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ONE-RELATION LANGUAGES AND CODE GENERATORS

Vinh Duc Tran[1,*] and Igor Litovsky[2]

Abstract. We investigate the open problem to characterize whether the infinite power of a given language is generated by an ω-code. In case the given language is a code (*i.e.* zero-relation language), the problem was solved. In this work, we solve the problem for the class of one-relation languages.

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1. Introduction

In this paper, we deal with the infinite power (ω-power) of languages. The infinite power of L denoted by L^ω, is the set of infinite concatenations of words in L. A language L having the property that any infinite word (ω-word) of L^ω has a unique infinite factorization in words in L is called an ω-code [12], thus ω-codes are for infinite concatenation, like usual codes for concatenation. Of course ω-codes are codes, but the converse does not hold. We investigate the open problem to characterize languages L such that $L^\omega = G^\omega$ for some code or ω-code G. See [4, 5, 8] for partial answers and various approaches. This question is still open even if the language L is a finite language.

Given a language L, there does not always exist a greatest language M such that $L^\omega = M^\omega$, however it is "often" the case, if L is a finite language, for example. Whenever such a greatest ω-generator M exists, M is a semigroup. We know [5] that if this greatest ω-generator is a free semigroup, that is if $M = L^+$ for some code L, then $L^\omega = C^\omega$ for some ω-code C if and only if the language L itself is an ω-code. This means that whenever the greatest ω-generator is a free semigroup, that is for the class of zero-relation languages, the problem is already solved. So we consider in this paper, a new class of languages, called *one-relation languages*.

For each language $L = \{u_0, u_1, u_2, \dots\} \subseteq A^+$, we consider the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \dots\}$ which is a labelling of L, and two words m, m' in $\Sigma^+ \cup \Sigma^\omega$ are *equivalent*, denoted by $m \cong m'$, if the corresponding words in $A^+ \cup A^\omega$ are equal. Thus L is a code if and only if the previous equivalence relation is the identity in Σ^+, and L is an ω-code if and only if the previous equivalence relation is the identity in Σ^ω. Here we consider languages L having *only one* relation $m \cong m'$ with $m \neq m'$. Of course, if $m \cong m'$, then $m_1 m m_2 \cong m_1 m' m_2$, for any word m_1 and m_2 (more precisely the relation $\cong$ is a congruence relation), and if $xu^n z \cong yv^n t$ for each integer n, then $xu^\omega \cong yv^\omega$ (more precisely the relation $\cong$ is closed by adherence). We say that L is a *one-relation language* where $m \cong m'$ is the *basic relation*, if there is "not any other" relation, that is all relations are

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[1] Hanoi University of Science and Technology, Vietnam.
[2] Université de Nice - Sophia Antipolis, Laboratoire I3S, B.P. 145, 06903 Sophia Antipolis Cedex, France.

[*] Corresponding author: ductv@soict.hust.edu.vn

obtained by finite applications of the rewriting rule $m \to m'$ or by closure by adherence. So we can see the class of one-relation languages as the simplest class after the class of codes.

The purpose of this paper is to prove the two following results.

Theorem 1.1. *Let L be a one-relation language such that L^+ is the greatest generator of L^ω. Then L^ω has no finite code generator.*

Theorem 1.2. *Let L be a one-relation language such that L^+ is the greatest generator of L^ω. Then L^ω has a code generator if and only if the basic relation $u \cong v$ of L is one of the following forms:*

(i) $u = \mathtt{0}^n w\mathtt{2}$ *and* $v = \mathtt{1}\mathtt{0}^n$ *with* $w \in \Sigma^*$ *and* $n \geq 1$;
(ii) $u = \mathtt{0}^n\mathtt{2}$ *and* $v = \mathtt{1}\mathtt{0}^m\mathtt{0}^n$ *with* $m \geq 1$ *and* $n \geq 1$;
(iii) $u = \mathtt{0}w\mathtt{2}$ *and* $v = \mathtt{1}^k\mathtt{0}$ *with* $w \in \Sigma^*$ *and* $k \geq 2$;
(iv) $u = \mathtt{02}$ *and* $v = (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}$ *with* $k \geq 2$ *and* $m \geq 1$.

Moreover, L^ω has an ω-code generator if and only if the basic relation is one of forms (i) or (ii).

The paper is structured as follows. Section 2 contains the preliminaries. In Section 3, we give the definition of one-relation language and useful lemmas. In Section 4, we consider some basic results needed in later proofs. Section 5 is devoted to prove Theorem 1.1. Sections 6 and 7 are devoted to prove Theorem 1.2.

2. Preliminaries

Let A be an alphabet and A^* (resp. A^ω) is the set of all finite (resp. infinite or ω) words. The empty word is denoted by ε and A^+ denotes $A^* \setminus \{\varepsilon\}$. Let $x \in A^*$, we denote by $|x|$ the length of x. The subsets of A^* (resp. A^ω) are called languages (resp. ω-languages).

We denote by $A^\infty = A^* \cup A^\omega$ the set of finite or infinite words. We make A^∞ a monoid equipping it with the product defined as:

$$xy = \begin{cases} x, & \text{if } x \in A^\omega,\ y \in A^\infty \\ xy, & \text{if } x \in A^*,\ y \in A^\infty \end{cases}$$

for any words $x, y \in A^\infty$. Clearly, the empty word ε is the identity element of A^∞.

A word $x \in A^\infty$ is called a prefix (resp. factor) of a word $y \in A^\infty$ if $y \in xA^\infty$ (resp. $y \in A^*xA^\infty$); and a word $x \in A^\infty$ is suffix of a word or ω-word y if $y \in A^*x$. The language $\mathrm{Pref}(x)$ is the set of all prefixes of x. Let $X \subseteq A^\infty$, we define $\mathrm{Pref}(X) = \bigcup_{x\in X} \mathrm{Pref}(x)$. In a similar manner we define $\mathrm{Fact}(X)$ and $\mathrm{Suff}(X)$ for the set of factors and of suffixes.

A word $x \in A^+$ is called *primitive* if $x = y^n$ for $y \in A^+$ implies $n = 1$. For a word $x \in A^+$, the shortest $y \in A^+$ such that $x = y^n$ for some $n \geq 1$ is called *primitive root* of x and is denoted by $\rho(x)$.

Now we formulate, in the form of lemmas, several facts which are useful in the sequel.

Lemma 2.1 (see [10])**.** *Let $x \in A^+$ and $y, z \in A^*$. If we have $xz = yx$, then there exist two words α, β and a positive integer k such that $x = (\alpha\beta)^k\alpha$, $y = \alpha\beta$ and $z = \beta\alpha$.*

Lemma 2.2 (see [10])**.** *Two words $u, v \in A^+$ commute, that is $uv = vu$, if and only if they have the same primitive root.*

Lemma 2.3 (see [1])**.** *Let $x, y \in A^+$ and let $z, t \in \{x, y\}^*$. If xzy and ytx are both prefix or suffix of a same word, then x and y commute.*

Lemma 2.4. *If two words x, y satisfy the relation $xzy = ytx$ for some $z \in \{x, y\}^*$ and $t \in A^*$, then x and y commute.*

Proof. The proof is by induction on $|xy|$. If $x = \varepsilon$ or $y = \varepsilon$, then x and y commute. Assume that Lemma is true for all x, y where $|xy| < n$. We prove it for $|xy| = n$. If $|x| = |y|$ then $x = y$, so x and y commute. Now, as the role played by x and y is symmetric by mirror, we can assume that $|x| < |y|$, and then x is a proper prefix of y, we write $y = xx'$ with $x' \in A^+$. We have $xzxx' = xx'tx$, then $zxx' = x'tx$. Since $zx \in \{x, xx'\}^*x \subset x\{x, x'\}^*$, it follows that x is a prefix of zx. We set $z' = x^{-1}(zx) \in \{x, x'\}^*$. We have thus the relation $xz'x' = x'tx$. Moreover, if $x \neq \varepsilon$, then $|xx'| < |xy|$. By the induction hypothesis then x and x' commute. Thus $y = xx'$ and x commute. □

Lemma 2.5. *Let $x \in A^*$ and $y, z \in A^+$. We have $y^\omega = xz^\omega$ if and only if there exist two positive integers i and j such that $y^i x = xz^j$.*

Proof. If $y^\omega = xz^\omega$ then there are a positive integer n and a word $t \in A^*$ such that $y^n = xt$. Therefore we have $y^\omega = (xt)^\omega = xz^\omega$. Thus $(tx)^\omega = z^\omega$. Hence there are two positive integers k and j such that $(tx)^k = z^j$. We have

$$x(tx)^k = xz^j.$$

As $xt = y^n$, we have $y^i x = xz^j$ where $i = nk$.

Conversely, if there exist two positive integers i and j such that $y^i x = xz^j$, then we have $y^{ni}x = xz^{nj}$ for $n = 0, 1, 2, \ldots$. Hence two words y^ω and xz^ω has an infinity of common prefixes. Thus $y^\omega = xz^\omega$. □

Given a language $L \subseteq A^+$. We define

$$L^\omega = \{u_0 u_1 \cdots \mid \forall i \geq 0, u_i \in L\}$$

the language of ω-words generated by L. An ω-language of the form L^ω is said to be an ω-power. A generator of an ω-power L^ω is a language $G \subseteq A^+$ such that $G^\omega = L^\omega$.

The following lemma is used frequently to prove the equality of two ω-powers.

Lemma 2.6 (see [9])**.** *Let L and R be languages. If $L^\omega \subseteq RL^\omega$ then $L^\omega \subseteq R^\omega$.*

An L-factorization of a word $w \in A^*$ is a finite sequence $(w_1, \ldots, w_n)$ of words of L such that $w = w_1 \ldots w_n$. An L-factorization of an ω-word $w \in A^\omega$ is an infinite sequence $(w_1, w_2, \ldots)$ of words of L such that $w = w_1 w_2 \ldots$.

A language $C \subseteq A^+$ is a *code* (resp. ω-code) if any word in A^* (resp. any ω-word in A^ω) has at most one C-factorization.

We now present a characterization of codes based on the factorizations of infinite periodic words.

Proposition 2.7 (see [2])**.** *Let $C \subseteq A^+$ be a language. Then C is a code if and only if for every $u \in C^+$, u^ω has a single C-factorization.*

3. One-relation languages

Given a language L in A^*. Let Σ be an alphabet with the same cardinality as L. A one-to-one mapping from Σ onto L is called a labelling of L, denoted as $\tilde{}: \Sigma \to L$. This mapping is extended in the canonical morphism from $(\Sigma^\infty, \cdot)$ over $(L^\infty, \cdot)$, where $\cdot$ denotes the concatenation operation. Thus each L-factorization of a word in A^∞ is presented by a word in Σ^∞. By abuse of language, the subsets of Σ^∞ are called *languages of factorizations*. For a language $C \subseteq \Sigma^\infty$, we denote $\tilde{C} = \{\tilde{x} \mid x \in C\}$.

Let x and y two words in Σ^∞ such that $\tilde{x} = \tilde{y}$, we write $x \cong y$ that is called a relation in Σ^∞. A relation $x \cong y$ is called *nontrivial* if $x \neq y$. A nontrivial relation $x \cong y$ is called *minimal* if for all nonempty proper prefix x' of x and for all nonempty proper prefix y' of y, we have $x' \ncong y'$. We denote by $E(L)$ and $E_{\min}(L)$ the set of nontrivial relations and minimal relations, respectively.

When the language L is finite, the sets $E_{\min}(L)$ and $E(L)$ can be easily computed by using domino graphs (see [3]) or finite automata (see [10], page 446). In order to compute easily with the examples, we write a program in Java to construct the domino graph of finite language. Source code is available online[1].

An equivalence relation R on Σ^∞ is called a congruence if $(x, y) \in R$ and $u, v \in \Sigma^\infty$ imply $(uxv, uyv) \in R$.

