ON THE COMPLEXITY OF THE GENERALIZED FIBONACCI WORDS

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Abstract. In this paper we undertake a general study of the complexity function of the generalized Fibonacci words which are generated by the morphism defined by $\sigma_l,m(a) = a^l b^m$ and $\sigma_l,m(a) = a$.

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1. Introduction

The complexity function $p$, which counts the number of factors of given length in an infinite word, is a central notion in the field of combinatorics on words. It was introduced in 1975 by Ehrenfeucht et al.\cite{8}. It allows one to measure diversity of patterns in an infinite word. It is often used in characterization of some words or families of words; for example eventually periodic words are the only words with bounded complexity function. For more details on this notion we refer the reader to \cite{4, 7}.

Let $\sigma$ be the morphism of the free monoid $\{a, b\}^*$ defined by $\sigma(a) = ab$ and $\sigma(b) = a$. By iterating infinitely many times the morphism $\sigma$ from $a$ we obtain an infinite word called the Fibonacci word $F = abaababaababaab\cdots$. This word was widely studied \cite{2, 9, 11, 12} and it is currently very famous for its numerous remarkable properties. The reader may consult \cite{3} for more details on it. Its complexity function is well-known: for any $n$ it admits exactly $n + 1$ factors of length $n$.

The generalized Fibonacci morphisms of the free monoid $\{a, b\}^*$ are the morphisms $\sigma_{l,m}$ defined by $\sigma_{l,m}(a) = a^l b^m$ and $\sigma_{l,m}(b) = a$, for $l \geq 1$ and $m \geq 2$. By iterating infinitely many times the morphism $\sigma_{l,m}$ from $a$ we obtain an infinite word $F_{l,m}$ called a generalized Fibonacci word (see \cite{1}, p. 336). In this paper we are interested in the complexity function of these words.

Precisely, we recall in Section 2 some basic definitions and notations. In Section 3 we describe weak and strong bispecial factors of $F_{l,m}$. These are specific factors which play an important role in the study of the complexity function of an infinite word. Section 4 is devoted to the complexity function of $F_{l,m}$. Then, we study asymptotic behavior of $p(n)$ (Sect. 5). We conclude the paper with some remarks and problems for further work. Note that a preliminary version of this paper was presented at CARI’2012 \cite{6}.

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2. Preliminaries

We recall here basic notions on words (see for instance [1, 10] for more details).

Let \( \mathcal{A} = \{a, b\} \) be a fixed alphabet. \( \mathcal{A}^* \), the set of finite words on \( \mathcal{A} \), is the free monoid generated by \( \mathcal{A} \); \( \varepsilon \) the empty word being the neutral element. For any \( u \in \mathcal{A}^* \), \(|u|\) is the length of \( u \) and represents the number of letters of \( u \) (\(|\varepsilon| = 0\)); and for each \( x \in \mathcal{A} \), \(|u|_x\) is the number of occurrences of the letter \( x \) in \( u \). A word \( u \) of length \( n \) written with a repeated single letter \( x \) is simply denoted \( u = x^n \), by extension \( x^0 = \varepsilon \).

An infinite word is a sequence of letters of \( \mathcal{A} \). The set of infinite words over \( \mathcal{A} \) is denoted \( \mathcal{A}^\infty \). A finite word \( v \) is a factor of a word \( u \) if there exist two words \( u_1 \) and \( u_2 \) on \( \mathcal{A} \) such that \( u = u_1 v u_2 \); we say also that \( u \) contains \( v \). The factor \( v \) is a prefix (resp. suffix) if \( u_1 \) (resp. \( u_2 \)) is the empty word. We denote by \( \text{pref}(u) \) (resp. \( \text{suf}(u) \)) the set of prefixes (resp. suffixes) of \( u \).

Let \( u \) be an infinite word on \( \mathcal{A} \), \( w \) a factor of \( u \) and \( x \) a letter of \( \mathcal{A} \). The set of factors of \( u \) of length \( n \) is denoted \( \mathcal{L}_n(u) \) and the set of all factors of \( u \), \( \mathcal{L}(u) \). The set \( \mathcal{L}(u) \) is usually called the language of \( u \). A letter \( x \) is a left (resp. right) extension of \( w \) in \( u \) if \( xw \) (resp. \( wx \)) is in \( \mathcal{L}(u) \). The factor \( w \) is a left (resp. right) special factor of \( u \) if \( aw \) and \( bw \) (respectively \( wa \) and \( wb \)) appear in \( u \). A factor of \( u \) which is both left special and right special in \( u \) is a bispecial factor.

The complexity function of an infinite word \( u \) is the map from \( \mathbb{N} \) to \( \mathbb{N}^* \) defined by \( p_u(n) = \# \mathcal{L}_n(u) \), where \( \# \mathcal{L}_n(u) \) designates the cardinality of the set of factors of \( u \) with length \( n \). In all the sequel, the complexity function \( p_u \) of a word \( u \) will be simply denoted \( p \).

We call the function denoted \( s(u) \), and defined by \( s(n) = p(n + 1) - p(n) \), the first difference of the complexity function of a word \( u \). So, we have the following formula

\[
p(n) = p(k_0) + \sum_{k=k_0}^{n-1} s(k).
\]

On a binary alphabet the function \( s \) counts the number of right special factors of a given length in \( u \). It happens that enumeration of some specific bispecial factors allows one to determine the function \( s \) (see [7]). We will come back to this in Sections 3 and 4.

