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CREATING A NETWORK-STATE HOMOMORPHISM THROUGH OPTIMIZATION

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Abstract. In graph theory, a mapping between two graphs that generally preserves the structure is called a graph homomorphism, which has been a fundamental notion and extensively studied in combinatorial and algebraic areas. Real-valued states are often assigned to the nodes of graphs (also called networks) in theory and applications underpinning the emerging science of networks. In this paper, we present a simple way to create homomorphisms between a network and its state space. The distance-induced structure in the state space is of practical relevance. We characterize the optimal homomorphism with minimum cost in terms of a constrained optimization problem, and demonstrate the calculation with concrete examples.

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1. Introduction

Consider a simple graph $G = (V, E)$ with the set of nodes $V = \{v_1, v_2, \cdots, v_n\}$ and the set of edges $E \subseteq \{\{v_i, v_j\} : v_i, v_j \in V, v_i \neq v_j\}$ connecting pairs of nodes. Let $|E| = m$ be the cardinality of E, which is called the size of G, and similarly $|V| = n$ is referred to as the order of G. Let $N_i = \{v_j \in V : \{v_i, v_j\} \in E\}$ be the neighborhood of $v_i \in V$. In graph theory, a graph homomorphism is a mapping between two graphs, say G and H, that preserves the edge structure [1, 2]. More specifically, given two graphs $G = (V, E)$ and $H = (V_H, E_H)$, a mapping $f : G \to H$ is a homomorphism if for any edge $\{v_i, v_j\} \in E$, the images also form an edge, namely $\{f(v_i), f(v_j)\} \in E_H$. The adjacency of nodes is preserved by the homomorphism f. Graph homomorphism can be viewed as a generalization of graph coloring [3, 4] and is closely related to important algebraic and combinatorial structures with swathes of applications in, for example, wireless networks [5], artificial intelligence [6, 7], and social network analysis [8, 9].

Dynamical processes and complex systems over graphs have gained prominence in the past few decades, where graphs are often interchangeably referred to as networks [10–12]. In the context of networked systems, each node v_i typically has a real-valued state denoted by $x_i \in \mathbb{R}$ representing quantities like position, speed, opinion, or cost in different scenarios. Let $X = \{x_1, x_2, \cdots, x_n\}$ be the state space associated with the graph or network G. Let $\|\cdot\|$ be the Euclidean distance, which is equivalent to the absolute value in the one-dimensional case. The natural mapping $f : G \to X$ associates every node v_i with its state x_i. By assigning an interval $I_i := [x_i - \underline{x}_i, x_i + \overline{x}_i]$ to each v_i with $\underline{x}_i, \overline{x}_i \geq 0$, we conveniently define the adjacency of the values in the

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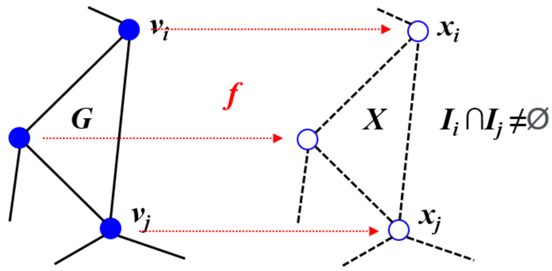

FIGURE 1. Schematic of a homomorphism $f : G \to X$.

state space X as follows: x_i and x_j are adjacent if and only if $I_i \cap I_j \neq \emptyset$. The interval I_i based at x_i can be interpreted as, for example, (a) a scanning range of frequencies in satellite communication, where adjacency indicates communicability between two vehicles, (b) acceptable opinions of an individual in social interactions, where adjacency indicates the two individuals are on speaking terms, or (c) a delivery region (in terms of a section of postcodes, for instance) of a store in grocery chains, where adjacency indicates a seamless coverage of the service, *etc.*

Inspired by the concept of graph homomorphism, we are interested in establishing a homomorphism in networked systems within the above framework. Specifically, we aim to determine the minimum cost that makes the mapping $f : G \to X$ a 'homomorphism'; *cf.* Figure 1 for an illustration. Let $\underline{x} = (\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n)^\mathsf{T} \in \mathbb{R}^n$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)^\mathsf{T} \in \mathbb{R}^n$, where T means transpose. The problem can be formulated as the following constrained optimization problem:

$$\begin{aligned} &\min \|\underline{x}\|^2 + \|\overline{x}\|^2 \\ &\text{s.t.} \quad x_i - x_j \le \overline{x}_j + \underline{x}_i \quad \text{if } x_i \ge x_j, \quad \forall \{v_i, v_j\} \in E, \\ &\qquad\ \ x_j - x_i \le \overline{x}_i + \underline{x}_j \quad \text{if } x_i \le x_j, \quad \forall \{v_i, v_j\} \in E, \\ &\qquad\ \ \underline{x}_i, \overline{x}_i \ge 0 \quad \forall v_i \in V. \end{aligned} \tag{1.1}$$