For a pair $(u, v) \in \Sigma^\infty \times \Sigma^\infty$, we denote by $\mathrm{Pref}(u, v) = \mathrm{Pref}(u) \times \mathrm{Pref}(v)$. Let $R \subseteq \Sigma^\infty \times \Sigma^\infty$, we define $\mathrm{Pref}(R) = \bigcup_{(u,v)\in R} \mathrm{Pref}(u, v)$ and the *adherence* of R is defined by (see [11]):

$$Adh(R) = \{(x, y) \in \Sigma^\omega \times \Sigma^\omega \mid \mathrm{Pref}(x, y) \subseteq \mathrm{Pref}(R)\}$$

A relation R is *closed* (topologically) if $Adh(R) \subseteq R$.

Proposition 3.1. *The relation $\cong$ is a closed congruence.*

Proof. According to definition, the relation $\cong$ is a congruence. Moreover, if

$$(x, y) = (x_0x_1 \dots,\ y_0y_1 \dots) \in Adh(\cong)$$

with all $x_i, y_i \in \Sigma$, then $\mathrm{Pref}(x, y) \subseteq \mathrm{Pref}(\cong)$. Therefore, for all integer $i \geq 0$, there exist $u_i, v_i \in \Sigma^\infty$ satisfying

$$x_0 \dots x_i u_i \ \cong\ y_0 \dots y_i v_i \qquad \text{for all } i \geq 0.$$

Setting

$$p_i = \begin{cases} \tilde{x_0} \dots \tilde{x_i} & \text{if } |\tilde{x_0} \dots \tilde{x_i}| < |\tilde{y_0} \dots \tilde{y_i}| \\ \tilde{y_0} \dots \tilde{y_i} & \text{otherwise.} \end{cases}$$

Then $\tilde{x}$ and $\tilde{y}$ have an infinity of common prefixes: $p_0, p_1, \dots$. Hence we have $\tilde{x} = \tilde{y}$, that is $x \ \cong\ y$. Thus the relation $\cong$ is closed. ☐

Motivated by this fact, we introduce the following notion.

Definition 3.2. A language $L = L \setminus LL^+ \subseteq A^+$ is a *one-relation language* if there is a pair $(u, v) \in \Sigma^+ \times \Sigma^+$, $u \neq v$ such that $\cong$ is the smallest closed congruence relation on Σ^∞ which contains (u, v). The relation $u \ \cong\ v$ is called the *basic* relation of L.

Regarding the words length, there is only one basic relation up to symmetry, in a given one-relation language. Furthermore, this basic relation must be minimal.

The following examples show the variety of the class of one-relation languages.

Example 3.3. Consider the language $L = \{a, ab, bc, c\}$, the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}$, and the labelling $\{\tilde{\mathtt{0}} = a,\ \tilde{\mathtt{1}} = ab,\ \tilde{\mathtt{2}} = bc,\ \tilde{\mathtt{3}} = c\}$. L has only one minimal relation $\mathtt{02} \ \cong\ \mathtt{13}$. Thus the language L is a one-relation language.

Example 3.4. Consider the language $L = \{a, ab, ba\}$, the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$, and the labelling $\{\tilde{\mathtt{0}} = a,\ \tilde{\mathtt{1}} = ab,\ \tilde{\mathtt{2}} = ba\}$. The set of minimal relations of L is exactly the following system:

$$\begin{cases} \mathtt{02}^n \ \cong\ \mathtt{1}^n\mathtt{0} & \text{for } n = 1, 2, \dots \\ \mathtt{02}^\omega \ \cong\ \mathtt{1}^\omega \end{cases}$$

[1] https://github.com/tranvinhduc/dominograph

The shortest minimal relation of L is $\mathtt{02} \cong \mathtt{10}$. From this relation, we get $\mathtt{022} \cong \mathtt{110}$ by applying the rewriting rule

$$\mathtt{022} \cong \mathtt{102} \cong \mathtt{110};$$

and by applying the rewriting rule several times, we obtain

$$\mathtt{02}^n \cong \mathtt{1}^n\mathtt{0}, \quad \text{for } n = 1, 2, \ldots.$$

By adherence, we get the infinitary relation $\mathtt{02}^\omega \cong \mathtt{1}^\omega$. Thus every relations of L are obtained from this shortest relation by rewriting or by adherence. Thus L is one-relation language with the basic relation $\mathtt{02} \cong \mathtt{10}$.

Example 3.5. Consider $L = \{a, ab, baba\}$ and the alphabet $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. The set of minimal relations of L is exactly the following system:

$$\begin{cases} \mathtt{02}^n \cong (\mathtt{11})^n\mathtt{0} & \text{for } n = 1, 2, \ldots \\ \mathtt{02}^\omega \cong \mathtt{1}^\omega & \\ \mathtt{02}^\omega \cong \mathtt{1}(\mathtt{11})^m\mathtt{02}^\omega & \text{for } m = 0, 1, \ldots. \end{cases}$$

We can verify that the relations $\mathtt{02}^n \cong (\mathtt{11})^n\mathtt{0}$ and $\mathtt{02}^\omega \cong \mathtt{1}^\omega$ are obtained from the relation $\mathtt{02} \cong \mathtt{110}$ by rewriting or by closure by adherence. Moreover, for any $m \geq 0$, we have

$$\mathtt{02}^\omega \cong \mathtt{1}^\omega = \mathtt{1}(\mathtt{11})^m\mathtt{1}^\omega \cong \mathtt{1}(\mathtt{11})^m\mathtt{02}^\omega.$$

Thus L is a one-relation language where the basic relation is $\mathtt{02} \cong \mathtt{110}$.

It is noticed that a one-relation language is not a code as the basic relation $u \cong v$ is such that u and v are finite words. If the language L has only one relation $w \cong w'$ with $w, w' \in \Sigma^\omega$ such that all relations are obtained from this relation by rewriting, then L is a code. In this case, the problem was solved (see [5]).

We denote by $\mathrm{First}(x)$ and $\mathrm{Last}(x)$, respectively, the first and the last letter of a nonempty word x.

Lemma 3.6. *Let L be a one-relation language. Then the basic relation of L is not in the form $uv \cong vu$ with $u, v \in \Sigma^*$.*

Proof. Assume the contrary, that L is a one-relation language where the basic relation is $uv \cong vu$. By Lemma 2.2, two words $\tilde{u}$ and $\tilde{v}$ have the same primitive root. Thus there are two positive integers p and q such that $u^p \cong v^q$. By definition of the one-relation languages, the pair (u^p, v^q) is in the smallest congruence containing (uv, vu). Thus u^p contains the factor uv or the factor vu, this mean that there are two words $x, y \in \Sigma^*$ such that $u^p = xuvy$ or $u^p = xvuy$.

- If $u^p = xuvy$ then $uu^p = uxuvy = u^pu = xuvyu$. Since $|xu| = |ux|$, we have $uvy = vyu$. It follows that $\mathrm{First}(u) = \mathrm{First}(v)$, which conflicts the fact that the relation basic $uv \cong vu$ is minimal.
- If $u^p = xvuy$ then $uu^p = uxvuy = u^pu = xvuyu$. Thus we have $uxv = xvu$. It follows that $\mathrm{Last}(u) = \mathrm{Last}(v)$, which conflicts again the minimality of the basic relation $uv \cong vu$.

In both cases we obtain a contradiction. □

Lemma 3.7. *A one-relation language can not contain two words which commute.*

Proof. Assume the contrary, that there is a one-relation language L contains two words $\tilde{\mathtt{0}}$ and $\tilde{\mathtt{1}}$ such that $\mathtt{01} \cong \mathtt{10}$. Regarding the word length, the basic relation of L must be $\mathtt{01} \cong \mathtt{10}$, which contradicts Lemma 3.6. □

4. Generators and codes

From this section to the end of this paper, we make the assumption: *L is a language such that L^+ is the greatest generator of L^ω and L is in one-to-one mapping with the alphabet Σ.*

Note that this assumption is satisfied by some interesting cases, for example the case where L^ω is an ω-power of a finite language (see [9] and [7]).

We denote by $Amb_\Sigma(L)$ the set of ω-words in Σ^ω such that the images of these ω-words in A^ω have at least two L-factorizations. That is,

$$\begin{aligned} Amb_\Sigma(L) &= \{x \in \Sigma^\omega \mid \tilde{x} \text{ has at least two } L\text{-factorizations}\} \\ &= \{x \in \Sigma^\omega \mid \exists y \in \Sigma^\omega,\ x \cong y \text{ and } x \neq y\}. \end{aligned}$$

According to Proposition 2.7, the language L is a code if and only if the set $Amb_\Sigma(L)$ has no periodic words.

Lemma 4.1. *Let $C \subseteq \Sigma^+$ such that $\widetilde{C}$ is a generator of L^ω and let $w \in \Sigma^\omega$. If $w \notin Amb_\Sigma(L)$, then $w \in C^\omega$.*

Proof. Since $\widetilde{C}$ is a generator of L^ω, it follows that for each $w \in \Sigma^\omega$ there is an ω-word $w' \in C^\omega$ such that $w \cong w'$. If $w \notin Amb_\Sigma(L)$ then $w = w'$. So $w \in C^\omega$. □

A language $P \subseteq \Sigma^+$ is a *prefix code* if no word in P is a proper prefix of another word in P.

Lemma 4.2. *Let $C \subseteq \Sigma^+$ such that the language $\widetilde{C}$ is a code generator of L^ω. Then the language C is a prefix code over Σ.*

Proof. Assume the contrary that there exist two nonempty words $u, v \in \Sigma^+$ such that $\{u, uv\} \subseteq C$. Since $\widetilde{C}$ is a generator of L^ω, there exists $w \in C^\omega$ such that $(vu)^\omega \cong w$. Then $u(vu)^\omega = (uv)^\omega \cong uw$. As $\tilde{uv} \neq \tilde{u}$, the periodic word $(\tilde{uv})^\omega$ has two factorizations on $\widetilde{C}$: one starts by $\tilde{u}$ and the other by $\tilde{uv}$. According to Proposition 2.7, $\widetilde{C}$ is not a code. □

We say that two words $u, v \in \Sigma^+$ are *incompatibles* if there exist $x, y \in \Sigma^+$ such that the relation $ux \cong vy$ is minimal.

Remark 4.3. By the minimality of relation $ux \cong vy$, two incompatible words u and v must have no common prefix.

Let $X \subseteq \Sigma^\infty$. We denote by $\mathrm{Pref}_*(x) = \mathrm{Pref}(x) \setminus \{\varepsilon\}$.

Lemma 4.4. *Let $C \subseteq \Sigma^+$ such that $\widetilde{C}$ is a code generator of L^ω. Let u and v be two incompatible words. Then for all $m \in \Sigma^*$, the set $m\mathrm{Pref}_*(\{u, v\}) \cap C$ is the empty set or a singleton.*

Proof. By Lemma 4.2, the language C is a prefix code. For each $m \in \Sigma^*$, each set of $m\mathrm{Pref}_*(u) \cap C$ and $m\mathrm{Pref}_*(v) \cap C$ is either the empty set or a singleton. Therefore, it is sufficient to show that $m\mathrm{Pref}_*(u) = \emptyset$ or $m\mathrm{Pref}_*(v) = \emptyset$.

Assume the contrary that there exist $p \in \mathrm{Pref}_*(u)$ and $q \in \mathrm{Pref}_*(v)$ such that $\{mp, mq\} \subseteq C$. As two words u and v are incompatible, there exist $x, y \in \Sigma^+$ such that $px \cong qy$ is a minimal. Since $\widetilde{C}$ is a generator of L^ω, the infinite words $(\tilde{x})^\omega$ has a ultimately periodic $\widetilde{C}$-factorization, that is there exist $z \in C^*$ and $t \in C^+$ such that $x^\omega \cong zt^\omega$. According to Lemma 2.5, there are $i, j > 0$ such that $x^i z \cong zt^j$. Now we have

$$\begin{aligned} (mpzt^j)^\omega &\cong (mpxx^{i-1}z)^\omega \\ &\cong (mqyx^{i-1}z)^\omega = mq(yx^{i-1}zmq)^\omega. \end{aligned}$$

Since $\widetilde{C}$ is a generator of L^ω, there exists $w \in C^\omega$ such that

$$(yx^{i-1}zmq)^\omega \cong w.$$

Thus we have

$$(mpzt^j)^\omega \;\cong\; (mq)w.$$

We recall that $\widetilde{C}$ is a code and that $\{mp, mq\} \subseteq C$. If $\widetilde{mp} \neq \widetilde{pq}$ then the infinite periodic word $(\widetilde{mpzt^j})^\omega$ has two $\widetilde{C}$-factorizations, which contradicts Proposition 2.7. If $mp \;\cong\; mq$ then $p \;\cong\; q$, which contradicts the fact that $px \;\cong\; qy$ is minimal. This completes the proof. □

Here is a corollary of Lemma 4.4.