A morphism \( f \) is a map from \( \mathcal{A}^* \) to itself such that \( f(uv) = f(u)f(v) \) for all \( u, v \in \mathcal{A}^* \).

It is said that an infinite word \( u \) is generated by a morphism \( f \) if there exists a letter \( x \in \mathcal{A} \) such that the words \( x, f(x), f^2(x), \ldots, f^n(x), \ldots \) are longer and longer prefixes of \( u \). Then we denote \( u = f^\omega(x) \).

Let \( u \) be an infinite word on \( \mathcal{A} \) and \( v \) a factor of \( u \). The Parikh vector of \( v \) is \( \chi(v) = ^t(|v|_a, |v|_b) \). We call the following matrix

\[
M_\varphi = \begin{pmatrix}
|\varphi(a)|_a & |\varphi(b)|_a \\
|\varphi(a)|_b & |\varphi(b)|_b
\end{pmatrix},
\]

the incidence matrix of a morphism \( \varphi \). Observe that \( \chi(\varphi(v)) = M_\varphi \chi(v) \).

3. Non-ordinary bispecial factors of \( F_{l,m} \)

Definition 3.1. Let \( u \) be an infinite word on \( \mathcal{A} \) and \( v \) a bispecial factor of \( u \).

- \( v \) is called strong bispecial if \( ava, avb, bva, bvb \) are factors of \( u \).
- \( v \) is called weak bispecial if uniquely \( ava \) and \( bvb \), or \( avb \) and \( bva \), are factors of \( u \).
- \( v \) is called ordinary bispecial if \( v \) is neither strong nor weak.

Definition 3.2. A factor of \( F_{l,m} \) is said to be short if it does not contain any of the three words \( a^l \), \( b^m \) and \( ba \). A factor of \( F_{l,m} \) which is not short will be called long.
Lemma 3.3. Let \( w \) be a long factor of \( F_{l,m} \). Then, there exists a unique triple of words \((p, s, v)\) verifying \( p \in \text{pref}(a^l b^{m-1})\), \( s \in \text{pref}(a^{l-1} b^m)\) and \( v \in \mathcal{L}(F_{l,m})\) such that \( w = s \alpha_{l,m}(v)p \) and \( (v \in \mathcal{A}^*b \implies |p| \geq 1) \).

Proof. Existence. Let \( w \) be a long factor of \( F_{l,m} \). Then, either \( w \) is factor of \( \sigma_{l,m}(x) \) where \( x \in \mathcal{A} \) or \( w = \sigma_{l,m}(v)p \) where \( s \) is a proper suffix of \( \sigma_{l,m}(x) \), \( p \) is a proper prefix of \( \sigma_{l,m}(y) \) with \( x, y \in \mathcal{A} \) and \( xyv \in \mathcal{L}(F_{l,m}) \). More precisely, \( p \in \text{pref}(a^l b^{m-1}) \), \( s \in \text{pref}(a^{l-1} b^m) \) if \( x = a \) and \( s = \varepsilon \) if \( x = b \). If \( v \notin \mathcal{A}^*b \) it is finished. Suppose \( v \in \mathcal{A}^*b \) and \( |p| < 1 \). So, \( p = a^{|p|} \). In this case one changes \( v \) and \( p \) as follows:

\[
v \leftarrow vb^{-1}, \quad p \leftarrow ap.
\]

We still have \( w = s \alpha_{l,m}(v)p \) with \( p \in \text{pref}(a^l b^{m-1}) \) and \( |p| \) has increased. We repeat this process until to get \( v \notin \mathcal{A}^*b \) or \( |p| \geq 1 \).

Uniqueness. Let \( w \) be a long factor of \( F_{l,m} \). Suppose \( w = s \alpha_{l,m}(v)p = s' \sigma_{l,m}(v')p' \) where

1. \( p, p' \in \text{pref}(a^l b^{m-1}) \)
2. \( s, s' \in \text{pref}(a^{l-1} b^m) \)
3. \( (s, v, p) \neq (s', v', p') \)

and verifying:

\[
v \in \mathcal{A}^*b \implies |p| \geq 1 \text{ and } v' \in \mathcal{A}^*b \implies |p'| \geq 1. \quad (\star)
\]

- Suppose \((v, p) = (v', p')\). Then, we have \( s = s' \). That is impossible.
- Suppose \( p = p' \) and \( v \neq v' \). Then, we have \( \sigma_{l,m}(v) = s' \alpha_{l,m}(v') \).
  - If \( v \) and \( v' \) are not empty then \( v \) and \( v' \) must end with the same letter. Then we change \( v \) to \( \text{pref}_{|v|-1}(v) \) and \( v' \) to \( \text{pref}_{|v'|-1}(v') \). We repeat the process while \( v \neq \varepsilon \) and \( v' \neq \varepsilon \).
  - If \( v \neq \varepsilon \) and \( v' = \varepsilon \) (or conversely) then we have \( \sigma_{l,m}(v) = s' \). Now, we have \( |s'| < l + m \). It follows that \( 0 < |\sigma_{l,m}(v)| < l + m \). Thus, we have \( v = b^k \) and \( sa^k = s' \). But \( s' \) ends with \( b \). That is impossible.

