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CORRIGENDUM: ON MULTIPERIODIC WORDS

Štěpán Holub[1]

Abstract. An algorithm is corrected here that was presented as Theorem 2 in [Š. Holub, *RAIRO-Theor. Inf. Appl.* **40** (2006) 583–591]. It is designed to calculate the maximum length of a nontrivial word with a given set of periods.

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The purpose of this contribution is to fill a gap in the algorithm presented in my paper [1] as Theorem 2. The theorem contains a formula that is supposed to yield the length $\mathcal{L}_P$ of the longest nontrivial multiperiodic word, that is, the longest word having a given set P of coprime periods and not the period one. The formula reads as follows:

$$\mathcal{L}_P = m_{n-1} - 1 + \sum_{i=0}^{n-1} m_i, \tag{1}$$

where m_i is the minimal element of the set Q_i, which is given by the following recursive formula: $Q_0 = P$, and

$$Q_{i+1} = \{q - m_i \mid q \in Q_i, q \neq m_i\} \cup \{m_i\}.$$

The number n is established as the smallest index such that $1 \in Q_n$.

Gwénaël Richomme [2] pointed out that the formula is not correct, giving the following counterexample:

Consider the set $P = \{5, 7, 8\}$ of coprime periods. We have

$$\begin{aligned} Q_0 &= \{5, 7, 8\}, \\ Q_1 &= \{2, 3, 5\}, \\ Q_2 &= \{1, 2, 3\}. \end{aligned}$$

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[1] Department of Algebra, Charles University, Sokolovská 83, 175 86 Praha, Czech Republic.
holub@karlin.mff.cuni.cz

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Therefore $n = 2$, and

$$m_{n-1} - 1 + \sum_{i=0}^{n-1} m_i = m_1 - 1 + m_0 + m_1 = 8.$$

However, the nontrivial word *aaaabaaaa* of length 9 has periods P.

The main idea of the proof of Theorem 2 is Lemma 2 claiming that (with an adjusted notation) for $k \geq m_i$, we have $[Q_i, k + m_i] = [Q_{i+1}, k]$. Recall that $[Q, \ell]$ denotes the maximum number of letters which can occur in a word of length ℓ having periods Q.

An additional observation is Lemma 3, according to which $[Q, 2m - 1] > 1$ if $m = \min Q > 1$.

To illustrate why Theorem 2 gives a wrong result for $P = \{5, 7, 8\}$ let us first look at an example where the formula works.

Let $P' = \{3, 5, 8\}$, whence

$$\begin{aligned} Q'_0 &= \{3, 5, 8\}, \\ Q'_1 &= \{2, 3, 5\}, \\ Q'_2 &= \{1, 2, 3\}. \end{aligned}$$

We can now deduce, from Lemmas 2 and 3, that

$$[Q'_0, 6] = [Q'_1, 3] > 1$$

while

$$[Q'_0, 7] = [Q'_1, 4] = [Q'_2, 2] = 1.$$

Therefore $\mathcal{L}_P = 6$.

Similar reasoning for $P = \{5, 7, 8\}$ would yield

$$[Q_0, 8] = [Q_1, 3] > 1$$

and

$$[Q_0, 9] = [Q_1, 4] = [Q_2, 2] = 1,$$

leading to the wrong answer $\mathcal{L}_P = 8$. The problem is that we cannot conclude $[Q_0, 9] = [Q_1, 4]$ due to the fact that the condition $k \geq m_0$ of Lemma 2 is not satisfied: we have $k = 4$ and $m_0 = 5$. In fact, $[Q_0, 9] \neq [Q_1, 4]$ holds in this case. (Similarly, $[Q_0, 8] \neq [Q_1, 3]$).

This is precisely the situation that has to be taken into account in order to obtain a correct algorithm, which we state and proof now. To simplify notation, consider further only one step of the reduction and denote $P = Q_0$, $Q = Q_1$ and $m = \min P$ (this notation conforms to [1]).

Theorem 1 (correction of Thm. 2 in [1]). *Let $P \subset \mathbb{N}_+$ be a set of positive integers such that* $\gcd(P) = 1$*, and* $m = \min(P) > 1$*. Let*

$$Q = \{q - m \mid q \in P, q \neq m\} \cup \{m\}. \tag{2}$$

Then the maximal length of a nontrivial word with periods P is given by the following recursive formula:

$$\mathcal{L}_P = m + \max\{\mathcal{L}_Q, m - 1\}$$

where $\mathcal{L}_Q$ is the maximal length of a nontrivial word with periods Q, and is defined as 0 *if* $1 \in Q$*.*

Proof. As in [1], we can verify that for any P (even infinite) the definition of $\mathcal{L}_P$ is correct, namely that the recursion terminates.

Let $\mathcal{L}_Q \geq m$. Lemma 2 now yields that $[P, \mathcal{L}_Q + m] = [Q, \mathcal{L}_Q] \neq 1$ while $[P, \mathcal{L}_Q + 1 + m] = [Q, \mathcal{L}_Q + 1] = 1$, which implies that $\mathcal{L}_P$ is equal to $\mathcal{L}_Q + m$.

Let $\mathcal{L}_Q < m$. Then $[Q, m] = 1$ and Lemma 2 implies $[P, 2m] = 1$. The proof is concluded by Lemma 3. $\square$

Note that the formula (1) is wrong if and only if $\mathcal{L}_{Q_{i+1}} < m_i - 1$ for some $i < n - 1$. The formula was formed under the (mistaken) assumption that this inequality holds only for $i = n - 1$.

To conclude, let us apply the corrected theorem to the above counterexample $P = \{5, 7, 8\}$. We have

$$\begin{aligned}
\mathcal{L}_{Q_2} &= 0,\\
\mathcal{L}_{Q_1} &= 2 + \max\{0, 1\} = 3,\\
\mathcal{L}_{Q_0} &= 5 + \max\{3, 4\} = 9.
\end{aligned}$$

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References

[1] Š. Holub, On multiperiodic words. *RAIRO-Theor. Inf. Appl.* **40** (2006) 583–591.
[2] G. Richomme, personal communication (July 2011).

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