When the network G is connected, an optimal solution of the problem (1.1) corresponds to the minimal cost or energy to create, for example, (a) a functional communication network, (b) a common ground for bargaining, or (c) an effective delivery service, *etc.* We note that the intersection mechanism proposed here may be reminiscent of the (random) intersection graphs [13, 14], where two nodes are connected if they share a common attribute. However, intersection graphs rely on a fixed discrete set of attributes, which are very different from the construction of our state space.

The rest of the paper is organized as follows. We characterize the optimal solution of the problem (1.1) in Section 2, and provide some examples in Section 3. The paper is concluded in Section 4.

2. MAIN RESULT

The main result is a characterization of the optimal homomorphism as follows.

Theorem 1. *The problem (1.1) has a unique global optimal solution* $\underline{x}^* = (\underline{x}_1^*, \underline{x}_2^*, \cdots, \underline{x}_n^*)^\mathsf{T}$ *and* $\overline{x}^* = (\overline{x}_1^*, \overline{x}_2^*, \cdots, \overline{x}_n^*)^\mathsf{T}$ *given by* $\underline{x}^* = \frac{1}{2}\Xi 1_n$ *and* $\overline{x}^* = \frac{1}{2}\Xi^\mathsf{T} 1_n$, *where* $1_n \in \mathbb{R}^n$ *is a vector of all ones and* $\Xi = (\xi_{ij}) \in \mathbb{R}^{n \times n}$ *is a non-negative matrix satisfying* $\xi_{ij} = \max\left\{0, x_i - x_j - \frac{1}{2}\sum_{v_k \in N_i \setminus \{v_j\}} \xi_{ik} - \frac{1}{2}\sum_{v_k \in N_j \setminus \{v_i\}} \xi_{kj}\right\}$ *for* $\{v_i, v_j\} \in E$ *and* $\xi_{ij} = 0$ *for* $\{v_i, v_j\} \notin E$.

Proof. First note that the objective function of (1.1) is strictly convex and continuous. Since the feasible set is a convex region, the optimization problem has a unique global optimal solution by the optimality condition; see *e.g.* [15], p. 87.

By a direct observation, the problem (1.1) is equivalent to the following

$$\begin{aligned} &\min \|\underline{x}\|^2 + \|\overline{x}\|^2 \\ &\text{s. t.} \quad x_i - \underline{x}_i \le \overline{x}_j + x_j \quad \forall \{v_i, v_j\} \in E, \\ &\qquad x_j - \underline{x}_j \le \overline{x}_i + x_i \quad \forall \{v_i, v_j\} \in E, \\ &\qquad \underline{x}_i, \overline{x}_i \ge 0 \quad \forall v_i \in V. \end{aligned} \tag{2.1}$$

Since the problem (2.1) only has inequality constraints, we introduce the Lagrangian function

$$\begin{aligned} L\big(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda}, \underline{\mu}, \overline{\mu}\big) = &\sum_{i=1}^{n} (\underline{x}_i^2 + \overline{x}_i^2) + \sum_{\{v_i, v_j\} \in E} \Big(\underline{\lambda}_{ij} (x_j - x_i - \underline{x}_j - \overline{x}_i) \\ &+ \overline{\lambda}_{ij} (x_i - x_j - \underline{x}_i - \overline{x}_j) \Big) - \sum_{i=1}^{n} (\underline{\mu}_i \underline{x}_i + \overline{\mu}_i \overline{x}_i), \end{aligned} \tag{2.2}$$

where the multipliers are represented by $\underline{\mu} = (\underline{\mu}_1, \underline{\mu}_2, \cdots, \underline{\mu}_n)^{\mathsf{T}} \in \mathbb{R}^n$, $\overline{\mu} = (\overline{\mu}_1, \overline{\mu}_2, \cdots, \overline{\mu}_n)^{\mathsf{T}} \in \mathbb{R}^n$, $\underline{\lambda} = (\underline{\lambda}_{ij}) \in \mathbb{R}^{n \times n}$, $\overline{\lambda} = (\overline{\lambda}_{ij}) \in \mathbb{R}^{n \times n}$ and $\underline{\lambda}_{ij} = \overline{\lambda}_{ij} = 0$ for all $\{v_i, v_j\} \notin E$. Since the Slater constraint qualification is satisfied, the Kuhn–Tucker conditions indicate that $\underline{x}^*$ and $\overline{x}^*$ become a global optimum if and only if they are feasible and the following conditions are true:

$$\frac{\partial}{\partial \underline{x}_i} L\big(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda}, \underline{\mu}, \overline{\mu}\big)\Big|_{\underline{x}=\underline{x}^*, \overline{x}=\overline{x}^*, \underline{\lambda}=\underline{\lambda}^*, \overline{\lambda}=\overline{\lambda}^*, \underline{\mu}=\underline{\mu}^*, \overline{\mu}=\overline{\mu}^*} = 0 \quad \forall v_i \in V, \tag{2.3}$$

$$\frac{\partial}{\partial \overline{x}_i} L\big(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda}, \underline{\mu}, \overline{\mu}\big)\Big|_{\underline{x}=\underline{x}^*, \overline{x}=\overline{x}^*, \underline{\lambda}=\underline{\lambda}^*, \overline{\lambda}=\overline{\lambda}^*, \underline{\mu}=\underline{\mu}^*, \overline{\mu}=\overline{\mu}^*} = 0 \quad \forall v_i \in V, \tag{2.4}$$

$$\underline{\lambda}_{ij}^* (x_j - x_i - \underline{x}_j^* - \overline{x}_i^*) = 0 \quad \forall \{v_i, v_j\} \in E, \tag{2.5}$$

$$\overline{\lambda}_{ij}^* (x_i - x_j - \underline{x}_i^* - \overline{x}_j^*) = 0 \quad \forall \{v_i, v_j\} \in E, \tag{2.6}$$

$$\underline{\mu}_i^* \underline{x}_i^* = \overline{\mu}_i^* \overline{x}_i^* = 0 \quad \forall v_i \in V, \tag{2.7}$$

$$\underline{\lambda}_{ij}^*, \overline{\lambda}_{ij}^* \ge 0 \quad \forall \{v_i, v_j\} \in E, \tag{2.8}$$

$$\underline{\mu}_i^*, \overline{\mu}_i^* \ge 0 \quad \forall v_i \in V. \tag{2.9}$$

We will first derive the optimum $\underline{x}^*$ and $\overline{x}^*$ from the expressions (2.3)–(2.9), and then check the feasibility. By a direct calculation using (2.2), (2.3) and (2.4), we obtain

$$\underline{x}_i^* = \frac{1}{2} \Big(\sum_{v_k \in N_i} (\overline{\lambda}_{ik}^* + \underline{\lambda}_{ki}^*) + \underline{\mu}_i^* \Big) \tag{2.10}$$

and

$$\overline{x}_i^* = \frac{1}{2} \Big(\sum_{v_k \in N_i} (\overline{\lambda}_{ki}^* + \underline{\lambda}_{ik}^*) + \overline{\mu}_i^* \Big) \tag{2.11}$$

for $v_i \in V$. It follows from (2.7) that $\overline{\mu}_i^* = 0$ or $\overline{x}_i^* = 0$ for any node $v_i \in V$. In the latter case, using (2.11) we derive $\overline{\mu}_i^* = -\sum_{v_k \in N_i} (\overline{\lambda}_{ki}^* + \underline{\lambda}_{ik}^*)$. Thanks to (2.8) and (2.9), we conclude that $\overline{\mu}_i^* = 0$ holds for every node $v_i \in V$. An analogous argument with (2.7) and (2.10) gives rise to $\underline{\mu}_i^* = 0$ for every $v_i \in V$. Plugging these back to (2.10) and (2.11), and singling out an edge $\{v_i, v_j\} \in E$, we obtain

$$\underline{x}_i^* = \frac{1}{2}\sum_{v_k\in N_i}(\overline{\lambda}_{ik}^* + \underline{\lambda}_{ki}^*) = \frac{1}{2}(\overline{\lambda}_{ij}^* + \underline{\lambda}_{ji}^*) + \frac{1}{2}\sum_{v_k\in N_i\backslash\{v_j\}}(\overline{\lambda}_{ik}^* + \underline{\lambda}_{ki}^*) \tag{2.12}$$