- Suppose \( p \neq p' \). Without loss generality let us assume that \( |p| > |p'| \). Then \( p \) can be written \( p = p''p' \) with \( p'' \neq \varepsilon \). So, it follows that \( \sigma_{l,m}(v)p'' = s' \alpha_{l,m}(v') \).
  - If \( u' \) is empty then \( \sigma_{l,m}(v)p'' = s' \). Now, we have \( |s'| < l + m \). So, \( v \) takes the form \( v = b^k \) and we have \( sa^k = s' \). Since \( p'' \neq \varepsilon \) then \( s' \neq \varepsilon \) and ends with \( b \). Therefore, \( p'' \) also ends with \( b \). So, we can write \( p'' = a^l b^i \), \( i \geq 1 \). Thus, \( s' \) contains \( a^i \). That is impossible since \( s' \in \text{pref}(a^{l-1} b^m) \).
  - If \( v' \neq \varepsilon \), let \( x \) be the last letter of \( p'' \) and of \( \alpha_{l,m}(v') \).
    - If \( x = a \), then the last letter of \( v' \) is \( b \) and by \((\star)\) we have \( |p'| \geq 1 \). So, we have \( p' = a^iz \) and \( p'' = ya^{i+1}z \). That is impossible.
    - If \( x = b \), then the last letter of \( v' \) is \( a \) and \( p'' = a^l b^i \) \( (0 < i < m) \). Now, the suffix of \( \sigma_{l,m}(v') \) of length \( m \) is \( b^m \). So, the word \( s'\alpha_{l,m}(v') = \sigma_{l,m}(v)p'' \) admits the two words \( b^m \) and \( ab^i \) as suffix. That is impossible.

\(\square\)

Lemma 3.4. 1. \( F_{l,m} \) admits exactly one short and weak bispecial factor: \( b^{m-1} \).
2. \( F_{l,m} \) admits exactly one short and strong bispecial factor: \( \varepsilon \).

Lemma 3.5. Let \( w \) be a factor of \( F_{l,m} \). The following assertions are equivalent:

1. \( w \) is a long bispecial factor of \( F_{l,m} \).
2. There exists a bispecial factor $v$ of $F_{l,m}$ such that $w = \hat{\sigma}_{l,m}(v)$ where $\hat{\sigma}_{l,m}(v) = \sigma_{l,m}(v)a^l$. Furthermore, $v$ and $w$ have the same type and $|v| < |w|$. 

Proof. Let $w$ be a long bispecial factor. Then $wa, wb, aw, bw$ appear in $F_{l,m}$. Furthermore with the synchronization Lemma there exists a unique triple of words $(p, s, v)$ verifying $p \in \text{pref}(a^l b^{m-1})$, $s \in \text{pref}(a^{l-1} b^m)$ and $v \in \mathcal{L}(F_{l,m})$ such that $w = s \sigma_{l,m}(v)p$ and $(v \in \mathcal{A}^* b \implies |p| \geq l)$. So, the words $s \sigma_{l,m}(v)pa$, $s \sigma_{l,m}(v)pb$, $bs \sigma_{l,m}(v)p$, $bsa \sigma_{l,m}(v)p$ appear in $F_{l,m}$.

Suppose $\sigma_{l,m}(v)p = \varepsilon$, i.e $w = s a^l b^m$ with $i < l$. Then, $bw = ba^l b^m$ appears in $F_{l,m}$. That is impossible.

Suppose $\sigma_{l,m}(v)p \neq \varepsilon$, $\sigma_{l,m}(v)p$ begin with $a$. Then, $asa$ and $bsa$ appear in $F_{l,m}$. If $s = b^j$ with $0 < j < m$, then $ab^ia$ appears in $F_{l,m}$, which is impossible. If $s = a^l b^i$ with $0 < i < l$, then $ba^l b^m$ appears in $F_{l,m}$, which is also impossible. Thus $s = \varepsilon$ and we have $w = \sigma_{l,m}(v)p$.

Let us now show that $p = a^l$.

Suppose $|p| > l$. Then, $p = a^l b^j$ with $0 < i < m$. So, $ab^ia$ appears in $F_{l,m}$, which is impossible.

Suppose $|p| < l$. Then $p = a^l$, with $0 \leq i < l$. So, $v \notin \mathcal{A}^a$. If $v = \varepsilon$ then $w = p = a^l$ and so $w$ is short in $F_{l,m}$. This is impossible because $w$ is assumed long. Otherwise if $v \neq \varepsilon$ then $v$ ends with $a$. So, the factor $wb = \sigma_{l,m}(v)pb$ of $F_{l,m}$ ends with $a^l b^m a^i b$. Thus, $ba^l b$ appears in $F_{l,m}$ with $0 \leq i < l$, which is again impossible.

It follows that $p = a^l$, so $w = \hat{\sigma}_{l,m}(v)$.

The inequality $|v| < |w|$ is obvious.

Conversely, assume that $v$ is a bispecial factor of $F_{l,m}$ and that $w = \hat{\sigma}_{l,m}(v)$. As the words $av$, $bv$, $va$ and $vb$ occur in $F_{l,m}$, it follows that $a^l b^m w$, $aw$, $wb a^l$ and $wa$ occur in $F_{l,m}$. So, $w$ is a bispecial factor of $F_{l,m}$, which is long since it contains $a^l$.