Corollary 4.5. *Let $C \subseteq \Sigma^+$ such that $\widetilde{C}$ is a code generator of L^ω. Let u and v be two incompatible words. Then for all $m \in \Sigma^*$, there exists $z \in \{u, v\}$ such that*

$$m\mathrm{Pref}_*(z) \cap C = \emptyset.$$

Proof. By Lemma 4.4, we have $m\mathrm{Pref}_*(u) \cap C = \emptyset$ or $m\mathrm{Pref}_*(v) \cap C = \emptyset$. □

Proposition 4.6. *Let $(\{u_i, v_i\})_{i\geq 0}$ be an infinite sequence of pairs of incompatible words. If*

$$Amb_\Sigma(L) \;\cap\; \prod_{i=0}^{\infty}\{u_i, v_i\} = \emptyset$$

where $\prod_{i=0}^{\infty} X_i$ represent the concatenation of languages X_i, then L^ω has no code generator.

Proof. Assume that there exists $C \subseteq \Sigma^+$ such that $\widetilde{C}$ is a code generator of L^ω. By induction, we build an infinite sequence $(z_i)_{i\geq 0}$ of words such that: for all $i \geq 0$, we have

$$\begin{cases} z_i \in \{u_i, v_i\} \\ \mathrm{Pref}_*(z_0 \ldots z_i) \cap C = \emptyset. \end{cases} \tag{4.1}$$

Indeed, according to Corollary 4.5, there exists $z_0 \in \{u_0, v_0\}$ such that

$$\mathrm{Pref}_*(z_0) \cap C = \emptyset.$$

Now assume that we have the sequence $(z_0, \ldots, z_{n-1})$ which verifies the condition (4.1). According to Corollary 4.5, there exists $z_n \in \{u_n, v_n\}$ such that

$$z_0 \ldots z_{n-1}\mathrm{Pref}_*(z_n) \cap C = \emptyset$$

and by induction hypothesis we have

$$\mathrm{Pref}_*(z_0 \ldots z_{n-1}) \cap C = \emptyset.$$

Then $z_0, \ldots, z_n$ verifies the condition (4.1).

Consider the ω-word

$$w = z_0 z_1 \ldots z_n \ldots \notin Amb_\Sigma(L),$$

according to Lemma 4.1, we have $w \in C^\omega$. However, by above construction we have $\mathrm{Pref}_*(w) \cap C = \emptyset$. This is a contradiction. □

By applying Proposition 4.6 for the infinite sequence of pair of incompatible words $(\{u,v\},\{u,v\},\ldots)$, we have.

Proposition 4.7. *Let u and v be two incompatible words. If*

$$Amb_\Sigma(L)\cap\{u,v\}^\omega=\emptyset,$$

then L^ω has no code generator.

Example 4.8. Let $L=\{a,ab,bc,c\}$ and $\Sigma=\{\mathtt{0},\mathtt{1},\mathtt{2},\mathtt{3}\}$. The language L has only one minimal relation $\mathtt{02}\cong\mathtt{13}$. Then $Amb_\Sigma(L)=\Sigma^*\{\mathtt{02},\mathtt{13}\}\Sigma^\omega$. We have

$$Amb_\Sigma(L)\ \cap\ \{\mathtt{0},\mathtt{1}\}^\omega=\emptyset.$$

As two words $\mathtt{0},\mathtt{1}$ are incompatibles, according to Proposition 4.7, L^ω has no code generator.

We say that two words $u,v\in\Sigma^+$ are ∞-*incompatibles* if there exists $x,y\in\Sigma^+\cup\Sigma^\omega$ such that the relation $ux\cong vy$ is minimal.

Remark 4.9. If two words u and v are incompatible, they are ∞-incompatible.

Lemma 4.10. *Let $C\subseteq\Sigma^+$ such that $\widetilde{C}$ is an ω-code generator of L^ω. Let u and v be two ∞-incompatible words. Then for all $m\in\Sigma^*$, the set $m\mathrm{Pref}_*(\{u,v\})\cap C$ is the empty set or a singleton.*

Proof. By Lemma 4.2, the language C is a prefix code. For each $m\in\Sigma^*$, each set of $m\mathrm{Pref}_*(u)\cap C$ and $m\mathrm{Pref}_*(v)\cap C$ is either the empty set or a singleton. Therefore, it is sufficient to show that $m\mathrm{Pref}_*(u)=\emptyset$ or $m\mathrm{Pref}_*(v)=\emptyset$.

Assume the contrary that there exist $p\in\mathrm{Pref}_*(u)$ and $q\in\mathrm{Pref}_*(v)$ such that $\{mp,mq\}\subseteq C$. As two words u and v are ∞-incompatible, there exist $x,y\in\Sigma^\omega$ such that $px\cong qy$ is a minimal relation. Then we have $(mp)x\cong(mq)y$. There are two cases:

- If $\tilde{mp}\neq\tilde{mq}$, then $\widetilde{C}$ is not an ω-code generator of L^ω;
- If $\tilde{mp}=\tilde{mq}$, that is $mp\cong mq$, then $p\cong q$ which contradicts the minimality of the relation $px\cong qy$.

In both cases we obtain a contradiction. □

The proof of following results is similar to the case of incompatible words.

Proposition 4.11. *Let $(\{u_i,v_i\})_{i\geq 0}$ be an infinite sequence of pairs of ∞-incompatible words. If*

$$Amb_\Sigma(L)\ \cap\ \prod_{i=0}^{\infty}\{u_i,v_i\}=\emptyset$$

then L^ω has no ω-code generator.

Proposition 4.12. *Let u and v be two ∞-incompatible words. If*

$$Amb_\Sigma(L)\cap\{u,v\}^\omega=\emptyset,$$

then L^ω has no ω-code generator.

Example 4.13. Let $L=\{a,ab,b^2\}$ be a suffix code and $\Sigma=\{\mathtt{0},\mathtt{1},\mathtt{2}\}$. The language L has only one minimal relation $\mathtt{02}^\omega\cong\mathtt{12}^\omega$. Thus $Amb_\Sigma(L)=\Sigma^*\{\mathtt{02}^\omega,\mathtt{12}^\omega\}$. We have therefore $Amb_\Sigma(L)\cap\{\mathtt{0},\mathtt{1}\}^\omega=\emptyset$. According to Proposition 4.12, the ω-language L^ω has no ω-code generator.

5. Proof of Theorem 1.1

Recall that L is a one-relation language with the basic relation $u \cong v$. Assume that there exists $C \subseteq \Sigma^+$ such that $\widetilde{C}$ is an ω-code generator of L^ω. According to Theorem 1.2, we consider two cases:

Case 1 (covers forms i and iii in Thm. 1.2): $u = \mathtt{0}^n w\mathtt{2}$ and $v = \mathtt{1}^k\mathtt{0}^n$ where $w \in \Sigma^*$ and $k, n \geq 1$.

Since $\mathtt{0}^\omega \notin Amb_\Sigma(L)$, there exists an integer $\ell \geq 0$ such that

$$\mathtt{0}^\ell\mathtt{0} \in C \tag{5.1}$$

It follows from the basic relation $\mathtt{0}^n w\mathtt{2} \cong \mathtt{1}^k\mathtt{0}^n$ that

$$\mathtt{0}^n(w\mathtt{2})^j \cong \mathtt{1}^{kj}\mathtt{0}^n \qquad \text{for} \quad j = 1, 2, \ldots.$$

Thus for each $j \geq 1$, two words $\mathtt{0}$ and $\mathtt{1}^j$ are incompatible. Combining Lemma 4.4 and (5.1) gives

$$\mathtt{0}^\ell \mathrm{Pref}_*\left(\{\mathtt{0}, \mathtt{1}^\omega\}\right) \cap C = \mathtt{0}^\ell\mathtt{0}.$$

Then

$$\mathrm{Pref}_*\left(\mathtt{0}^\ell\mathtt{1}^\omega\right) \cap C = \emptyset. \tag{5.2}$$

We show that $\mathtt{1}$ is not a prefix of w. Assume the contrary that $w = \mathtt{1}w'$ for some $w' \in \Sigma^*$. Thus $\widetilde{\mathtt{0}^n\mathtt{1}}$ and $\widetilde{\mathtt{1}^k\mathtt{0}}$ are both prefix of $\widetilde{\mathtt{1}^k\mathtt{0}^n}$. According to Lemma 2.3, two words $\tilde{\mathtt{0}}$ and $\tilde{\mathtt{1}}$ commute, which contradicts Lemma 3.7.

Thus we have

$$\mathtt{0}^\ell\mathtt{1}^+\mathtt{2}^\omega \cap Amb_\Sigma(L) = \emptyset.$$

It follows from Lemma 4.1 and (5.2) that for each integer $p > 0$, there exists an integer $q > 0$ such that $\mathtt{0}^\ell\mathtt{1}^p\mathtt{2}^q \in C$. Consequently, $\widetilde{C}$ is infinite.

Case 2 (covers forms ii and iv in Thm. 1.2): $u = \mathtt{0}^n\mathtt{2}$ and $v = (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n$ where $k, m, n \geq 1$.

Since $\mathtt{0}^\omega \notin Amb_\Sigma(L)$, there exists an integer $\ell \geq 0$ such that

$$\mathtt{0}^\ell\mathtt{0} \in C \tag{5.3}$$

It follows from the basic relation $\mathtt{0}^n\mathtt{2} \cong (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n$ that

$$\mathtt{0}^n\mathtt{2}^j \cong (\mathtt{1}\mathtt{0}^m)^{kj}\mathtt{0}^n \quad \text{for} \quad j = 1, 2, \ldots.$$

Thus for each word $x \in \mathrm{Pref}_*((\mathtt{1}\mathtt{0}^m)^\omega)$, the words $\mathtt{0}$ and x are incompatible. Combining Lemma 4.4 and (5.3), gives

$$\mathtt{0}^\ell \mathrm{Pref}_*\left(\{\mathtt{0}, (\mathtt{1}\mathtt{0}^m)^\omega\}\right) \cap C = \mathtt{0}^\ell\mathtt{0}.$$

Then

$$\mathrm{Pref}_*\left(\mathtt{0}^\ell(\mathtt{1}\mathtt{0}^m)^\omega\right) \cap C = \emptyset. \tag{5.4}$$

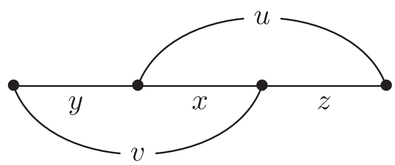


FIGURE 1. $x \in \mathrm{OVL}(u,v)$ and $(y,z) \in \mathrm{Border}(u,v)$.

Since $m \geq 1$, we get

$$\mathtt{0}^{\ell}(\mathtt{1}\mathtt{0}^{m})^{+}\mathtt{1}^{\omega} \cap Amb_{\Sigma}(L) = \emptyset.$$

It follows from Lemma 4.1 and (5.4) that for each integer $p > 0$, there exists an integer $q > 1$ such that $\mathtt{0}^{\ell}(\mathtt{1}\mathtt{0}^{m})^{p}\mathtt{1}^{q} \in C$. Consequently, $\widetilde{C}$ is infinite.

6. Technical lemmas

In this section, we establish the technical lemmas which we need to prove Theorem 1.2 in the following section.