and

$$\overline{x}_i^* = \frac{1}{2}\sum_{v_k\in N_i}(\overline{\lambda}_{ki}^* + \underline{\lambda}_{ik}^*) = \frac{1}{2}(\overline{\lambda}_{ji}^* + \underline{\lambda}_{ij}^*) + \frac{1}{2}\sum_{v_k\in N_i\backslash\{v_j\}}(\overline{\lambda}_{ki}^* + \underline{\lambda}_{ik}^*) \tag{2.13}$$

for $v_i \in V$ and $v_j \in N_i$. Likewise, we have the following expressions for v_j:

$$\underline{x}_j^* = \frac{1}{2}(\overline{\lambda}_{ji}^* + \underline{\lambda}_{ij}^*) + \frac{1}{2}\sum_{v_k\in N_j\backslash\{v_i\}}(\overline{\lambda}_{jk}^* + \underline{\lambda}_{kj}^*) \tag{2.14}$$

and

$$\overline{x}_j^* = \frac{1}{2}(\overline{\lambda}_{ij}^* + \underline{\lambda}_{ji}^*) + \frac{1}{2}\sum_{v_k\in N_j\backslash\{v_i\}}(\overline{\lambda}_{kj}^* + \underline{\lambda}_{jk}^*). \tag{2.15}$$

Plugging (2.12) and (2.15) into (2.6), we obtain

$$\overline{\lambda}_{ij}^*\big(x_i - x_j - \overline{\lambda}_{ij}^* - \underline{\lambda}_{ji}^* - \eta_{ij}\big) = 0 \quad \forall\{v_i, v_j\} \in E, \tag{2.16}$$

where $\eta_{ij} := \frac{1}{2}\sum_{v_k\in N_i\backslash\{v_j\}}(\overline{\lambda}_{ik}^* + \underline{\lambda}_{ki}^*) + \frac{1}{2}\sum_{v_k\in N_j\backslash\{v_i\}}(\overline{\lambda}_{kj}^* + \underline{\lambda}_{jk}^*) \geq 0$. Exchanging i and j in (2.16) yields

$$\overline{\lambda}_{ji}^*\big(x_j - x_i - \overline{\lambda}_{ji}^* - \underline{\lambda}_{ij}^* - \eta_{ji}\big) = 0 \quad \forall\{v_i, v_j\} \in E. \tag{2.17}$$

Similarly, plugging (2.13) and (2.14) into (2.5), we have

$$\underline{\lambda}_{ij}^*\big(x_j - x_i - \overline{\lambda}_{ji}^* - \underline{\lambda}_{ij}^* - \eta_{ji}\big) = 0 \quad \forall\{v_i, v_j\} \in E, \tag{2.18}$$

where $\eta_{ji} := \frac{1}{2}\sum_{v_k\in N_j\backslash\{v_i\}}(\overline{\lambda}_{jk}^* + \underline{\lambda}_{kj}^*) + \frac{1}{2}\sum_{v_k\in N_i\backslash\{v_j\}}(\overline{\lambda}_{ki}^* + \underline{\lambda}_{ik}^*) \geq 0$. Exchanging i and j in (2.18) yields

$$\underline{\lambda}_{ji}^*\big(x_i - x_j - \overline{\lambda}_{ij}^* - \underline{\lambda}_{ji}^* - \eta_{ij}\big) = 0 \quad \forall\{v_i, v_j\} \in E. \tag{2.19}$$

If $\overline{\lambda}_{ij}^* > 0$, we have $x_i - x_j - \overline{\lambda}_{ij}^* - \underline{\lambda}_{ji}^* - \eta_{ij} = 0$ by (2.16). This implies $x_i - x_j > \eta_{ij}$. [In fact, if $x_i - x_j = \eta_{ij}$, then $\overline{\lambda}_{ij}^* + \underline{\lambda}_{ji}^* = 0$. This means $\overline{\lambda}_{ij}^* = 0$, which is a contradiction. If $x_i - x_j < \eta_{ij}$, then $\overline{\lambda}_{ij}^* + \underline{\lambda}_{ji}^* < 0$, which contradicts with the condition that both of these two terms are non-negative.] Applying the same argument to (2.19), we show that if $\underline{\lambda}_{ji}^* > 0$ then $x_i - x_j - \overline{\lambda}_{ij}^* - \underline{\lambda}_{ji}^* - \eta_{ij} = 0$ and hence $x_i - x_j > \eta_{ij}$. Therefore,

$$\overline{\lambda}_{ij}^* + \underline{\lambda}_{ji}^* = \max\{0, x_i - x_j - \eta_{ij}\} \quad \forall\{v_i, v_j\} \in E. \tag{2.20}$$

