(x_1) = \delta(x_k) = \delta(z)$, with only one border $s = d$. But then $\delta(x_2)$ begins with a, which is impossible.

Thus, $u' \neq \epsilon$, u' is a nonempty prefix of $\delta(x_k)$, which starts with d. Since u' has a later occurrence in $\delta(x_k)$ and $\delta(x_2), \dots, \delta(x_{k-1}) = b$ or c, that occurrence is only in the position 6, therefore $s = ad$ or $s = d$ and

$$\delta(x_2)\dots\delta(x_{k-1}) = bcb = \delta(x)\delta(y)\delta(x).$$

Hence $k = 5$ and

$$x_1 \dots x_k = zxyxz.$$

The other case

$$\delta(x_1) = u'u''\delta(x_2)\dots\delta(x_{k-1})pu'',$$

where p is a nonempty prefix of $\delta(x_k)$, is similarly treated that completes the verification.

Finally, we give a more elaborate construction on three letters. We exploit the square-freeness of Lallement's morphism. Let $\Delta = \{x, y, z\}$ and $A = \{a, b, c\}$. We search for $S = \{\delta(x), \delta(y), \delta(z)\}$ in the form

$$\delta(x) = \phi(b), \delta(y) = \phi(c), \delta(z) = ps\phi(c)\phi(b)\phi(c)\phi(a)$$

where p is a prefix and s is a suffix of $\phi(a)$. As such, p and s at least satisfy
(i) ps is not a prefix of $\phi(a)$;
(ii) $ps\phi(c)$ is square-free.

A preliminary examination reveals that the only candidates are

$$p = abcbacbca, s = cba.$$

So we present

Example 3.3. Let

$$\delta(x) = \phi(b) = bcacbacabcacb, \delta(y) = \phi(c) = cabacbabcabac$$

and

$$\delta(z) = ps\phi(c)\phi(b)\phi(c)\phi(a) =$$

$$(abcbacbca)(cba)(cabacbabcabac)(bcacbacabcacb)(cabacbabcabac)(abcbacbcabcba).$$

Then $S = \{\phi(x), \phi(y), \phi(z)\}$ has a unique relator $zyxyz$.

Verification. First, S has no ps-relator because any two words of it have no common initial or terminal letter. Further, note that the word of S has two borders, of length 1 and 5, for which it is routine to verify that there is no i- or o-relator either. It assumes a bit of more time to treat the k-relators.

First, S has no 1- or 2-relator. It is enough to check that

$$\delta(xzy), \delta(yzx)$$

are square-free. The reader may do it by hand, but we demonstrate it using Proposition 2.2.

Consider for instance

$$\delta(yzx) = \phi(c)ps\phi(c)\phi(b)\phi(c)\phi(a)\phi(b) = [\phi(c)p][s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)],$$

and suppose the contrary that it contains a square. The words included in brackets

$$\phi(c)p = (cabacbabcabac)(abcbacbca)$$

and

$$s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b) =$$

$$(cba)(cabacbabcabac)(bcacbacabcacb)(cabacbabcabac)(abcbacbcabcba)(bcacbacabcacb)$$

are square-free in view of the square-freeness of ϕ.

Let

$$s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b) = uv$$

and

$$u = u'u''$$

for a common prefix u' of u and v and a common suffix u'' of $\phi(b)p$ and u, be factorizations, for which (∗∗) of Proposition 2.2 holds true.

Since the prefix *cbacaba* of $s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)$ does not occur again in itself,

$$|u'| < |cbacaba| = 7$$

and since the suffix *cbacbca* of $\phi(c)p$ does not occur in $s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)$,

$$|u''| < |cbacbca| = 7$$

we obtain

$$|u| = |u'| + |u''| < 14.$$

Now, u'' is not empty as $s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)$ is square-free and we can easily verify that $\phi(c)p$ does not overlap $s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)$, therefore u' is not empty either. Moreover, u has the last letter a, the common last letter of $v, v'', \phi(b)p$ and v has the first letter c, the common first letter of u, u', v. Observe that ac occurs in $s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)$ in the positions

$$2, 6, 14, \ldots$$

that correspond to

$$|u| = 3, 7, 15, \ldots.$$

Consequently, we need to consider only $u = cba$ and $u = cbaacba$, but for them an instant calculation shows that (∗∗) fails.

Now for the other issue, let

$$\phi(c)p = uv$$

and

$$v = v'v''$$

for a common prefix v' of v and $s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)$ and a common suffix v'' of $\phi(c)p$ and u, be factorizations, for which (∗) of Proposition 2.2 holds true.

Observe that the suffix *cbca* of $\phi(c)p$ has no more occurrence in it and the prefix *cbaca* of $s\phi(c)\phi(b)\phi(c)\phi(a)\phi(b)$ has no occurrence in $\phi(c)p$, hence $|v'| < 5, |v''| < 4$ and

$$|v| = |v'v''| < 9.$$

Like above, we see that u ends with a and v begins with c and that ac occurs in $\phi(c)p$ in the positions

$$-3, -9, \ldots$$

that correspond to

$$|v| = 4, 10, \ldots.$$

That is, we need consider only $v = cbca$, for which ($*$) obviously does not hold. Thus $\delta(yzx)$ is square-free.

The verification of square-freeness of $\delta(xzy)$ is completely analogous.