Let u and v be two non-empty words. We denote by $\mathrm{OVL}(u,v)$ the set of overlapping words of the words u with the word v :

$$\mathrm{OVL}(u,v) = \{x \in \Sigma^{+} \mid u = xz,\ v = yx \ \text{ for some } y, z \in \Sigma^{+}\}.$$

A pair $(y,z) \in \Sigma^{+} \times \Sigma^{+}$ is a *border* of the pair of words (u,v) if there is a word $x \in \mathrm{OVL}(u,v)$ such that

$$u = xz \qquad \text{and} \qquad v = yx.$$

We denote by $\mathrm{Border}(u,v)$ the set of borders of the pair (u,v). This is illustrated in Figure 1.

Note that these notations are not symetric.

Example 6.1. Let $u = \mathtt{0102}$ and $v = \mathtt{1010}$. We have

$$\begin{aligned} &\mathrm{OVL}(u,v) = \{\mathtt{010}, \mathtt{0}\}, && \mathrm{Border}(u,v) = \{(\mathtt{1},\mathtt{2}), (\mathtt{101},\mathtt{102})\},\\ &\mathrm{OVL}(v,u) = \emptyset, && \mathrm{Border}(v,u) = \emptyset. \end{aligned}$$

Lemma 6.2. *Let L be a one-relation language where $u \cong v$ is the basic relation. The set of relations of L is exactly $M^{\#}$ where $M^{\#}$ is the smallest congruence containing*

$$M = (\varepsilon, u) \cdot B^{*} \cdot (v, \varepsilon) \ \cup \ (\varepsilon, u) \cdot B^{\omega},$$

with

$$B = \bigcup_{(x,y) \in \{u,v\}^{2}} \mathrm{Border}(x,y).$$

Proof. First, we show the inclusion $M^{\#} \subseteq \cong$. If $B = \emptyset$, the proof is immediate. Assume that $B \neq \emptyset$. We show that for each $(y,z) \in B$, we have

$$uz \ \cong \ yv. \tag{6.1}$$

We verify (6.1) in four cases:

(1) If $(y,z) \in \mathrm{Border}(u,u)$. By definition, there exists $x \in \Sigma^+$ such that $u = yx = xz$. We have thus $uz = yxz = yu \cong yv$.
(2) If $(y,z) \in \mathrm{Border}(u,v)$. By definition, there exists $x \in \Sigma^+$ such that $v = yx$ and $u = xz$. We have thus $uz \cong vz = yxz = yu \cong yv$.
(3) If $(y,z) \in \mathrm{Border}(v,u)$. By definition, there exists $x \in \Sigma^+$ such that $u = yx$ and $v = xz$. We have thus $uz = yxz = yv$.
(4) If $(y,z) \in \mathrm{Border}(v,v)$. By definition, there exists $x \in \Sigma^+$ such that $v = yx = xz$. We have thus $uz \cong vz = yxz = yv$.

Now we consider an infinite sequence of borners in B:

$$(y_1, z_1),\ (y_2, z_2),\ \ldots.$$

By (6.1), we have $uz_i \cong y_i v$ for $i = 1, 2, \ldots$. Thus for any positive integer n, we have

$$\begin{aligned}(uz_1)z_2 \ldots z_n &\cong (y_1 v) z_2 z_3 \ldots z_n \\ &\cong (y_1 u) z_2 z_3 \ldots z_n = y_1 (u z_2) z_3 \ldots z_n \\ &\vdots \\ &\cong y_1 \ldots y_{n-1} y_n v.\end{aligned}$$

Since $\cong$ is closed (by adherence), we obtain

$$uz_1 z_2 \ldots \cong y_1 y_2 \ldots.$$

Thus $M \subseteq \cong$. Since $\cong$ is a congruence, we have $M^\# \subseteq \cong$.

Now we show $\cong \subseteq M^\#$. It is clear that the congruence $M^\#$ is closed. As $(u,v) \in M^\#$, by definition of one-relation language, we have the inclusion. □

We denote by $\mathrm{LB}(u,v)$ the first projection of $\mathrm{Border}(u,v)$. By Lemma 6.2, we have

Lemma 6.3. *Let L be a one-relation language where the basic relation is $u \cong v$. We have*

$$Amb_\Sigma(L) = \Sigma^*\{u,v\}\Sigma^\omega \cup \Sigma^*\Big(\mathrm{LB}(u,u) \cup \mathrm{LB}(v,v) \cup \mathrm{LB}(u,v) \cup \mathrm{LB}(v,u)\Big)^\omega.$$

If $\mathrm{OVL}(u,v) \neq \emptyset$ then $\mathrm{OVL}(u,v)$ has a unique greatest (by the length) element which will denote by $O_{u,v}$, and $\mathrm{LB}(u,v)$ has a unique smallest element which will denote by $b_{u,v}$. We have thus

$$v = b_{u,v} O_{u,v}.$$

Example 6.4. Let $u = \mathtt{0102}$ and $v = \mathtt{1010}$. We have

$$\mathrm{OVL}(u,v) = \{\mathtt{010}, \mathtt{0}\}, \qquad \mathrm{LB}(u,v) = \{\mathtt{1}, \mathtt{101}\},$$
$$O_{u,v} = \mathtt{010}, \qquad b_{u,v} = \mathtt{1}.$$

For a word $u \in \Sigma^\infty$, we denote by $\mathrm{Alph}(u)$ the set of letters of Σ appearing in u.

Lemma 6.5. *Let $u \in \Sigma^+$. For any $y \in \mathrm{LB}(u,u)$, we have $\mathrm{Alph}(y) = \mathrm{Alph}(u)$.*

Proof. For any $y \in \mathrm{LB}(u,u)$, there exist two non-empty word x and z such that $u = xz = yx$. By Lemma 2.1, there exist two words α, β and an integer k such that $x = (\alpha\beta)^k\alpha$, $y = \alpha\beta$, and $z = \beta\alpha$. Thus $\mathrm{Alph}(u) = \mathrm{Alph}(\alpha\beta) = \mathrm{Alph}(y)$. □

Lemma 6.6. *For any word $u \in \Sigma^+$, we have $(\mathrm{LB}(u,u))^\omega \subseteq u\Sigma^\omega$.*

Proof. If $\mathrm{LB}(u,u) = \emptyset$, the claim is trivial. Assume $\mathrm{LB}(u,u) \neq \emptyset$. By 6.1, for any $y \in \mathrm{LB}(u,u)$, there exists a word z such that $uz = yu$. By induction, for any sequence $y_1, y_2, \dots \in \mathrm{LB}(u,u)$, there are $z_1, z_2, \dots$ such that

$$y_1 y_2 \dots y_n u = u z_1 z_2 \dots z_n, \qquad \text{for } n = 1, 2, \dots.$$

By adhrence, we have

$$y_1 y_2 \dots = u z_1 z_2 \dots.$$

Thus $(\mathrm{LB}(u,u))^\omega \subseteq u\Sigma^\omega$. □

Lemma 6.7. *Let $u, v \in \Sigma^+$. If $\mathrm{OVL}(u,v) \neq \emptyset$ then for any $w \in \mathrm{LB}(u,v) \setminus \{b_{u,v}\}$, we have $\mathrm{Alph}(w) = \mathrm{Alph}(v)$.*

Proof. By definition, we have

$$\mathrm{OVL}(u,v) = \{O_{u,v}\} \cup \mathrm{OVL}(O_{u,v}, O_{u,v}).$$

Thus

$$\mathrm{LB}(u,v) = \{b_{u,v}\} \cup \{b_{u,v}\}\mathrm{LB}(O_{u,v}, O_{u,v}).$$

By Lemma 6.5, for any $y \in \mathrm{LB}(O_{u,v}, O_{u,v})$, we have $\mathrm{Alph}(y) = \mathrm{Alph}(O_{u,v})$; we obtain thus

$$\mathrm{Alph}(b_{u,v}y) = \mathrm{Alph}(b_{u,v}O_{u,v}) = \mathrm{Alph}(u),$$

which completes the proof. □

Lemma 6.8. *Let $\mathtt{0}, \mathtt{1}$ be two distinct letters and let x be a word. Then*

$$x\mathtt{0} \notin \mathrm{Fact}(\{x\mathtt{1}, \mathtt{1}\}^\omega).$$

Proof. We denote by $|x|_{\mathtt{1}}$ the number of letters $\mathtt{1}$ in x. Let $y \in \mathrm{Fact}(\{x\mathtt{1}, \mathtt{1}\}^\omega)$ and $|y| = |x| + 1$, we have $|y|_{\mathtt{1}} > |x|_{\mathtt{1}} = |x\mathtt{0}|_{\mathtt{1}}$. Thus $x\mathtt{0} \notin \mathrm{Fact}(\{x\mathtt{1}, \mathtt{1}\}^\omega)$. □

7. Proof of Theorem 1.2

Recall that L is a one-relation language where the basic relation is $u \cong v$. We set $\mathtt{0} = \mathrm{First}(u)$ and $\mathtt{1} = \mathrm{First}(v)$. Two letters $\mathtt{0}$ and $\mathtt{1}$ are distinct because the relation $u \cong v$ is minimal. Since the roles of u and v are symmetric, we consider three cases.

7.1. When $\mathrm{OVL}(u,v) = \emptyset$ and $\mathrm{OVL}(v,u) = \emptyset$

We prove that L^ω has no code generator in this case. Indeed, we have $\mathrm{LB}(u,v) = \mathrm{LB}(v,u) = \emptyset$. By Lemma 6.3, we have

$$Amb_\Sigma(L) = \Sigma^*\{u,v\}\Sigma^\omega \cup \Sigma^* \left(\mathrm{LB}(u,u) \cup \mathrm{LB}(v,v)\right)^\omega. \tag{7.1}$$

As the roles of u and v are symmetric, we consider three cases.

Case 1: $u \notin \{0,1\}^+$ and $v \notin \{0,1\}^+$.
By Lemma 6.5, we have

$$\mathrm{LB}(u,u) \cap \{0,1\}^+ = \emptyset \quad \text{and} \quad \mathrm{LB}(v,v) \cap \{0,1\}^+ = \emptyset.$$

By (7.1), we have $Amb_\Sigma(L) \cap \{0,1\}^\omega = \emptyset$. Since two words 0 and 1 are incompatible, it follows from Proposition 4.7 that L^ω has no code generator.

Case 2: $u \in \{0,1\}^+$ and $v \notin \{0,1\}^+$.
By (7.1) we have

$$\begin{aligned} Amb_\Sigma(L) \cap \{0,1\}^\omega &= \{0,1\}^* u \{0,1\}^\omega \cup \{0,1\}^* \left(\mathrm{LB}(u,u)\right)^\omega \\ &= \{0,1\}^* u \{0,1\}^\omega && \text{(by Lem. 6.6)} \end{aligned}$$

Since $\mathrm{OVL}(v,u) = \emptyset$ and $\mathrm{First}(v) = 1$, we have $\mathrm{Last}(u) = 0$. Because $|u| > 1$, we can write $u = 0z0$ with $z \in \{0,1\}^*$. The basic relation becomes $0z0 \;\cong\; v$. Then we have

$$0zv \;\cong\; 0z0z0 \;\cong\; vz0.$$

Since $\mathrm{First}(v) = 1$, it follows that two words $0z1$ and 1 are incompatible. By Lemma 6.8, we have

$$u = 0z0 \notin \mathrm{Fact}(\{0z1, 1\}^\omega).$$

Thus $Amb_\Sigma(L) \cap \{0z1, 1\}^\omega = \emptyset$. By the Proposition 4.7, L^ω has no code generator.

Case 3: $u \in \{0,1\}^+$ and $v \in \{0,1\}^+$.
By Lemma 2.4, two words $\tilde{0}$ and $\tilde{1}$ commute. This contradicts Lemma 3.7.

Example 7.1. The following languages show that there exist one-relation languages where their basic relations as in Case 1 and Case 2.

1. Let $L = \{a, ab, bc, c\}$ and $\Sigma = \{0,1,2,3\}$. The language L is a one-relation language where $02 \;\cong\; 13$ is the basic relation.
2. Let $H = \{ab, aba, b\}$ and $\Sigma = \{0,1,2\}$. The language H is a one-relation language where the basic relation is $00 \;\cong\; 12$.