Finally, if $w = \hat{\sigma}_{l,m}(v)$, then

$$\# \{ (x, y) \in \mathcal{A}^2 : x w y \in \mathcal{L}(F_{l,m}) \} = \# \{ (x', y') \in \mathcal{A}^2 : x' v y' \in \mathcal{L}(F_{l,m}) \}. $$

So, $v$ and $w$ have the same type. \hfill \square

As a consequence, we have:

1. The weak bispecial factors of $F_{l,m}$ are given by the sequence $(y_n)$ defined by $y_1 = b^{n-1}$ and $y_{n+1} = \hat{\sigma}_{l,m}(y_n)$, for $n \geq 1$.

2. The strong bispecial factors of $F_{l,m}$ are given by the sequence $(x_n)$ defined by $x_0 = \varepsilon$ and $x_{n+1} = \hat{\sigma}_{l,m}(x_n)$, for $n \geq 0$.

4. COMPLEXITY OF $F_{l,m}$

In order to understand the complexity function of $F_{l,m}$, we begin this section with a review of some properties of sequences of weak bispecial and strong bispecial factors of $F_{l,m}$.

**Definition 4.1.** Let $v, w \in \mathcal{A}^*$ and $\chi(v), \chi(w)$ be their Parikh vectors. One says that $\chi(v)$ is less than $\chi(w)$ and one writes $\chi(v) \leq \chi(w)$ when $|v|x \leq |w|x$ for all $x \in \mathcal{A}$. Moreover, if $\chi(v) \neq \chi(w)$, one writes $\chi(v) < \chi(w)$.

Note that this define a partial order on words.

**Proposition 4.2.** Let $v, w, v', w'$ be four finite words such that $v' = \hat{\sigma}_{l,m}(v)$ and $w' = \hat{\sigma}_{l,m}(w)$. Then,

$$\chi(v) < \chi(w) \implies \chi(v') < \chi(w').$$

Proof. On the one hand, we have $|v'|_a = l |v|_a + |v|_b + l$ and $|w'|_a = l |w|_a + |w|_b + l$; so $|v'|_a < |w'|_a$. On the other hand, we have $|v'|_b = m |v|_a$ and $|w'|_b = m |w|_a$; so $|v'|_b \leq |w'|_b$. \hfill \square
Proposition 4.3. For all \( l \geq 1 \) and \( m \geq 2 \) one has:

\[
\forall n \geq 0, \chi(x_n) < \chi(y_{n+1}) < \chi(x_{n+2})
\]

Proof. Since \( x_0 = \varepsilon, y_1 = b^{m-1} \) and \( x_2 = (a^l b^m)^l a^l \) we have \( \chi(x_0) < \chi(y_1) < \chi(x_2) \). Suppose these inequalities stay valid until rank \( n \), i.e.,

\[
\chi(x_n) < \chi(y_{n+1}) < \chi(x_{n+2}).
\]

By Proposition 4.2, we obtain \( \chi(x_{n+1}) < \chi(y_{n+2}) < \chi(x_{n+3}). \) \( \square \)

The following Lemma describes the function \( s \).

Lemma 4.4. Let \( n \in \mathbb{N} \). One has:

- \( s(n) = 1 \) for \( n = 0 \).
- if \( n \in \mathbb{N}^* \), take \( k \) the largest integer such that \( n > |x_k| \).
  1. If \( k = 0 \), one has: \( s(n) = \begin{cases} 2 & \text{if } 1 \leq n \leq \min(l, m-1) \\ 1 & \text{if } m \leq n \leq l \end{cases} \).
  2. Otherwise, one has: \( s(n) = \begin{cases} 3 & \text{if } |x_k| < n \leq |y_k| \\ 2 & \text{if } |y_k| < n \leq |y_{k+1}| \\ 1 & \text{if } |y_{k+1}| < n \leq |x_{k+1}| \end{cases} \).

Proof. The function \( s \) is given by the following formula

\[
s(n) = 1 + \# \{ w \text{ strong bispecial} : |w| < n \} - \# \{ w \text{ weak bispecial} : |w| < n \}.
\]

Now \( s(0) = 1 \). Also, let us observe that \( s(n) = 2 \) if \( 1 \leq n \leq \min(l, m-1) \) and \( s(n) = 1 \) if \( m \leq n \leq l \).

Suppose \( n \geq |x_1| \). Take \( k \) the largest integer such that \( n > |x_k| \). Then, it follows:

\[
s(n) = 1 + (k + 1) - \begin{cases} k - 1 & \text{if } |x_k| < n \leq |y_k| \\ k & \text{if } |y_k| < n \leq |y_{k+1}| \\ k + 1 & \text{if } |y_{k+1}| < n \leq |x_{k+1}| \end{cases}.
\]

The proof is complete. \( \square \)

Theorem 4.5. The complexity function of \( F_{l,m} \) satisfies the following inequalities:

\[
n + 1 \leq p(n) \leq 3n + 1.
\]

Proof. By Lemma 4.4, we have

\[
\forall n \geq 0, 1 \leq s(n) \leq 3.
\]

It follows that:

\[
1 + \sum_{k=0}^{n-1} 1 \leq p(n) \leq 1 + \sum_{k=0}^{n-1} 3.
\]

Lemma 4.6. We have the following equivalences.
1. \( m \in \left[ 2, 2l^2 + 1 \right] \iff \exists k_0 \in \mathbb{N} : \forall k \geq k_0, |x_k| - |y_k| > 0. \)

2. \( m \in \left[ 2l^2 + 2, \infty \right[ \iff \exists k_0 \in \mathbb{N} : \forall k \geq k_0, |x_k| - |y_k| < 0. \)