Concluding, we attend to the case of the k-relator on Δ. Consider

$$x_1 x_2 \dots x_{k-1} x_k$$

with $k \geq 3$. Since $k - 2 \geq 1$, only $\delta(z)$ has a representation, required by definition, so $x_1 = z$ or $x_k = z$.

First, let it be $x_1 = z$ and

$$\delta(x_1) = \delta(z) = ps\phi(c)\phi(b)\phi(c)\phi(a) = u'u''\delta(x_2)\dots\delta(x_{k-1})p'u''$$

where p' is a nonempty prefix of $\delta(x_k)$ and $u', u'' \in A^*$. Note that $x_2, \dots, x_{k-1}$ are different from z.

Consider the occurrence $\delta(x_{k-1})$, which is $\phi(b)$ or $\phi(c)$. First, it does not overlap the occurrence $\phi(c)$, $\phi(b)$, $\phi(c)$ or $\phi(a)$, otherwise it results in a square inside $\delta(z)$. Also, ps does not overlap it, because $|ps| = 12$, it would overlap $\phi(c)$, otherwise. It does not coincide with $\phi(a)$ as $|p'u''| > 0$. Consequently, it should coincide with one of the occurrences $\phi(c), \phi(b), \phi(c)$, so

$$p'u'' \in \{\phi(a), \phi(c)\phi(a), \phi(b)\phi(c)\phi(a)\}. \tag{3.1}$$

Analogously, consider the occurrence $\delta(x_2)$, we see that it may only coincide also with one of the occurrences $\phi(c), \phi(b), \phi(c)$, so

$$u'u'' \in \{ps, ps\phi(c), ps\phi(c)\phi(b)\}. \tag{3.2}$$

If $u'' = \epsilon$ then $\delta(z)$ overlaps $\delta(x_k)$, hence $\delta(z) = \delta(x_k)$, and $x_k = z$, and, as a prefix of $\delta(z)$, p' begins with a, hence, by (3.1)

$$p' = p'u'' = \phi(a)$$

that is $\phi(a)$ is a prefix of $\delta(z)$. But ps is also a prefix of $\delta(z)$, of length 12, which implies that ps is a prefix of $\phi(a)$, a contradiction.

Thus $u'' \neq \epsilon$. Then by (3.1) u'' ends with a, which implies, by (3.2)

$$u'u'' = ps$$

and so

$$\phi(c)\phi(b)\phi(c)\phi(a) = \delta(x_2)\dots\delta(x_{k-1})p'u''.$$

If $p'u'' \neq \phi(a)$, then $|p'u''| \geq |\phi(c)\phi(a)|$. Hence, if p' is a prefix of $\delta(x)$ or $\delta(y)$, we have $|p'| \leq 13$ and

$$12 = |ps| = |u'u''| \geq |u''| \geq |\phi(c)\phi(a)| - |p'| = 26 - |p'| \geq 26 - 13 = 13,$$

a contradiction. Consequently, p' is a prefix of $\delta(z)$ and p' begins with a. Therefore, by (3.2),

$$p'u'' = \phi(a)$$

which implies that, first,

$$\delta(x_2)\dots\delta(x_{k-1}) = \phi(c)\phi(b)\phi(c) = \delta(y)\delta(x)\delta(y),$$

$k = 5$ and, second, as p' is a prefix of $\delta(z)$ only, $\delta(x_k) = \delta(z)$, $x_k = z$ and

$$x_1 \ldots x_k = zyxyz$$

is a unique 5-relator.

Finally, the other alternative

$$\delta(x_k) = \delta(z) = ps\phi(c)\phi(b)\phi(c)\phi(a) = u's'\delta(x_2)\ldots\delta(x_{k-1})u'u''$$

where s' is a nonempty prefix of $\delta(x_1)$ and $u', u'' \in A^*$.

Analogously as before, considering the starting position of the occurrence $\delta(x_2)$ and the occurrence $\delta(x_{k-1})$, both of which differ from $\phi(a)$, we get

$$u's' \in \{ps, ps\phi(c), ps\phi(c)\phi(b)\} \tag{3.3}$$

and

$$u'u'' \in \{\phi(a), \phi(c)\phi(a), \phi(b)\phi(c)\phi(a)\}. \tag{3.4}$$

If $u' = \epsilon$, then, as above, $\delta(x_1) = \delta(z)$, s' ends with a, therefore, $ps = u's' = s'$ by (3.3). This shows that ps is a suffix of $\delta(z)$ of length 12, therefore, of $\phi(a)$, which is not.

Thus, $u' \neq \epsilon$ and u' begins with a, hence,

$$u'u'' = \phi(a)$$

by (3.4). On the other hand, if s' ends with a then

$$u's' = ps$$

by (3.3). Otherwise, s' ends by c or b, that is, s' is a suffix of $\delta(y)$ or $\delta(x)$ and $|s'| \leq 13$.

Further, u' must be a proper prefix of ps by (3.3), if not ps would be a prefix of u', and of $\phi(a)$, which is not. It follows $|u'| < |ps| = 12$. Sum up, $|u's'| < 12 + 13 = 25$, which shows

$$u's' = ps$$

again. Consequently,

$$\delta(x_2)\ldots\delta(x_{k-1}) = \phi(c)\phi(b)\phi(c) = \delta(y)\delta(x)\delta(y),$$

and

$$x_2 \ldots x_{k-1} = yxy,$$

$k = 5$. Moreover, s' ends with a, so $x_1 = z$, and

$$x_1 \ldots x_k = zyxyz$$

is a unique relator again. This concludes the verification.

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