Likewise, if $\overline{\lambda}_{ji}^* > 0$, we have $x_j - x_i - \overline{\lambda}_{ji}^* - \underline{\lambda}_{ij}^* - \eta_{ji} = 0$ by (2.17). This implies $x_j - x_i > \eta_{ji}$. [In fact, if $x_j - x_i = \eta_{ji}$, then $\overline{\lambda}_{ji}^* + \underline{\lambda}_{ij}^* = 0$. This means $\overline{\lambda}_{ji}^* = 0$, which is a contradiction. If $x_j - x_i < \eta_{ji}$, then $\overline{\lambda}_{ji}^* + \underline{\lambda}_{ij}^* < 0$, which contradicts again with the condition that both terms are non-negative.] Applying the same argument to (2.18), we obtain that if $\underline{\lambda}_{ij}^* > 0$ then $x_j - x_i - \overline{\lambda}_{ji}^* - \underline{\lambda}_{ij}^* - \eta_{ji} = 0$ and hence $x_j - x_i > \eta_{ji}$. Therefore,

$$\overline{\lambda}_{ji}^* + \underline{\lambda}_{ij}^* = \max\{0, x_j - x_i - \eta_{ji}\} \quad \forall \{v_i, v_j\} \in E. \tag{2.21}$$

Define $\xi_{ij} = \overline{\lambda}_{ij}^* + \underline{\lambda}_{ji}^*$ and $\xi_{ji} = \overline{\lambda}_{ji}^* + \underline{\lambda}_{ij}^*$ for $\{v_i, v_j\} \in E$, and $\xi_{ij} = 0$ for $\{v_i, v_j\} \notin E$. By our definition of η_{ij} and η_{ji}, we can express them in terms of $\{\xi_{ij}\}_{i,j\in V}$ as follows

$$\eta_{ij} = \frac{1}{2} \sum_{v_k \in N_i \setminus \{v_j\}} \xi_{ik} + \frac{1}{2} \sum_{v_k \in N_j \setminus \{v_i\}} \xi_{kj} \quad \forall \{v_i, v_j\} \in E \tag{2.22}$$

and

$$\eta_{ji} = \frac{1}{2} \sum_{v_k \in N_j \setminus \{v_i\}} \xi_{jk} + \frac{1}{2} \sum_{v_k \in N_i \setminus \{v_j\}} \xi_{ki} \quad \forall \{v_i, v_j\} \in E. \tag{2.23}$$

Invoking (2.20) and (2.21), the matrix $\Xi = (\xi_{ij}) \in \mathbb{R}^{n\times n}$ can be formally given by $\xi_{ij} = \max\{0, x_i - x_j - \eta_{ij}\}$ and $\xi_{ji} = \max\{0, x_j - x_i - \eta_{ji}\}$ for $\{v_i, v_j\} \in E$, and $\xi_{ij} = 0$ otherwise. Using (2.12) and (2.13), we have

$$\underline{x}_i^* = \frac{1}{2} \sum_{v_k \in N_i} \xi_{ik} \quad \text{and} \quad \overline{x}_i^* = \frac{1}{2} \sum_{v_k \in N_i} \xi_{ki}. \tag{2.24}$$

Therefore, $\underline{x}^* = \frac{1}{2}\Xi 1_n$ and $\overline{x}^* = \frac{1}{2}\Xi^\mathsf{T} 1_n$ as desired.

Finally, we will verify the feasibility of this solution. Since $\{\xi_{ij}\}_{i,j\in V}$ are non-negative by definition, we have $\underline{x}_i^*, \overline{x}_i^* \geq 0$ for $v_i \in V$ *via* (2.24). Using (2.13) and (2.14), we have $\overline{x}_i^* + \underline{x}_j^* = \xi_{ji} + \eta_{ji}$. Similarly, we have $\overline{x}_j^* + \underline{x}_i^* = \xi_{ij} + \eta_{ij}$ by (2.12) and (2.15). In view of (2.20) and (2.21), we obtain

$$\begin{aligned} \overline{x}_i^* + \underline{x}_j^* &= \xi_{ji} + \eta_{ji} = \max\{0, x_j - x_i - \eta_{ji}\} + \eta_{ji} \\ &= \max\{\eta_{ji}, x_j - x_i\} \geq x_j - x_i \quad \forall \{v_i, v_j\} \in E \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} \overline{x}_j^* + \underline{x}_i^* &= \xi_{ij} + \eta_{ij} = \max\{0, x_i - x_j - \eta_{ij}\} + \eta_{ij} \\ &= \max\{\eta_{ij}, x_i - x_j\} \geq x_i - x_j \quad \forall \{v_i, v_j\} \in E. \end{aligned} \tag{2.26}$$

This completes the proof of Theorem 1. □

Some quick remarks are in order.