**Proof.** Consider the sequence \((V_k)_{k \geq 1}\) defined by \( V_k = \chi(x_k) - \chi(y_k). \) We have:

\[
V_1 = \begin{pmatrix} l \\ -m + 1 \end{pmatrix} \quad \text{and} \quad V_{k+1} = AV_k
\]

where \( A = M_{\sigma} = \begin{pmatrix} l & 1 \\ m & 0 \end{pmatrix} \) is the incidence matrix of \( \sigma_{l,m}. \) The eigenvalues of the matrix \( A, \) being the roots of \( X^2 - lX - m, \) are

\[
\lambda_1 = \frac{l + \sqrt{l^2 + 4m}}{2} \quad \text{and} \quad \lambda_2 = \frac{l - \sqrt{l^2 + 4m}}{2}.
\]

Observe that \( \lambda_1 > l \geq 1 \) and \( -\lambda_1 < \lambda_2 < 0. \) Moreover we have

\[
\forall k \geq 1, \ |x_k| - |y_k| = \begin{pmatrix} 1 & 1 \end{pmatrix} V_k.
\]

Thus,

\[
|x_k| - |y_k| = \begin{pmatrix} 1 & 1 \end{pmatrix} A^{k-1} \begin{pmatrix} l \\ -m + 1 \end{pmatrix} = \alpha_1 \lambda_1^{k-1} + \alpha_2 \lambda_2^{k-1}
\]

with \( \alpha_1 \) and \( \alpha_2 \) verifying the following system of equations

\[
\begin{cases}
\alpha_1 + \alpha_2 = l - m + 1 \\
\alpha_1 \lambda_1 + \alpha_2 \lambda_2 = l^2 + lm - m + 1
\end{cases}.
\]

We have

\[
\alpha_1 = \frac{l^2 + lm - m + 1 - \lambda_2 (l - m + 1)}{\lambda_1 - \lambda_2},
\]

\[
\alpha_2 = \frac{l^2 + lm - m + 1 - \lambda_1 (l - m + 1)}{\lambda_2 - \lambda_1}.
\]

Since \( |\lambda_2| < \lambda_1, \) then \( |x_k| - |y_k| \) has the same sign as \( \alpha_1 \) for \( k \) sufficiently large.

**Case 1.** \( l - m + 1 \geq 0. \) Then, we have \( -\lambda_2 (l - m + 1) \geq 0 \) since \( \lambda_2 < 0. \) So, \( \alpha_1 > 0 \) since \( \lambda_1 - \lambda_2 > 0 \) and \( l^2 + lm - m + 1 = l^2 + l(m - 1) + 1 > 0. \)

**Case 2.** \( l - m + 1 < 0. \) We have:

\[
\alpha_1 > 0 \iff \lambda_2 > \frac{l^2 + lm - m + 1}{l - m + 1} \iff \frac{l - \sqrt{l^2 + 4m}}{2} > \frac{l^2 + lm - m + 1}{l - m + 1}.
\]
\[ \iff l^2 + 4m < \frac{(-l^2 - 3lm + l + 2m - 2)^2}{(l - m + 1)^2} \]

since \(-l^2 - 3lm + l + 2m - 2 = m(2 - 3l) - l(l - 1) - 2 < 0\) and \(l - m + 1 < 0\). The last inequality can be turned into the following one:

\[ P_l(m) = m^3 + m^2 (-2l^2 + l - 3) + m (-2l^3 + 3l^2 - 2l + 3) + l^3 - l^2 + l - 1 < 0. \]

If \(l = 1\) then \(P_l(m) = m(m^2 - 4m + 2)\). So,

\[ P_l(m) < 0 \iff m \in \{2, 3\}. \]

Suppose from now on that \(l \geq 2\). The derivative \(P'_l(m) = 3m^2 + 2m (-2l^2 + l - 3) - 2l^3 + 3l^2 - 2l + 3\) admits two roots of opposite signs

\[ \beta_1 = \frac{2l^2 - l + 3 - \sqrt{4l^4 + 2l^3 + 4l^2}}{3} < 0, \]

\[ \beta_2 = \frac{2l^2 - l + 3 + \sqrt{4l^4 + 2l^3 + 4l^2}}{3} > 0 \]

and is negative between these two roots. So, \(P_l\) is decreasing on \([\beta_1, \beta_2]\) which contains 0, and is increasing on \([-\infty, \beta_1] \cup [\beta_2, +\infty]\). Furthermore, one verifies that \(P_l(0) > 0, P_l(1) < 0, P_l(2l^2 + 1) < 0\) and \(P_l(2l^2 + 2) > 0\). It results that \(P_l\) admits two positive roots \(m_1, m_2\) and one negative root \(m_3\) with \(0 < m_1 < 1, 2l^2 + 1 < m_2 < 2l^2 + 2\) and \(m_1 \leq \beta_2 \leq m_2\). Thus, \(P_l\) is negative on \([m_1, m_2]\) and positive on \([m_2, +\infty]\). Then, \(\alpha_1 > 0\) if \(m \in \{2, 2l^2 + 1\}\) and \(\alpha_1 < 0\) if \(m \geq 2l^2 + 2\).