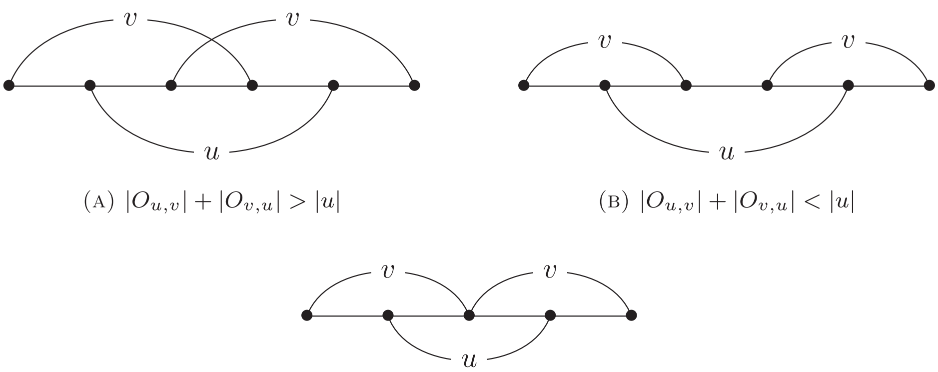
7.2. When $\mathrm{OVL}(u,v) \neq \emptyset$ and $\mathrm{OVL}(v,u) \neq \emptyset$

We prove that L^ω has no code generator in this case. The situations are illustrated in Figure 2.
We first prove that

$$|u| > |O_{u,v}| + |O_{v,u}| \quad \text{and} \quad |v| > |O_{u,v}| + |O_{v,u}|. \tag{7.2}$$

Conversely (to obtain a contradiction), suppose $|u| \leq |O_{u,v}| + |O_{v,u}|$. Since $|\tilde{u}| = |\tilde{v}|$, it follows that $|v| \leq |O_{u,v}| + |O_{v,u}|$. Then the basic relation $u \cong v$ can be rewritten in the form

$$u_1 u_2 u_3 \;\cong\; v_1 v_2 v_3,$$



(A) $|O_{u,v}| + |O_{v,u}| > |u|$ (B) $|O_{u,v}| + |O_{v,u}| < |u|$

(C) $|O_{u,v}| + |O_{v,u}| = |u|$

FIGURE 2. Situations occur when $\mathrm{OVL}(u, v) \neq \emptyset$ and $\mathrm{OVL}(v, u) \neq \emptyset$.

where $u_i, v_i \in \Sigma^*$; $u_1u_2 = v_2v_3 = O_{u,v}$; and $u_2u_3 = v_1v_2 = O_{v,u}$. Hence

$$u_1v_1v_2 \;\cong\; v_1u_1u_2.$$

As $|\widetilde{u_1v_1}| = |\widetilde{v_1u_1}|$, we have $u_1v_1 \cong v_1u_1$ and $v_2 \cong u_2$. Since $u_1v_1v_2 \cong v_1u_1u_2$ is a minimal relation, it follows that $u_2 = v_2 = \varepsilon$ or $u_1v_1 = v_1u_1 = \varepsilon$. However, if $u_2 = v_2 = \varepsilon$ then the basic relation becomes $u_1v_1 \cong v_1u_1$ which contradicts Lemma 3.6. If $u_1v_1 = v_1u_1 = \varepsilon$ then $O_{u,v} = u_2 = v$, a contradiction. We get (7.2).

Now the basic relation can be rewritten of the form

$$xzt \;\cong\; tyx$$

where $x, y, z, t \in \Sigma^+$; $x = O_{u,v}$; and $t = O_{v,u}$. Recall that $\mathrm{First}(x) = \mathtt{0}$ and $\mathrm{First}(t) = \mathtt{1}$. There are three cases:

Case 1: $xz \in \{\mathtt{0},\mathtt{1}\}^+$ and $ty \in \{\mathtt{0},\mathtt{1}\}^+$.
It follows from Lemma 2.4 that two words $\tilde{0}$ and $\tilde{1}$ commute. This contradicts Lemma 3.7.

Case 2: $xz \notin \{\mathtt{0},\mathtt{1}\}^+$ and $ty \notin \{\mathtt{0},\mathtt{1}\}^+$.
By Lemma 6.3, we have

$$Amb_\Sigma(L) \cap \{\mathtt{0},\mathtt{1}\}^\omega = \emptyset.$$

By Proposition 4.7, L^ω has no code generator.

Case 3: $xz \in \{\mathtt{0},\mathtt{1}\}^+$ and $ty \notin \{\mathtt{0},\mathtt{1}\}^+$.
We first prove that $t \notin \{\mathtt{0},\mathtt{1}\}^+$. Conversely (to obtain a contradiction), suppose $t \in \{\mathtt{0},\mathtt{1}\}^+$. If $\mathrm{Last}(t) = \mathtt{0}$ then by Lemma 2.3, two words $\tilde{\mathtt{0}}$ and $\tilde{\mathtt{1}}$ commute. Otherwise $\mathrm{Last}(t) = \mathtt{1}$, by minimality of the basic relation, we have $\mathrm{Last}(x) = \mathtt{0}$; according to Lemma 2.4, two words $\tilde{\mathtt{0}}$ and $\tilde{\mathtt{1}}$ commute. Both cases yield a contradiction with Lemma 3.7.

Now we have $t \notin \{\mathtt{0},\mathtt{1}\}^+$, then

$$\mathrm{LB}(xzt, xzt) \cap \{\mathtt{0},\mathtt{1}\}^+ = \emptyset, \qquad \mathrm{LB}(tyx, tyx) \cap \{\mathtt{0},\mathtt{1}\}^+ = \emptyset,$$

$$\mathrm{LB}(xzt, tyx) \cap \{\mathtt{0},\mathtt{1}\}^+ = \emptyset$$

and by Lemma 6.7, we have $\mathrm{LB}(tyx, xzt) \cap \{\mathtt{0}, \mathtt{1}\}^+ = xz$. We have thus

$$Amb_\Sigma(L) \cap \{\mathtt{0}, \mathtt{1}\}^\omega = \{\mathtt{0}, \mathtt{1}\}^*(xz)^\omega.$$

By Lemma 6.8, we have $xz\mathtt{0} \notin \mathrm{Fact}(\{xz\mathtt{1}, \mathtt{1}\}^\omega)$. As $\mathrm{First}(x) = \mathtt{0}$, we have $(xz)^\omega \notin \mathrm{Suff}(\{xz\mathtt{1}, \mathtt{1}\}^\omega)$. Therefore

$$Amb_\Sigma(L) \cap \{xz\mathtt{1}, \mathtt{1}\}^\omega = \emptyset$$

Since $\mathrm{First}(t) = \mathtt{1}$, two words $xz\mathtt{1}$ and $\mathtt{1}$ are incompatible. According to Lemma 4.7, L^ω has no code generator.

The following example show that there exist languages as in Case 2.

Example 7.2. Let $L = \{a, aba, bac, cab\}$ and let $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}$. It can be verified that L is a one-relation language where the basic relation is $\mathtt{021} \cong \mathtt{130}$.

7.3. When $\mathrm{OVL}(u, v) \neq \emptyset$ and $\mathrm{OVL}(v, u) = \emptyset$

We deal this case by some claims which are proven later.

Claim 7.3. *Assume that the basic relation of L is of the form*

$$\mathtt{0}xz \ \cong \ \mathtt{1}y\mathtt{0}x \tag{7.3}$$

where $z \in \Sigma^+$; $x, y \in \Sigma^*$; and $\mathtt{0}x = O_{u,v}$ is the greatest (by the length) element of $\mathrm{OVL}(u, v)$. *If $x \notin \mathtt{0}^*$ or $\rho(\mathtt{1}y) \notin \mathtt{10}^*$, then L^ω has no code generator.*

Now there remain the cases: $u = \mathtt{0}^n z$ and $v = (\mathtt{10}^m)^k \mathtt{0}^n$ with $k \geq 1, m \geq 0$, $n \geq 1$, and $\mathtt{0}^n = O_{u,v}$. It is to be noticed that $\mathrm{Last}(z) \notin \{\mathtt{0}, \mathtt{1}\}$ as $\mathrm{OVL}(v, u) = \emptyset$.

Claim 7.4. *Assume that the basic relation of L is of the form*

$$\mathtt{0}^n z \ \cong \ (\mathtt{10}^m)^k \mathtt{0}^n$$

with $\mathtt{0}^n = O_{u,v}$ and $\mathrm{Last}(z) \notin \{\mathtt{0}, \mathtt{1}\}$.

If the parameters z, k, m, n of the basic relation satisfies one of two following conditions:

1. *$k \geq 2, m \geq 0$, and $n \geq 2$;*
2. *$|z| \geq 2$, $k \geq 1, m \geq 1$, and $n \geq 1$.*

Then L^ω has no code generator.

According to Claims 7.3 and 7.4, there remains to prove that L^ω has always a code generator in the cases where the basic relation is in the form

$$\mathtt{0}^n z \ \cong \ (\mathtt{10}^m)^k \mathtt{0}^n$$

with

(i) $k = 1, m = 0, n \geq 1$, and $z \in \Sigma^*\mathtt{2}$;
(ii) $k = 1, m \geq 1, n \geq 1$, and $z = \mathtt{2}$;
(iii) $k > 1, m = 0, n = 1$, and $z \in \Sigma^*\mathtt{2}$; or
(iv) $k > 1, m \geq 1, n = 1$, and $z = \mathtt{2}$.

Claim 7.5 (Case i)**.** *Assume that the basic relation of L is of the form*

$$\mathtt{0}^n z \ \cong \ \mathtt{10}^n$$

with $n \geq 1$ and $z \in \Sigma^*\mathtt{2}$. Then $\widetilde{C}$ is an infinite ω-code generator of L^ω, where

$$C = \{\mathtt{0}\} \ \cup \ \left(\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i\right)^* (\Sigma \setminus \{\mathtt{0},\mathtt{1}\}).$$

Claim 7.6 (Case ii)**.** *Assume that the basic relation of L is of the form*

$$0^n 2 \ \cong \ 10^m 0^n$$

with $m \geq 1$ and $n \geq 1$. Then $\widetilde{C}$ is an infinite ω-code generator of L^ω, where

$$C = \{0,2\} \cup \left(\bigcup_{i=0}^{n-1} 10^m 0^i\right)^* \left(\bigcup_{i=0}^{m-1} 10^i (\Sigma \setminus \{0,2\}) \cup \bigcup_{i=0}^{n-1} 10^i 2 \ \cup \Sigma \setminus \{0,1,2\}\right).$$

Claim 7.7 (Case iii)**.** *Assume that the basic relation of L is of the form*

$$\mathtt{0}z \ \cong \ \mathtt{1}^k\mathtt{0}$$

with $z \in \Sigma^*\mathtt{2}$ and $k > 1$. Then $\widetilde{C}$ is an infinite code generator of L^ω, where

$$C = \bigcup_{i=0}^{k-1} \mathtt{1}^i\mathtt{0} \ \cup \ \mathtt{1}^* \left(\Sigma \setminus \{\mathtt{0},\mathtt{1}\}\right).$$

Moreover, L^ω has no ω-code generator.

Claim 7.8 (Case iv)**.** *Assume that the basic relation of L is of the form*

$$\mathtt{02} \ \cong \ (\mathtt{10}^m)^k\mathtt{0}$$

with $k > 1$ and $m \geq 1$. Then $\widetilde{C}$ is an infinite code generator of L^ω, where

$$C = \{\mathtt{2}\} \ \cup \ \bigcup_{i=0}^{k-1} (\mathtt{10}^m)^i\mathtt{0} \cup \ (\mathtt{10}^m)^* \left(\{\mathtt{12}\} \ \cup \ \mathtt{1}\left(\bigcup_{i=0}^{m-1} \mathtt{0}^i\right) (\Sigma \setminus \{\mathtt{0},\mathtt{2}\}) \ \cup \ \Sigma \setminus \{\mathtt{0},\mathtt{1},\mathtt{2}\}\right).$$

Moreover, L^ω has no ω-code generator.