Firstly, the non-negative matrix Ξ by construction is in line with the pattern of the adjacency matrix of G, namely, if $\{v_i, v_j\} \notin E$ then $\xi_{ij} = 0$. Moreover, we observe from (2.20) and (2.21) that, for any $\{v_i, v_j\} \in E$, at most one of the two symmetric entries ξ_{ij} and ξ_{ji} can be positive. Secondly, let $B \in \mathbb{R}^{n\times m}$ be the incidence matrix [2] of G associated with a given order of edges. By stacking one of the two symmetric entries corresponding to each edge $\{v_i, v_j\} \in E$ into a vector $\underline{\zeta} \in \mathbb{R}^m$ in the same order of the edges as specified in B, we can rewrite the solution as $\underline{x}^* = \frac{1}{2}B\underline{\zeta}$. Analogously, stacking the other entries into a vector $\overline{\zeta} \in \mathbb{R}^m$ in the same order of the edges as specified in B will give rise to $\overline{x}^* = \frac{1}{2}B\overline{\zeta}$. Finally, Theorem 1 presents a characterization for the optimum through construction but does not address the algorithmic aspects for the solution, which would be useful especially for a large-scale network G. Many efficient network algorithms have been investigated in the literature adopting different ideas from evolutionary process [16], machine learning [17], and distributed

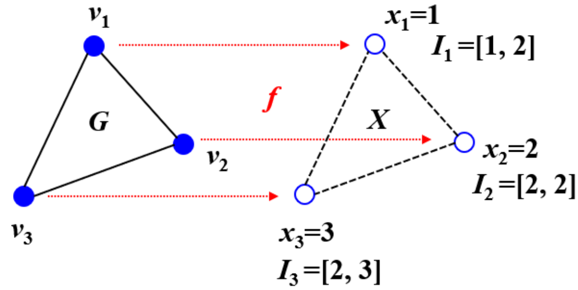

FIGURE 2. Optimal homomorphism for $G = K_3$ with $x_i = i$ for $i = 1, 2, 3$.

computing [18] *etc.* Our future work will be along this direction. In the next section, we will present some small-scale computational examples.

3. EXAMPLES

We first consider a complete graph $G = K_3$ with $V = \{v_1, v_2, v_3\}$ taking values $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$; see Figure 2. To figure out the matrix $\Xi = \begin{pmatrix} 0 & \xi_{12} & \xi_{13} \\ \xi_{21} & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{pmatrix}$, we apply (2.20) and (2.21) to obtain

$$\begin{aligned} \xi_{12} &= \max\Big\{0, -1 - \frac{1}{2}(\xi_{13} + \xi_{32})\Big\}, & \xi_{21} &= \max\Big\{0, 1 - \frac{1}{2}(\xi_{23} + \xi_{31})\Big\}, \\ \xi_{13} &= \max\Big\{0, -2 - \frac{1}{2}(\xi_{12} + \xi_{23})\Big\}, & \xi_{31} &= \max\Big\{0, 2 - \frac{1}{2}(\xi_{32} + \xi_{21})\Big\}, \\ \xi_{23} &= \max\Big\{0, -1 - \frac{1}{2}(\xi_{21} + \xi_{13})\Big\}, & \xi_{32} &= \max\Big\{0, 1 - \frac{1}{2}(\xi_{31} + \xi_{12})\Big\}. \end{aligned}$$

Since Ξ is non-negative, we immediately have $\xi_{12} = \xi_{13} = \xi_{23} = 0$. Plugging these back to the above expressions, we obtain $\xi_{21} = \xi_{32} = 0$ and $\xi_{31} = 2$. Therefore, $\underline{x}^* = \frac{1}{2}\Xi 1_3 = (0, 0, 1)^\top$ and $\overline{x}^* = \frac{1}{2}\Xi^T 1_3 = (1, 0, 0)^\top$. The optimum is $\|\underline{x}^*\|^2 + \|\overline{x}^*\|^2 = 2$ and the corresponding intervals are given by $I_1 = [1, 2]$, $I_2 = [2, 2]$ and $I_3 = [2, 3]$. Here, the homomorphism $f : K_3 \to X$ is isomorphic as displayed in Figure 2. Moreover, we observe that any pair of intervals only meets at a boundary point. We will see that it is not always the case.