So, we have:

1. \(m \in [2, 2l^2 + 1] \iff \exists k_0 \in \mathbb{N} : \forall k \geq k_0, |x_k| - |y_k| > 0. \)
2. \(m \in [2l^2 + 2, +\infty] \iff \exists k_0 \in \mathbb{N} : \forall k \geq k_0, |x_k| - |y_k| < 0. \)

**Theorem 4.7.**

1. If \(m \in [2, 2l^2 + 1]\), then there exists a constant \(\delta\) and an integer \(n_0\) such that for all \(n > n_0\)

\[ n + 1 \leq p(n) \leq 2n + \delta. \]

2. If \(m \in [2l^2 + 2, +\infty]\), then there exists a constant \(\delta\) and an integer \(n_0\) such that for all \(n > n_0\)

\[ 2n + \delta \leq p(n) \leq 3n + 1. \]

**Proof.** Suppose \(m \in [2, 2l^2 + 1]\). Then, there exists \(k_0\) such that for all \(k \geq k_0, |x_k| - |y_k| > 0\). In this case, for \(n \in [|x_k|, |x_{k+1}|]\) we have by Lemma 4.4

\[ s(n) = \begin{cases} 2 & \text{if } |y_k| < n \leq |y_{k+1}| \\ 1 & \text{if } |y_{k+1}| < n \leq |x_{k+1}| \end{cases} \]
as the case $|x_k| < n \leq |y_k|$ is empty. Thus,

$$\forall n \geq |x_{k_0}|, \ 1 \leq s(n) \leq 2.$$ 

By summation, it follows that

$$\sum_{k=|x_{k_0}|}^{n-1} 1 \leq \sum_{k=|x_{k_0}|}^{n-1} s(k) \leq \sum_{k=|x_{k_0}|}^{n-1} 2.$$ 

Therefore

$$p(|x_{k_0}|) + n - |x_{k_0}| \leq p(n) \leq p(|x_{k_0}|) + 2(n - |x_{k_0}|).$$ 

It follows that

$$n + 1 \leq p(n) \leq 2n + \delta, \text{ with } \delta \in \mathbb{Z}.$$ 

Suppose $m \geq 2l^2 + 2$. Then, by Lemma 4.6, there exists $k_0$ such that for all $k \geq k_0, |x_k| - |y_k| < 0$. In this case, for $n \in [|x_k|, |x_{k+1}|]$ we have

$$s(n) = \begin{cases} 3 & \text{if } |x_k| < n \leq |y_k| \\ 2 & \text{if } |y_k| < n \leq |x_{k+1}| \end{cases}.$$ 

It follows that,

$$\forall n \geq |x_{k_0}|, \ 2 \leq s(n) \leq 3.$$ 

Thus, in the similar way as previously, we get

$$2n + \delta \leq p(n) \leq 3n + 1, \text{ with } \delta \in \mathbb{Z}.$$ 

5. Asymptotic behavior of $\frac{p(n)}{n}$

Before the statement of the main result we need some technical lemmas.

**Lemma 5.1.** Let $(r_k), (s_k)$ be two strictly increasing sequences of integers and $l$ be a real number such that:

$$\lim_{k \to \infty} \frac{s_{k+1} - s_k}{r_{k+1} - r_k} = l. \text{ Then, } \lim_{k \to \infty} \frac{s_k}{r_k} = l.$$ 

**Proof.** Let $(r_k), (s_k)$ be two strictly increasing sequences of integers and $l$ be a real number such that

$$\lim_{k \to \infty} \frac{s_{k+1} - s_k}{r_{k+1} - r_k} = l. \text{ Write } m_k = \frac{s_{k+1} - s_k}{r_{k+1} - r_k}. \text{ Let } \varepsilon > 0. \text{ There exists } k_0 \in \mathbb{N} \text{ such that:}$$

$$\forall k \geq k_0, \ l - \varepsilon \leq m_k \leq l + \varepsilon. \quad (5.1)$$
Lemma 5.2. For all $k > k_0$ we have

$$s_k = s_{k_0} + \sum_{j=k_0}^{k-1} (s_{j+1} - s_j)$$

(5.2)

where $R_k = \sum_{j=k_0}^{k-1} (m_j - l)(r_{j+1} - r_j)$. Thereby $\frac{s_k}{r_k} = l + \frac{s_{k_0} - lr_{k_0}}{r_k} + \frac{R_k}{r_k}$. From (5.1) and since $(r_k)$ is increasing, it follows that $|R_k| \leq \varepsilon (r_k - r_{k_0})$ and so $\frac{R_k}{r_k} \leq \frac{\varepsilon}{r_k}$.

Thus, $\lim_{k \to \infty} \frac{s_k}{r_k} = l$. \hfill \qed

Lemma 5.2. For all $k \geq 1$, $|x_k| = \beta_1 \lambda_1^k + \beta_2 \lambda_2^k + \beta_3$ and $|y_k| = \gamma_1 \lambda_1^k + \gamma_2 \lambda_2^k + \gamma_3$ where $\lambda_1$ and $\lambda_2$ are the eigenvalues of matrix $A$, and $(\beta_1, \beta_2, \beta_3), (\gamma_1, \gamma_2, \gamma_3)$ are some triples of real numbers.

Proof. Let $w, w'$ be two words such that $w' = \sigma_{l,m}(w) = \sigma_{l,m}(w) a'$. Put $W = \begin{pmatrix} |w|_a \\ |w|_b \\ 1 \end{pmatrix}$. Observe that

$$|w| = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} W$$

and $W' = BW$ where $B = \begin{pmatrix} l & 1 & 1 \\ m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. With these relations we are able to determine the two sequences $(x_k)_{k \geq 0}, (y_k)_{k \geq 1}$, and the length of $x_k$ and $y_k$ for all $k$.