Example 7.9. The following languages show that there exist one-relation languages where their basic relations as in Cases i-iv.

(i) Let $L = \{a, ab, ba\}$ and $\Sigma = \{\mathtt{0},\mathtt{1},\mathtt{2}\}$. The language L is a one-relation language where the basic relation is $\mathtt{02} \ \cong \ \mathtt{10}$. By Claim 7.5, the ω-language L^ω has an infinite ω-code generator $\widetilde{C}$ with $C = \mathtt{0} \cup \mathtt{1}^*\mathtt{2}$.

(ii) Let $H = \{a, a^2b, ba^4\}$ and $\Sigma = \{\mathtt{0},\mathtt{1},\mathtt{2}\}$. It can be verified that H is a one-relation language where the basic relation is $\mathtt{0}^2\mathtt{2} \ \cong \ \mathtt{10}^2\mathtt{0}^2$. By Claim 7.6, the ω-language H^ω has an infinite ω-code generator $\widetilde{C}$ with

$$C = \{\mathtt{0},\mathtt{2}\} \cup \{\mathtt{10}^2, \mathtt{10}^2\mathtt{0}\}^*\{\mathtt{11},\mathtt{101},\mathtt{12},\mathtt{102}\}.$$

(iii) Let $I = \{a, ab, baba\}$ and $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. It can be verified that I is a one-relation language where the basic relation is $\mathtt{02} \cong \mathtt{110}$. By Claim 7.7, the ω-language I^ω has an infinite code generator $\widetilde{C}$ with $C = \{\mathtt{0}, \mathtt{10}\} \cup \mathtt{1}^*\mathtt{2}$.

(iv) Let $J = \{a, ab, ba^3ba^3\}$ and $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. It can be verified that J is a one-relation language where the basic relation is $\mathtt{02} \cong (\mathtt{100})^2\mathtt{0}$. By Claim 7.8, the ω-language J^ω has an infinite code generator $\widetilde{C}$ with

$$C = \{\mathtt{0}, \mathtt{1000}, \mathtt{2}\} \cup (\mathtt{100})^*\{\mathtt{12}, \mathtt{11}, \mathtt{101}\}.$$

7.3.1. Proof of Claim 7.3

Assume that the basic relation of L is $\mathtt{0}xz \cong \mathtt{1}y\mathtt{0}x$. First, we prove that

$$xz \notin \{\mathtt{0}, \mathtt{1}\}^+. \tag{7.4}$$

Conversely (to obtain a contradiction), suppose $x \in \{\mathtt{0}, \mathtt{1}\}^*$ and $z \in \{\mathtt{0}, \mathtt{1}\}^+$. By minimality of relation $\mathtt{0}xz \cong \mathtt{1}y\mathtt{0}x$, we have $\mathrm{Last}(z) \neq \mathrm{Last}(\mathtt{0}x)$.

- If $\mathrm{Last}(\mathtt{0}x) = \mathtt{0}$ and $\mathrm{Last}(z) = \mathtt{1}$, then $\mathtt{1} \in \mathrm{OVL}(v, u)$, this contradicts $\mathrm{OVL}(v, u) = \emptyset$.
- Otherwise $\mathrm{Last}(\mathtt{0}x) = \mathtt{1}$ and $\mathrm{Last}(z) = \mathtt{0}$, then we can rewrite the basic relation in the form

$$\mathtt{0}x'\mathtt{1}z'\mathtt{0} \cong \mathtt{1}y\mathtt{0}x'\mathtt{1}$$

where $x'\mathtt{1} = x$ and $z'\mathtt{0} = z$. Thus $\widetilde{\mathtt{1}z'\mathtt{0}}$ and $\widetilde{\mathtt{0}x'\mathtt{1}}$ are both suffix of the word $\widetilde{\mathtt{0}x'\mathtt{1}z'\mathtt{0}} = \widetilde{\mathtt{1}y\mathtt{0}x'\mathtt{1}}$. By Lemma 2.3, two words $\tilde{\mathtt{0}}$ and $\tilde{\mathtt{1}}$ commute. This contradicts Lemma 3.7.

Thus, we get (7.4).

Now, we consider three cases:

Case 1: $y \notin \{\mathtt{0}, \mathtt{1}\}^*$.
Since $xz \notin \{\mathtt{0}, \mathtt{1}\}^+$ and by Lemma 6.3, we have

$$Amb_\Sigma(L) \cap \{\mathtt{0}, \mathtt{1}\}^\omega = \emptyset.$$

According to Proposition 4.7, L^ω has no code generator.

The following example shows that there exists one-relation languages with the basic relation as in *Case 1*.

Example 7.10. Let $L = \{a, ab, bca, c\}$ and let $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}$. The language L is one-relation language where the basic relation is $\mathtt{02} \cong \mathtt{130}$.

Case 2: $y \in \{\mathtt{0}, \mathtt{1}\}^*$ and $x \notin \mathtt{0}^*$.
First, we show that $x \notin \{\mathtt{0}, \mathtt{1}\}^+$. Indeed, if $x \in \{\mathtt{0}, \mathtt{1}\}^+$ then, by Lemma 2.3 and the relation $\mathtt{0}xz \cong \mathtt{1}y\mathtt{0}x$, two words $\tilde{\mathtt{0}}$ and $\tilde{\mathtt{1}}$ commute, which contradicts Lemma 3.7.

Now, using Lemma 6.3, we have

$$Amb_\Sigma(L) \cap \{\mathtt{0}, \mathtt{1}\}^\omega = \{\mathtt{0}, \mathtt{1}\}^*(\mathtt{1}y)^\omega.$$

By Lemma 6.8, we have $\mathtt{1}y\mathtt{1} \notin \mathrm{Fact}(\{\mathtt{0}, \mathtt{1}y\mathtt{0}\}^\omega)$. Thus

$$Amb_\Sigma(L) \cap \{\mathtt{0}, \mathtt{1}y\mathtt{0}\}^\omega = \emptyset.$$

Two words $\mathtt{0}$ and $\mathtt{1}y\mathtt{0}$ are incompatible as $\mathtt{0}xz \cong \mathtt{1}y\mathtt{0}x$. By Proposition 4.7, L^ω has no code generator.

The following example shows that there exists one-relation language with the basic relation as in *Case 2*.

Example 7.11. Let $L = \{ab, abc, bcaba, caba\}$ and let $\Sigma = \{0, 1, 2, 3\}$. It can be verified that L is a one-relation language where the basic relation is $032 \cong 1003$.

Case 3: $y \in \{0,1\}^*$, $\rho(1y) \notin 10^*$, and $0x = 0^n$ with $n \geq 1$.
Because $xz \notin \{0,1\}^*$ we can write $z = z_{01}2z'$ where $z_{01} \in \{0,1\}^*$ and $z' \in \Sigma^*$. Now the basic relation is in the form $0^n z_{01}2z' \cong 1y0^n$. By Lemma 3.7, L cannot contain two words commute, and then by Lemma 2.3, we have $z_{01} \in 0^*$. Thus the basic relation can be written in the form

$$0^n 0^k 2z' \;\cong\; 1y0^n \tag{7.5}$$

with $k \geq 0$.

By Lemma 6.3, we have

$$Amb_\Sigma(L) \cap \{0,1\}^\omega = \{0,1\}^* 1y0^n \{0,1\}^\omega \cup \{0,1\}^* \left(\mathrm{LB}(0^n 0^k 2z', 1y0^n) \cup \mathrm{LB}(1y0^n, 1y0^n)\right)^\omega . \tag{7.6}$$

Consider two subcases:

Subcase 3.1: $\mathrm{OVL}(1y0^n, 1y0^n) \neq \emptyset$.
That is, there are $f \in \{0,1\}^*$ and $p \in \{0,1\}^*$ such that $1y0^n = 1pf1p$. The basic relation (7.5) becomes

$$0^n 0^k 2z' \;\cong\; 1pf1p.$$

And we have

$$0^n 0^k 2z' f1p \;\cong\; 1pf1pf1p \;\cong\; 1pf0^n 0^k 2z'.$$

Thus two words 0 and $1pf0$ are incompatible.

By Lemma 6.8, we have $1pf1 \notin \mathrm{Fact}(\{0, 1pf0\}^\omega)$, then $1y \notin \mathrm{Fact}(\{0, 1pf0\}^\omega)$. Thus

$$Amb_\Sigma(L) \cap \{0, 1pf0\}^\omega = \emptyset.$$

By Proposition 4.7, L^ω has no code generator.

Subcase 3.2: $\mathrm{OVL}(1y0^n, 1y0^n) = \emptyset$.
From the equation (7.6), we have

$$Amb_\Sigma(L) \cap \{0,1\}^\omega = \{0,1\}^* 1y0^n \{0,1\}^\omega \cup \{0,1\}^* \{1y, 1y0, \cdots, 1y0^{n-1}\}^\omega.$$

From the basic relation (7.5), we have

$$0^n 0^k 2z' 0^k 2z' \;\cong\; 1y0^n 0^k 2z' \;\cong\; 1y1y0^n.$$

Then two words 0 and $1y1$ are incompatible. Now we show that

$$Amb_\Sigma(L) \cap \{0, 1y1\}^\omega = \emptyset \tag{7.7}$$

and thus, by Proposition 4.7, L^ω has no code generator.

To obtain (7.7), it is sufficient to verify that

$$1y0^n \notin \mathrm{Fact}(\{0, 1y1\}^\omega) \tag{7.8}$$

and

$$\left(\bigcup_{i=0}^{n-1} \mathtt{1}y\mathtt{0}^i\right)^{\omega} \cap \mathrm{Suff}(\{\mathtt{0}, \mathtt{1}y\mathtt{1}\}^{\omega}) = \emptyset. \tag{7.9}$$

If $\mathtt{1}y\mathtt{0}^n \in \mathrm{Fact}(\{\mathtt{0}, \mathtt{1}y\mathtt{1}\}^{\omega})$ then there are a word $p \in \mathrm{Pref}(\mathtt{1}y) \setminus \{\mathtt{1}y\}$, a word $s \in \mathrm{Suff}(\mathtt{1}y)$, and an integer $m \geq 0$ such that

$$\mathtt{1}y\mathtt{0}^n = s\mathtt{1}\mathtt{0}^m p.$$

But since $\mathrm{OVL}(\mathtt{1}y\mathtt{0}^n, \mathtt{1}y\mathtt{0}^n) = \emptyset$, it follows that $p = \varepsilon$. Hence

$$\mathtt{1}y\mathtt{0}^n \in \mathrm{Suff}(\mathtt{1}y)\mathtt{1}\mathtt{0}^m.$$

But this show that $\rho(\mathtt{1}y) \in \mathtt{1}\mathtt{0}^*$, a contradiction. We get (7.8).

If

$$\left(\bigcup_{i=0}^{n-1} \mathtt{1}y\mathtt{0}^i\right)^{\omega} \cap \mathrm{Suff}(\{\mathtt{0}, \mathtt{1}y\mathtt{1}\}^{\omega}) = \emptyset$$

then there exists a word $s \in \mathrm{Suff}(\mathtt{1}y\mathtt{1}) \setminus \{\mathtt{1}y\mathtt{1}\}$ and two sequences $(i_j), (k_j)$ of integers such that

$$\mathtt{1}y\mathtt{0}^{i_1}\mathtt{1}y\mathtt{0}^{i_2}\ldots = s\mathtt{0}^{k_1}\mathtt{1}y\mathtt{1}\mathtt{0}^{k_2}\ldots.$$

Let $\ell = |1y|_1 - |s|_1 \geq 0$. We have

$$\begin{aligned} |s|_{\mathtt{1}} + |(\mathtt{1}y\mathtt{1})^{\ell}|_{\mathtt{1}} &= |s|_{\mathtt{1}} + \ell + |(\mathtt{1}y)^{\ell}|_{\mathtt{1}} \\ &= |\mathtt{1}y|_{\mathtt{1}} + |(\mathtt{1}y)^{\ell}|_{\mathtt{1}} = |(\mathtt{1}y)^{\ell+1}|_{\mathtt{1}}. \end{aligned}$$

Observe, further, that: if two words $\alpha\mathtt{1}$ and $\beta\mathtt{1}$ are both prefix of a same word and $|\alpha|_{\mathtt{1}} = |\beta|_{\mathtt{1}}$, then $\alpha = \beta$. We have thus

$$\mathtt{1}y\mathtt{0}^{i_1}\ldots\mathtt{1}y\mathtt{0}^{i_{\ell+1}} = s\mathtt{0}^{k_1}\mathtt{1}y\mathtt{1}\mathtt{0}^{k_2}\ldots\mathtt{1}y\mathtt{1}\mathtt{0}^{k_\ell}.$$

Thus

$$\mathtt{1}y\mathtt{0}^{i_{\ell+1}} \in \mathrm{Suff}(\mathtt{1}y)\mathtt{1}\mathtt{0}^{k_\ell}.$$

But this show that $\rho(\mathtt{1}y) \in \mathtt{1}\mathtt{0}^*$, a contradiction. We get (7.9).