Next, consider a 4-cycle $G = C_4$ with $V = \{v_1, v_2, v_3, v_4\}$ taking values $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$; see Figure 3. In this case, Ξ is a 4 by 4 non-negative matrix, which can be similarly derived as $\Xi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$. Therefore, $\underline{x}^* = \frac{1}{2}\Xi 1_4 = (0, 0, 0.5, 0.5)^\top$ and $\overline{x}^* = \frac{1}{2}\Xi^T 1_4 = (1.5, 0.5, 0, 0)^\top$. The optimum is $\|\underline{x}^*\|^2 + \|\overline{x}^*\|^2 = 5$ and the corresponding intervals are given by $I_1 = [1, 2.5]$, $I_2 = [2, 2.5]$, $I_3 = [2.5, 3]$, and $I_4 = [2.5, 4]$. The graph induced by X is a complete graph since $I_i \cap I_j \neq \emptyset$ for any $i \neq j$; see Figure 3. The optimal homomorphism $f : C_4 \to X$ is not an isomorphism in this example.

In the above two examples, the induced graphs by X are always complete. This is not necessarily true as we will see in the next example. Consider the network shown in Figure 4 by simply rewiring two edges $\{v_2, v_3\}$ and $\{v_4, v_1\}$ in the previous 4-cycle example. The new network G is still a 4-cycle with the same states assigned. The matrix $\Xi = (\xi_{ij}) \in \mathbb{R}^{4\times4}$ can be calculated as $\Xi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$. Hence, $\underline{x}^* = \frac{1}{2}\Xi 1_4 = (0, 0, 1, 1)^\top$ and

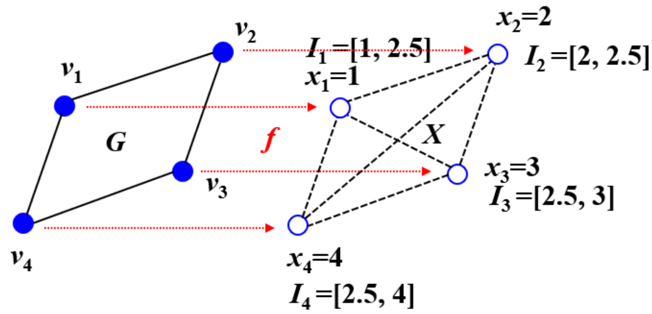

FIGURE 3. Optimal homomorphism for $G = C_4$ with $x_i = i$ for $i = 1, 2, 3, 4$.

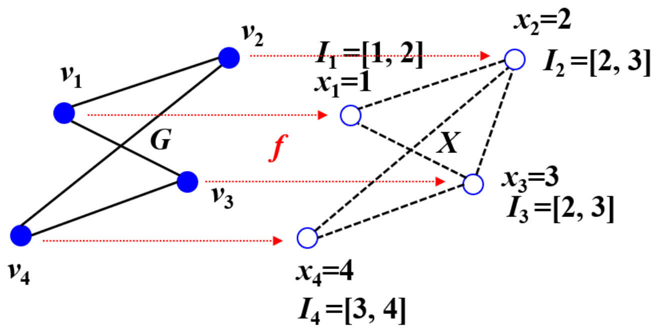

FIGURE 4. Optimal homomorphism for a rewired $G = C_4$ with $x_i = i$ for $i = 1, 2, 3, 4$.

$\overline{x}^* = \frac{1}{2}\Xi^T 1_4 = (1, 1, 0, 0)^\mathsf{T}$. The optimum is $\|\underline{x}^*\|^2 + \|\overline{x}^*\|^2 = 4$ and the corresponding intervals are given by $I_1 = [1, 2]$, $I_2 = I_3 = [2, 3]$, and $I_4 = [3, 4]$. Note that the homomorphism creation 'cost' in this case is lower than that in the previous example, namely, 5. This is somewhat expected because the maximum state gap $\delta := \max_{\{v_i,v_j\}\in E} |x_i - x_j| = 2$ which is smaller than $\delta = 3$ in the previous example. As a result, the graph induced by X is no longer a complete graph.