Namely, since $x_{k+1} = \sigma_{l,k}(x_k), y_{k+1} = \sigma_{l,k}(y_k), x_0 = \varepsilon$ and $y_1 = b^{-m-1}$ we have $X_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, Y_1 = \begin{pmatrix} 0 \\ m-1 \\ 1 \end{pmatrix}$, $X_{k+1} = BX_k$ and $Y_{k+1} = BY_k$, and so $X_k = b^k X_0$ and $Y_k = b^{k-1} Y_1$. The eigenvalues of the matrix $B$ are $\lambda_1, \lambda_2$ and 1. Then, it follows that

$$|x_k| = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} B^k X_0$$

(5.3)

$$= \beta_1 \lambda_1^k + \beta_2 \lambda_2^k + \beta_3$$

(5.4)

where $(\beta_1, \beta_2, \beta_3)$ is the solution of the following system of equations

$$\begin{cases} 
\beta_1 + \beta_2 + \beta_3 = 0 \\
\beta_1 \lambda_1 + \beta_2 \lambda_2 + \beta_3 = l \\
\beta_1 \lambda_1^2 + \beta_2 \lambda_2^2 + \beta_3 = l^2 + lm + l
\end{cases}$$

and

$$|y_k| = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} B^{k-1} Y_1$$

(5.5)

$$= \gamma_1 \lambda_1^k + \gamma_2 \lambda_2^k + \gamma_3$$

(5.6)
where $(\gamma_1, \gamma_2, \gamma_3)$ is given by the following system of equations

$$
\begin{align*}
\gamma_1 \lambda_1 + \gamma_2 \lambda_2 + \gamma_3 &= m - 1 \\
\gamma_1 \lambda_1^2 + \gamma_2 \lambda_2^2 + \gamma_3 &= m - 1 + l \\
\gamma_1 \lambda_1^3 + \gamma_2 \lambda_2^3 + \gamma_3 &= (m + l)^2 - m
\end{align*}
$$

\[ \blacksquare \]

**Theorem 5.3.**
1. If $m \in [1, 2l^2 + 1]$, we have

$$
\lim \inf \frac{p(n)}{n} = 1 + \frac{\gamma_1 \lambda_1 - \beta_1}{\beta_1 (\lambda_1 - 1)}, \quad \text{and} \quad \lim \sup \frac{p(n)}{n} = 1 + \frac{\gamma_1 \lambda_1 - \beta_1}{\gamma_1 (\lambda_1 - 1)}.
$$

2. If $m \geq 2l^2 + 2$ we have

$$
\lim \inf \frac{p(n)}{n} = 2 + \frac{\gamma_1 - \beta_1}{\beta_1 (\lambda_1 - 1)} \quad \text{and} \quad \lim \sup \frac{p(n)}{n} = 2 + \frac{\lambda_1 (\gamma_1 - \beta_1)}{\gamma_1 (\lambda_1 - 1)}.
$$

**Proof.**

**Case 1.** $m \in [1, 2l^2 + 1]$. From Lemma 4.6, there exists $k_0$ such that for all $k \geq k_0$,

$$
|y_k| < |x_k| < |y_{k+1}| < |x_{k+1}|.
$$

In this case, we have

$$
\forall n \in |x_k|, |x_{k+1}|, \ s(n) = \begin{cases} 
2 & \text{if } |x_k| < n \leq |y_{k+1}| \\
1 & \text{if } |y_{k+1}| < n \leq |x_{k+1}|.
\end{cases}
$$

So, $\lim \inf \frac{p(n)}{n} = \lim \inf \frac{p(|x_{k+1}| + 1)}{|x_{k+1}| + 1}$ and $\lim \sup \frac{p(n)}{n} = \lim \sup \frac{p(|y_{k+1}|)}{|y_{k+1}|}$. Furthermore, with Lemma 5.2 we have

$$
p(|x_{k+1}| + 1) - p(|x_k| + 1) = \sum_{n=|x_k|+1}^{|x_{k+1}|} s(n) = 2 (|y_{k+1}| - |x_k|) + (|x_{k+1}| - |y_{k+1}|) = 2 (\gamma_1 \lambda_1^{k+1} - \beta_1 \lambda_1^k) + (\beta_1 \lambda_1^{k+1} - \gamma_1 \lambda_1^{k+1}) + o (\lambda_1^k) = \gamma_1 \lambda_1^{k+1} + \beta_1 \lambda_1^{k+1} - 2 \beta_1 \lambda_1^k + o (\lambda_1^k).
$$

Moreover

$$
|x_{k+1}| - |x_k| = \beta_1 \lambda_1^{k+1} - \beta_1 \lambda_1^k + o (\lambda_1^k).
$$

So

$$
\frac{p(|x_{k+1}| + 1) - p(|x_k| + 1)}{|x_{k+1}| - |x_k|} = 1 + \frac{\gamma_1 \lambda_1 - \beta_1}{\beta_1 (\lambda_1 - 1)} + o(1).
$$
By Lemma 5.1 it follows that $\lim_{k \to \infty} \frac{p(|x_k| + 1)}{|x_k| + 1} = 1 + \frac{\gamma_1 \lambda_1 - \beta_1}{\beta_1 (\lambda_1 - 1)}$. In a similar way, we have

$$
p(|y_{k+1}| + 1) - p(|y_k| + 1) = \sum_{n=|y_k|+1}^{\gamma_1 \lambda_1^{k+1} - \gamma_1 \lambda_1^k + o(\lambda_1^k)} s(n)
$$

(5.8)

and

$$
|y_{k+1}| - |y_k| = \beta_1 \lambda_1^{k+1} - \beta_1 \lambda_1^k + o(\lambda_1^k).
$$

So, $\lim_{k \to \infty} \frac{p(|y_k| + 1)}{|y_k| + 1} = 1 + \frac{\gamma_1 \lambda_1 - \beta_1}{\gamma_1 (\lambda_1 - 1)}$.