Example 7.12 (for Subcase 3.1)**.** Let $L = \{a, ab, baaba\}$ and let $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. It can be verified that L is a one-relation language where the basic relation is $\mathtt{02} \cong \mathtt{1010}$.

Example 7.13 (for Subcase 3.2)**.** Let $L = \{a, ab, babaa\}$ and let $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. It can be verified that L is a one-relation language where the basic relation is $\mathtt{02} \cong \mathtt{1100}$.

7.3.2. Proof of Claim 7.4

Assume that the basic relation of L is of the form

$$\mathtt{0}^n z \cong (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n \tag{7.10}$$

with $\mathtt{0}^n = O_{u,v}$ and $\text{Last}(z) \notin \{\mathtt{0},\mathtt{1}\}$. We will show that L^ω has no code generator in both following cases:

Case 1: $k \geq 2, m \geq 0$, and $n \geq 2$.
By Lemma 6.3, we have

$$Amb_\Sigma(L) \cap \{\mathtt{0},\mathtt{1}\}^\omega = \{\mathtt{0},\mathtt{1}\}^*(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n\{\mathtt{0},\mathtt{1}\}^\omega \cup \{\mathtt{0},\mathtt{1}\}^* \left(\bigcup_{i=0}^{n-1}(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^i\right)^\omega$$

From the relation (7.10), it follows that

$$\mathtt{0}^n z\mathtt{0}z \;\cong\; (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n\mathtt{0}z \;\cong\; (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n.$$

Hence two words $\mathtt{0}$ and $(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}\mathtt{1}$ are incompatible.
Because $k \geq 2$ and $n \geq 2$, we have

$$(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n \notin \text{Fact}\left(\{\mathtt{0},(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}\mathtt{1}\}^\omega\right).$$

It can be verified that

$$\text{Suff}\left(\{\mathtt{0},(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}\mathtt{1}\}^\omega\right) \;\cap\; \left(\bigcup_{i=0}^{n-1}(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^i\right)^\omega = \emptyset.$$

Thus we have

$$Amb_\Sigma(L) \;\cap\; \{\mathtt{0},(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}\mathtt{1}\}^\omega = \emptyset.$$

By Proposition 4.7, L^ω has no code generator.

Example 7.14 (for Case 1)**.** Let $L = \{a, aab, baabaa\}$ and let $\Sigma = \{\mathtt{0},\mathtt{1},\mathtt{2}\}$. It can be verified that L is a one-relation language where the basic relation is $\mathtt{002} \;\cong\; \mathtt{1100}$.

Case 2: $|z| \geq 2, k \geq 1$, and $n \geq 1$.
As $O_{u,v} = \mathtt{0}^n$, we have $\text{First}(z) \neq \mathtt{0}$. Since L cannot contains two words commute, it follows from Lemma 2.3 that $\text{First}(z) \neq \mathtt{1}$. Thus the basic relation can be written in the form

$$\mathtt{0}^n\mathtt{2}z' \;\cong\; (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n$$

where $z' \neq \varepsilon$ and $\text{Last}(z') \notin \{\mathtt{0},\mathtt{1}\}$, and $\mathtt{2} \in \Sigma \setminus \{\mathtt{0},\mathtt{1}\}$. Then two words $\mathtt{0}^n\mathtt{2}$ and $\mathtt{1}$ are incompatible.
By Lemma 6.3, we have

$$Amb_\Sigma(L) = \Sigma^*\left\{\mathtt{0}^n\mathtt{2}z',(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n\right\}\Sigma^\omega \cup \Sigma^*\left(\text{LB}\left(\mathtt{0}^n\mathtt{2}z',\mathtt{0}^n\mathtt{2}z'\right) \cup \text{LB}\left(\mathtt{0}^n\mathtt{2}z',(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n\right)\right)^\omega. \tag{7.11}$$

Thus we have

$$Amb_\Sigma(L) \cap \{\mathtt{0}^n\mathtt{2},\mathtt{1}\}^\omega \subseteq \Sigma^*\{\mathtt{0}^n\mathtt{2}z'\}\Sigma^\omega. \tag{7.12}$$

If $Amb_\Sigma(L) \cap \{\mathtt{0}^n\mathtt{2},\mathtt{1}\}^\omega = \emptyset$, then by Proposition 4.7, L^ω has no code generator. We now assume that $Amb_\Sigma(L) \cap \{\mathtt{0}^n\mathtt{2},\mathtt{1}\}^\omega \neq \emptyset$. This mean that

$$\mathtt{0}^n\mathtt{2}z' \in \text{Fact}(\{\mathtt{0}^n\mathtt{2},\mathtt{1}\}^\omega).$$

Since $\mathrm{Last}(z') \notin \{\mathtt{0}, \mathtt{1}\}$, it follow that $z' \in \Sigma^*\mathtt{0}^n\mathtt{2}$. Thus the basic relation can be written in the form

$$\mathtt{0}^n\mathtt{2}f\mathtt{0}^n\mathtt{2} \ \cong \ (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n$$

where $f \in \mathrm{Pref}(z')$. We have:

$$\mathtt{0}^n\mathtt{2}f(\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n \ \cong \ \mathtt{0}^n\mathtt{2}f\mathtt{0}^n\mathtt{2}f\mathtt{0}^n\mathtt{2} \cong \ (\mathtt{1}\mathtt{0}^m)^k\mathtt{0}^n f\mathtt{0}^n\mathtt{2}.$$

Thus two words $\mathtt{0}^n\mathtt{2}f\mathtt{1}$ and $\mathtt{1}$ are incompatible.

By Lemma 6.8, we have $\mathtt{0}^n\mathtt{2}f\mathtt{0} \notin \mathrm{Fact}(\{\mathtt{0}^n\mathtt{2}f\mathtt{1}, \mathtt{1}\}^\omega)$. Therefore

$$\mathtt{0}^n\mathtt{2}f\mathtt{0}^n\mathtt{2} \notin \mathrm{Fact}(\{\mathtt{0}^n\mathtt{2}f\mathtt{1}, \mathtt{1}\}^\omega).$$

Thus

$$\mathit{Amb}_\Sigma(L) \cap \{\mathtt{0}^n\mathtt{2}f\mathtt{1}, \mathtt{1}\}^\omega = \emptyset.$$

By Proposition 4.7, L^ω has no code generator.

Example 7.15 (for Case 2)**.**
1. Let $L = \{ab, abc, b, caba\}$ and $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}$. It can be verified that L is a one-relation language where that basic relation is $\mathtt{032} \ \cong \ \mathtt{100}$.
2. Let $L' = \{a, aba^3b, ba^2\}$ and $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. It can be verified that L' is a one-relation language where the basic relation is $\mathtt{0202} \ \cong \ \mathtt{100}$.

7.3.3. Proof of Claim 7.5

Assume that the basic relation of L is of the form

$$\mathtt{0}^n z \ \cong \ \mathtt{1}\mathtt{0}^n$$

with $n \geq 1$ and $z \in \Sigma^*\mathtt{2}$. We will show that $\widetilde{C}$ is an ω-code generator of L^ω, where

$$C = \{\mathtt{0}\} \ \cup \ (\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i)^*(\Sigma \setminus \{\mathtt{0}, \mathtt{1}\}).$$

To prove that $\widetilde{C}$ is a generator of L^ω, we show first assertion: for every $x \in \Sigma^\omega \setminus C\Sigma^\omega$, there exists $y \in C\Sigma^\omega$ such that $x \ \cong \ y$. Indeed, direct computation show that

$$\Sigma^\omega \setminus C\Sigma^\omega = (\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i)^\omega \ \cup \ (\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i)^*\mathtt{1}\mathtt{0}^n\Sigma^\omega.$$

For each $x \in (\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i)^\omega$, we write

$$x = \mathtt{1}\mathtt{0}^{i_1}\mathtt{1}\mathtt{0}^{i_2}\dots$$

where $n > i_j \geq 0$. Then we have

$$\mathtt{1}\mathtt{0}^{i_1}\mathtt{1}\mathtt{0}^{i_2}\dots \ \cong \ \mathtt{0}^n z\mathtt{0}^{i_1}z\mathtt{0}^{i_2}\dots \in C\Sigma^\omega.$$

And for each $x \in (\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i)^* \mathtt{1}\mathtt{0}^n \Sigma^\omega$, we write

$$x = \mathtt{1}\mathtt{0}^{i_1} \ldots \mathtt{1}\mathtt{0}^{i_p} \mathtt{1}\mathtt{0}^n w$$

where $p \geq 0$, $n > i_j \geq 0$ and $w \in \Sigma^\omega$. We have thus

$$\mathtt{1}\mathtt{0}^{i_1} \ldots \mathtt{1}\mathtt{0}^{i_p} \mathtt{1}\mathtt{0}^n w \;\cong\; \mathtt{0}^n z \mathtt{0}^{i_1} z \ldots z \mathtt{0}^{i_p} w \in C\Sigma^\omega.$$

Thus we have $L^\omega \subseteq \widetilde{C} L^\omega$. According to Lemma 2.6, we have $L^\omega \subseteq \widetilde{C}^\omega$. Thus $\widetilde{C}^\omega = L^\omega$.

Now we show that $\widetilde{C}$ is an ω-code. Indeed, it is clear that C is a prefix code. Observe, further, that

$$\mathtt{1}\mathtt{0}^n \notin \mathrm{Fact}(C^\omega) \qquad \text{and} \qquad (\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i)^\omega \cap \mathrm{Suff}(C^\omega) = \emptyset.$$

Hence we have $R(L) \cap (C^\omega \times C^\omega) = \emptyset$. Thus each ω-word in $\widetilde{C}^\omega$ has only one factorization on $\widetilde{C}$. The proof is completed.