4. CONCLUSIONS

We have presented a new, natural optimization problem called "network-state homomorphism", where the input is an unoriented graph G, together with a real-value state x_i associated with each node i. The objective is to transform each state x_i into an interval $I_i = [x_i - \underline{x}_i, x_i + \overline{x}_i]$, such that for each pair of adjacent nodes of i and j of G, the corresponding intervals intersect. We aim to achieve this condition while minimizing the cost $\|\underline{x}\|^2 + \|\overline{x}\|^2$, where $\underline{x}$ and $\overline{x}$ are the vectors of coordinates $\underline{x}_i$ and $\overline{x}_i$, respectively.

We have approached this problem through classic tools for optimization namely the Lagrangian method, and the main result characterizes the unique optimum, by an elegant expression depending on the graph G and the vector x. As mentioned in Section 2, we do not provide at this stage an algorithm for computing the optimum. Another natural direction is to explore the optimum when the state of a node is a vector instead of a real value. In this case, the intervals would be replaced by some higher-dimensional objects such as cubes or balls. We believe that an analytical treatment along the line of Theorem 1 is also possible.

REFERENCES

[1] G. Hahn and C. Tardif, Graph homomorphisms: structure and symmetry, in Graph Symmetry, edited by G. Hahn, G. Sabidussi. Springer, Dordrecht (1997) 107–166.

[2] C. Godsil and G.F. Royle, Algeraic Graph Theory. Springer, New York, NY (2001).

[3] A. Dochtermann and A. Singh, Homomorphism complexes, reconfiguration, and homotopy for directed graphs. *Eur. J. Combin.* **110** (2023) 103704.

[4] S. Brandt, Y.-J. Chang, J. Grebík, C. Grunau, V. Rozhoň and Z. Vidnyánszky, On homomorphism graphs. *Forum Math. Pi* **12** (2024) e10.

[5] B. Alaya, L. Laouamer and N. Msilini, Homomorphic encryption systems statement: trends and challenges. *Comput. Sci. Rev.* **36** (2020) 100235.

[6] H. Nguyen and T. Maehara, Graph homomorphism convolution, in *Proceedings of the 37th International Conference on Machine Learning*, PMLR **119** (2020) 7306–7316.

[7] L. Ruiz, F. Gama and A. Ribeiro, Graph neural networks: architectures, stability, and transferability. *Proc. IEEE* **109** (2021) 660–682.

[8] C. Gao, Q. Cheng, X. Li and S. Xia, Cloud-assisted privacy-preserving profile-matching scheme under multiple keys in mobile social network. *Clust. Comput.* **22** (2019) S1655–S1663.

[9] B. Miao, S. Wang, L. Fu and X. Lin, De-anonymizability of social network: through the lens of symmetry, in *Mobihoc'20: Proceedings of the 21st International Symppsium on Theory, Algorithmic Foundations, and Protocol Design for Mobile Networks and Mobile Computing*, Boston, MA (2020) 71–80.

[10] M.E.J. Newman, The structure and function of complex networks. *SIAM Rev.* **45** (2003) 167–256.

[11] F. Menczer, S. Fortunato and C.A. Davis, A First Course in Network Science. Cambridge University Press, Cambridge (2020).

[12] Y. Shang, A system model of three-body interactions in complex networks: consensus and conservation. *Proc. R. Soc. A* **478** (2022) 20210564.

[13] M. Karoński, E.R. Scheinerman and K.B. Singer-Cohen, On random intersection graphs: the subgraph problem. *Combin. Prob. Comput.* **8** (1999) 131–159.

[14] R. van der Hofstad, J. Komjáthy and V. Vadoon, Phase transition in random intersection graphs with communities. *Random Struct. Alg.* **60** (2022) 406–461.

[15] D.P. Bertsekas, A. Nedić and A.E. Ozdaglar, Convex Analysis and Optimization. Athena Scientific, Belmont, MA (2003)

[16] N. Olver, F. Schalekamp, S. van der Ster, L. Stougie and A. van Zuylen, A duality based 2-approximation algorithm for maximum agreement forest. *Math. Program.* **198** (2023) 811–853.

[17] M. Karimi-Mamaghan, M. Mohammadi, P. Meyer, A.M. Karimi-Mamaghan and E.-G. Talbi, Machine learning at the service of meta-heuristics for solving combinatorial optimization problems: a state-of-the-art. *Eur. J. Oper. Res.* **296** (2022) 393–422.

[18] G. Oliva, A.I. Rikos, A. Gasparri and C.N. Hadjicostis, Distributed negotiation for reaching agreement among reluctant players in cooperative multi-agent systems. *IEEE Trans. Autom. Contr.* **67** (2022) 4838–4845.