Case 2. $m > 2l^2 + 1$. From Lemma 4.6, there exists $k_0$ such that for all $k \geq k_0$,

$$
|x_k| < |y_k| < |x_{k+1}| < |y_{k+1}|.
$$

In this case, we have

$$
\forall n \in [|x_k|, |x_{k+1}|], s(n) = \begin{cases} 
3 & \text{if } |x_k| < n \leq |y_k| \\
2 & \text{if } |y_k| < n \leq |x_{k+1}|.
\end{cases}
$$

So, $\liminf \frac{p(n)}{n} = \liminf \frac{p(|x_k| + 1)}{|x_k| + 1}$ and $\limsup \frac{p(n)}{n} = \limsup \frac{p(|y_k| + 1)}{|y_k| + 1}$. Furthermore

$$
p(|x_{k+1}| + 1) - p(|x_k| + 1) = \sum_{n=|x_k|+1}^{\gamma_1 \lambda_1^{k+1} - \gamma_1 \lambda_1^k + o(\lambda_1^k)} s(n)
$$

(5.9)

and

$$
|x_{k+1}| - |x_k| = \beta_1 \lambda_1^{k+1} - \beta_1 \lambda_1^k + o(\lambda_1^k).
$$

So

$$
\frac{p(|x_{k+1}| + 1) - p(|x_k| + 1)}{|x_{k+1}| - |x_k|} = 2 + \frac{\gamma_1 - \beta_1}{\beta_1 (\lambda_1 - 1)} + o(1).
$$

By Lemma 5.1 again it follows that

$$
\lim_{k \to \infty} \frac{p(|x_k| + 1)}{|x_k| + 1} = 1 + \frac{\gamma_1 \lambda_1 - \beta_1}{\beta_1 (\lambda_1 - 1)}.
$$
In a similar way, we have

\[ p(\|y_{k+1}\| + 1) - p(\|y_k\| + 1) = \sum_{n = \|y_k\| + 1}^{\|y_{k+1}\|} s(n) \]

\[ = 2 (\|x_{k+1}\| - \|y_k\|) + 3 (\|y_{k+1}\| - \|x_{k+1}\|) \]

\[ = 3 \gamma_1 \lambda_1^{k+1} - 2 \gamma_1 \lambda_1^{k} - \beta_1 \lambda_1^{k+1} + o(\lambda_1^k) \]

(5.10)

and

\[ |y_{k+1}| - |y_k| = \gamma_1 \lambda_1^{k+1} - \gamma_1 \lambda_1^{k} + o(\lambda_1^k) \]

So, \( \lim_{k \to \infty} \frac{p(\|y_k\| + 1)}{\|y_k\| + 1} = 2 + \beta_1 \frac{(\gamma_1 - \beta_1)}{\gamma_1 (\lambda_1 - 1)}. \)

6. Concluding remarks and further work

It results from Theorem 4.7 that:

if \( m \in [2, 2l^2 + 1] \), then \( 1 \leq \liminf \frac{p(n)}{n} \leq \limsup \frac{p(n)}{n} \leq 2 \); \hspace{1cm} (6.1)

if \( m > 2l^2 + 1 \), then \( 2 \leq \liminf \frac{p(n)}{n} \leq \limsup \frac{p(n)}{n} \leq 3 \). \hspace{1cm} (6.2)

By Theorem 5.3 one observes that for \( F_{l,m} \), the values of \( \liminf \frac{p(n)}{n} \) and \( \limsup \frac{p(n)}{n} \) are strictly dependent with those of the parameters \( l \) and \( m \). Indeed, we check that (6.1) and (6.2) become

if \( m \in [2, 2l^2 + 1] \), then \( 1 < \liminf \frac{p(n)}{n} < \limsup \frac{p(n)}{n} < 2 \); \hspace{1cm} (6.3)

if \( m > 2l^2 + 1 \), then \( 2 < \liminf \frac{p(n)}{n} < \limsup \frac{p(n)}{n} < 3 \). \hspace{1cm} (6.4)

For \( l \geq 1 \), \( m \geq 2 \), let us write \( \alpha_{l,m} = \liminf \frac{p(n)}{n} \) and \( \beta_{l,m} = \limsup \frac{p(n)}{n} \). In (6.1) the value 1 is reached when \( m = 1 \). In this case \( F_{l,m} \) is Sturmian and we have excluded it by taking \( m \geq 2 \). The values 2 and 3 are never reached, but we can prove that they are accumulation points for \( \alpha_{l,m} \) and \( \beta_{l,m} \).

In further work, it will be interesting to describe the region covered by the cloud of points \( (\alpha_{l,m}, \beta_{l,m}) \) in the first quadrant of the plane.

Another problem is to undertake a similar study in the case of \( S \)-adic words where morphisms are all generalized Fibonacci morphisms.

References


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