7.3.4. *Proof of Claim 7.6*

Assume that the basic relation of L is of the form

$$\mathtt{0}^n \mathtt{2} \;\cong\; \mathtt{1}\mathtt{0}^m \mathtt{0}^n$$

with $m \geq 1$ and $n \geq 1$. We will show that $\widetilde{C}$ is an ω-code generator of L^ω, where

$$C = \{\mathtt{0}, \mathtt{2}\} \cup \left(\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^m \mathtt{0}^i \right)^* \left(\bigcup_{i=0}^{m-1} \mathtt{1}\mathtt{0}^i (\Sigma \setminus \{\mathtt{0}, \mathtt{2}\}) \cup \bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^i \mathtt{2} \;\cup \Sigma \setminus \{\mathtt{0}, \mathtt{1}, \mathtt{2}\} \right).$$

To prove that $\widetilde{C}$ is a generator of L^ω, we show first assertion: for each $x \in \Sigma^\omega \setminus C\Sigma^\omega$, there exists $y \in C\Sigma^\omega$ such that $x \;\cong\; y$. Indeed, direct computation show that

$$\Sigma^\omega \setminus C\Sigma^\omega = \left(\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^m \mathtt{0}^i \right)^\omega \;\cup\; \left(\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^m \mathtt{0}^i \right)^* (\mathtt{1}\mathtt{0}^m \mathtt{0}^n \cup \mathtt{1}\mathtt{0}^n \mathtt{0}^* \mathtt{2}) \Sigma^\omega.$$

For every $x \in \left(\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^m \mathtt{0}^i \right)^\omega$, we can write

$$x = \mathtt{1}\mathtt{0}^m \mathtt{0}^{i_1} \mathtt{1}\mathtt{0}^m \mathtt{0}^{i_2} \ldots$$

where $n > i_j \geq 0$, we have

$$\mathtt{1}\mathtt{0}^m \mathtt{0}^{i_1} \mathtt{1}\mathtt{0}^m \mathtt{0}^{i_2} \ldots \;\cong\; \mathtt{0}^n \mathtt{2} \mathtt{0}^{i_1} \mathtt{2} \mathtt{0}^{i_2} \ldots \in C\Sigma^\omega.$$

For every $x \in \left(\bigcup_{i=0}^{n-1} \mathtt{1}\mathtt{0}^m \mathtt{0}^i \right)^* \mathtt{1}\mathtt{0}^m \mathtt{0}^n \Sigma^\omega$, we write

$$x = \mathtt{1}\mathtt{0}^m \mathtt{0}^{i_1} \ldots \mathtt{1}\mathtt{0}^m \mathtt{0}^{i_p} \mathtt{1}\mathtt{0}^m \mathtt{0}^n w$$

with $p \geq 0$, $n > i_j \geq 0$, and $w \in \Sigma^\omega$, we have

$$\mathtt{10}^m\mathtt{0}^{i_1}\ldots\mathtt{10}^m\mathtt{0}^{i_p}\mathtt{10}^m\mathtt{0}^n w \;\cong\; \mathtt{0}^n\mathtt{20}^{i_1}\mathtt{2}\ldots\mathtt{20}^{i_p}w \in C\Sigma^\omega.$$

For every $x \in \left(\bigcup_{i=0}^{n-1}\mathtt{10}^m\mathtt{0}^i\right)^*\mathtt{10}^n\mathtt{0}^*\mathtt{2}\Sigma^\omega$, we write

$$x = \mathtt{10}^m\mathtt{0}^{i_1}\ldots\mathtt{10}^m\mathtt{0}^{i_p}\mathtt{10}^n\mathtt{0}^q\mathtt{2}w$$

where $q \geq 0$, $p \geq 0$, $n > i_j \geq 0$, and $w \in \Sigma^\omega$. If $q \geq m$, we have

$$x = \mathtt{10}^m\mathtt{0}^{i_1}\ldots\mathtt{10}^m\mathtt{0}^{i_p}\mathtt{10}^n\mathtt{0}^m\mathtt{0}^{q-m}\mathtt{2}w \;\cong\; \mathtt{0}^n\mathtt{20}^{i_1}\mathtt{2}\ldots\mathtt{20}^{i_p}\mathtt{20}^{q-m}\mathtt{2}w \in C\Sigma^\omega,$$

if $q < m$, we have

$$\begin{aligned}\mathtt{10}^m\mathtt{0}^{i_1}\ldots\mathtt{10}^m\mathtt{0}^{i_p}(\mathtt{10}^n\mathtt{0}^q)\mathtt{2}w &= \mathtt{10}^m\mathtt{0}^{i_1}\ldots\mathtt{10}^m\mathtt{0}^{i_p}(\mathtt{10}^q\mathtt{0}^n)\mathtt{2}w\\ &\cong\; \mathtt{10}^m\mathtt{0}^{i_1}\ldots\mathtt{10}^m\mathtt{0}^{i_p}\mathtt{10}^q(\mathtt{10}^m\mathtt{0}^n)w\\ &\in (\mathtt{10}^m\mathtt{0}^{i_1}\ldots\mathtt{10}^m\mathtt{0}^{i_p})\mathtt{10}^q\mathtt{1}\Sigma^\omega \subseteq C\Sigma^\omega\end{aligned}$$

that prove the assertion. Thus $L^\omega \subseteq \widetilde{C}L^\omega$. According to Lemma 2.6, we have a $L^\omega \subseteq \widetilde{C}^\omega$. Thus $\widetilde{C}^\omega = L^\omega$.

Now we prove that $\widetilde{C}$ is an ω-code. Indeed, it is clear that C is a prefix code. Observe, further, that

$$\left(\bigcup_{i=0}^{n-1}\mathtt{10}^m\mathtt{0}^i\right)^\omega \cap \mathrm{Suff}(C^\omega) = \emptyset,$$

and if there exists $u \in \Sigma^*$ and $w \in \Sigma^\omega$ such that $u\mathtt{10}^m\mathtt{0}^n w \in C^\omega$, then $u\mathtt{1} \in C^+$. We have thus $R(L) \cap (C^\omega \times C^\omega) = \emptyset$. Thus each ω-word in $\widetilde{C}^\omega$ has only one factorization on $\widetilde{C}$. The proof is completed.

7.3.5. Proof of Claim 7.7

Assume that the basic relation of L is of the form

$$\mathtt{0}z \;\cong\; \mathtt{1}^k\mathtt{0}$$

with $z \in \Sigma^*\mathtt{2}$ and $k > 1$. We will show that $\widetilde{C}$ is a code generator of L^ω, where

$$C = \bigcup_{i=0}^{k-1}\mathtt{1}^i\mathtt{0} \;\cup\; \mathtt{1}^*\left(\Sigma \setminus \{\mathtt{0},\mathtt{1}\}\right).$$

To prove that $\widetilde{C}$ is a generator of L^ω, we show first assertion: for every $x \in \Sigma^\omega \setminus C\Sigma^\omega$, there exists $y \in C\Sigma^\omega$ such that $x \;\cong\; y$. Indeed, direct computation show that

$$\Sigma^\omega \setminus C\Sigma^\omega = \mathtt{1}^\omega \;\cup\; \mathtt{1}^*\mathtt{1}^k\mathtt{0}\Sigma^\omega.$$

For $x = \mathtt{1}^\omega$, we have

$$\mathtt{1}^\omega \;\cong\; \mathtt{0}z^\omega \in C\Sigma^\omega,$$

and for each $x \in 1^*1^k0\Sigma^\omega$, we can write

$$x = 1^q1^{pk}0w \qquad \text{with} \quad p \geq 0, k > q \geq 0, \text{ and } w \in \Sigma^\omega$$

we have then

$$1^q1^{pk}0w \cong 1^q0z^pw \in C\Sigma^\omega.$$

Thus we have $L^\omega \subseteq \widetilde{C}L^\omega$. By Lemma 2.6, we have $L^\omega \subseteq \widetilde{C}^\omega$. Thus $\widetilde{C}^\omega = L^\omega$.

Now we prove that $\widetilde{C}$ is a code. Indeed, it is clear that C is a prefix code. Moreover we have $1^k0 \notin \text{Fact}(C^+)$, we obtain $R(L) \cap (C^+ \times C^+) = \emptyset$. Thus each words in $\widetilde{C}^*$ has only one factorization on $\widetilde{C}$. The proof is completed.

It is to be noticed that $\widetilde{C}$ is not an ω-code because $10z^\omega \cong 0z^\omega$ and $\{0, 10\} \subseteq C$.

Now we show that L^ω has no ω-code generator. Indeed, for every $i > 0$, we have

$$0z^i \cong 1^k0z^{i-1} \cong 1^k1^k0z^{i-2} \cong \cdots \cong 1^{ki}0,$$

By passing to adherence, we obtain the relation $0z^\omega \cong 1^\omega$. Thus

$$10z^\omega \cong 11^\omega = 1^\omega \cong 0z^\omega.$$

Since $k > 1$, it follows that $10z^\omega \cong 0z^\omega$ is a minimal relation. Thus two words 0 and 10 are ∞-incompatible. Moreover, it is easy to see that

$$Amb_\Sigma(L) \cap \{0,1\}^\omega = \{0,1\}^*\{1^k0\}\{0,1\}^\omega \cup \{0,1\}^*1^\omega$$

Since $1^k \notin \text{Fact}(\{10,0\}^\omega)$ with $k > 1$, it follows that $Amb_\Sigma(L) \cap \{10,0\}^\omega = \emptyset$. By Proposition 4.12, L^ω has no ω-code generator.

7.3.6. Proof of Claim 7.8

Assume that the basic relation of L is of the form

$$02 \cong (10^m)^k0$$

with $k > 1$ and $m \geq 1$. We will show that $\widetilde{C}$ is a code generator of L^ω, where

$$C = \{2\} \cup \bigcup_{i=0}^{k-1}(10^m)^i0 \cup (10^m)^* \left(\{12\} \cup 1\left(\bigcup_{i=0}^{m-1} 0^i\right)(\Sigma \setminus \{0,2\}) \cup \Sigma \setminus \{0,1,2\}\right).$$

To prove that $\widetilde{C}$ is a generator of L^ω, we show first assertion: for every $x \in \Sigma^\omega \setminus C\Sigma^\omega$, there exists $y \in C\Sigma^\omega$ such that $x \cong y$. Indeed, direct computation show that

$$\Sigma^\omega \setminus C\Sigma^\omega = \{(10^m)^\omega\} \cup (10^m)^* \left(\{(10^m)^k0\} \cup 10^+2\right)\Sigma^\omega.$$

For $x = (10^m)^\omega$, we have

$$(10^m)^\omega \cong 02^\omega \in C\Sigma^\omega.$$

For every $x \in (10^m)^*(10^m)^k0\Sigma^\omega$, we can write $x = (10^m)^q(10^m)^{pk}0w$ with $p \geq 0, k > q \geq 0$, and $w \in \Sigma^\omega$, we have therefore

$$(10^m)^q(10^m)^{pk}0w' \ \cong\ (10^m)^q02^p0w' \in C\Sigma^\omega.$$

And for every $x \in (10^m)^*10^+2\Sigma^\omega$, we can write

$$x = (10^m)^q10^p02w$$

with $q \geq 0, m > p \geq 0$ and $w \in \Sigma^\omega$. We have therefore

$$(10^m)^q10^p02w' \ \cong\ (10^m)^q10^p(10^m)^k0w' \in (10^m)^q10^p1\Sigma^\omega \subseteq C\Sigma^\omega.$$

Thus $L^\omega \subseteq \widetilde{C}L^\omega$. By Lemma 2.6, we have $L^\omega \subseteq \widetilde{C}^\omega$. Thus $\widetilde{C}^\omega = L^\omega$.

Now we prove that $\widetilde{C}$ is a code. Indeed, it is clear that C is a prefix code. Moreover, since $(10^m)^k0 \notin \text{Fact}(C^+)$, we obtain $R(L) \cap (C^+ \times C^+) = \emptyset$. This means that every words in $\widetilde{C}^*$ has only one factorization on $\widetilde{C}$. Hence $\widetilde{C}$ is a code.

It is to be noticed that that $\widetilde{C}$ is not an ω-code because $10^m02^\omega \ \cong\ 02^\omega$ and $\{0, 10^m0, 2\} \subseteq C$.

Now we show that L^ω has no ω-code generator. Indeed, for every $i > 0$, we have

$$02^i \ \cong\ (10^m)^k02^{i-1} \ \cong\ (10^m)^k(10^m)^k02^{i-2} \ \cong\ \cdots \ \cong\ (10^m)^{ki}0.$$

By passing to adherence, we get $02^\omega \ \cong\ (10^m)^\omega$. So

$$10^m02^\omega \ \cong 10^m(10^m)^\omega = (10^m)^\omega \ \cong\ 02^\omega.$$

Because $k > 1$, we have $10^m02^\omega \ \cong\ 02^\omega$ is a minimal relation. Thus two words 0 and 10^m0 are ∞-incompatible. Moreover, it is easy to see that

$$Amb_\Sigma(L) \cap \{0,1\}^\omega = \{0,1\}^*\{(10^m)^k0\}\{0,1\}^\omega \cup \{0,1\}^*(10^m)^\omega$$

By Lemma 6.8, we have $10^m1 \notin \text{Fact}(\{10^m0, 0\}^\omega)$. Hence

$$Amb_\Sigma(L) \cap \{10^m0, 0\}^\omega = \emptyset.$$

By Proposition 4.12, L^ω has no ω-code generator.

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