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ON THE SEMIDIRECT PRODUCT OF THE PSEUDOVARIETY OF SEMILATTICES BY A LOCALLY FINITE PSEUDOVARIETY OF GROUPS (*)

by F. Blanchet-Sadri (1) (2)

Abstract — In this paper, we give a sequence of identities defining the product pseudovariety $J_1 \star H$ generated by all semidirect products of the form $M \star N$ with $M \in J_1$ and $N \in H$ (here $J_1$ is the pseudovariety of semilattice monoids and $H$ is a locally finite pseudovariety of groups) A sequence of sets of identities ultimately defining $J_1 \star G_p$ results (here $G_p$ is the pseudovariety of $p$-groups).

Résumé — Dans cet article, nous donnons une suite d’identités définissant la pseudovariété $J_1 \star H$ engendrée par les produits semidirects de la forme $M \star N$ où $M \in J_1$ et $N \in H$ (ici $J_1$ est la pseudovariété des demi-treillis et $H$ une pseudovariété de groupes localement finie). Une suite d’ensembles d’identités définissant ultimement $J_1 \star G_p$ en résulte (ici $G_p$ est la pseudovariété des $p$-groupes).

1. INTRODUCTION

In this paper, we discuss a technique to produce identities for the semidirect product pseudovariety $J_1 \star H$ generated by all semidirect products of the form $M \star N$ with $M \in J_1$ and $N \in H$, where $J_1$ is the pseudovariety of all semilattice monoids and $H$ is a locally finite pseudovariety of groups.

The notion of congruence plays a central role in our approach. For any finite alphabet $A$ denote by $A^*$ the free monoid generated by $A$. We say that a monoid $M$ is $A$-generated if there exists a congruence $\beta$ on $A^*$ such that $M$ is isomorphic to $A^*/\beta$. A pseudovariety of monoids $V$ is locally finite if

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for any \(A\) there are finitely many \(A\)-generated monoids in \(V\). Equivalently, there exists for each \(A\) a congruence \(\beta_A\) such that an \(A\)-generated monoid \(M\) is in \(V\) if and only if \(M\) is a morphic image of \(A^*/\beta_A\).

Let \(H\) be a locally finite pseudovariety of groups. Let \(\gamma\) be the congruence generating \(H\) for the finite alphabet \(A\). The idea is to associate with \(J_1 \ast H\) a congruence \(\sim_\gamma\) on \(A^*\). Section 3 gives a criterion to determine when an identity on \(A\) is satisfied in \(J_1 \ast H\) with the help of \(\sim_\gamma\). This leads to a proof that such \(J_1 \ast H\) are locally finite and hence decidable. This criterion follows from Almeida’s semidirect product representation of the free objects in \(V \ast W\) in case both \(V\) and \(W\) have finite free objects [1] (Almeida’s representation is stated in Section 2.1). In Section 5, we give a basis of identities for \(J_1 \ast H\) which follows mainly from a result on graphs due to Simon [8] (Simon’s result is stated in Section 4) and the identity criterion of Section 3. In Section 6, we give a sequence of sets of identities ultimately defining the pseudovariety \(J_1 \ast G_p\), where \(p\) is a prime number and \(G_p\) is the pseudovariety of all \(p\)-groups, that is the pseudovariety of all groups of order \(p^k\) for some nonnegative integer \(k\).

Related known results include the following. The product \(J_1 \ast G\) is generated by the inverse monoids (Margolis and Pin [11]) and is the class of finite monoids in which the idempotents commute (Ash [4]) (here \(G\) is the pseudovariety of groups). Blanchet-Sadri and Zhang [6] give identities ultimately defining the pseudovariety \(J_1 \ast G_{com}\) where \(G_{com}\) denotes the pseudovariety of commutative groups. Irastorza [10] shows that if the pseudovarieties \(V\) and \(W\) are finitely based, their product may not be.

The techniques in this paper were used in particular by Pin [13] to give a basis of identities for \(J_1 \ast J_1\), by Almeida [2] to generalize Pin’s result to iterated semidirect products of finite semilattices, and by Blanchet-Sadri [5] to give a basis of identities for \(J_1 \ast J_k\) where \(J_k\) denotes the pseudovariety of \(J\)-trivial monoids of height \(k\).

2. PRELIMINAIRES

We refer the reader to [3, 7, 8, 12] for terms not explicitly defined here.

2.1. Pseudovarieties of monoids

A nonempty class of finite monoids is called a pseudovariety if it is closed under submonoids, morphic images, and finitary direct products. A nonempty
class of monoids is called a variety if it is closed under submonoids, morphic images, and direct products.

As the intersection of a class of pseudovarieties of monoids is again a pseudovariety, and as all finite monoids form a pseudovariety, we can conclude that for every class $C$ of finite monoids there is a smallest pseudovariety containing $C$, called the pseudovariety generated by $C$. Now, if $C$ is a class of monoids, the smallest variety containing $C$ is called the variety generated by $C$.

For a pseudovariety $V$ and a set $A$, $F_V(A)$ denotes the free object on $A$ (or generated by $A$) in the variety generated by $V$. If $A$ is finite, say $A = \{a_1, \ldots, a_r\}$, we often write $F_V(a_1, \ldots, a_r)$ for $F_V(A)$. In case $V$ is the pseudovariety of all finite semigroups (respectively all finite monoids), the semigroup (respectively monoid) $F_V(A)$ is usually denoted by $A^+$ (respectively $A^*$). Elements of $A^+$ are viewed as nonempty words of elements of $A$, and the multiplication is given by concatenation of words. The monoid $A^*$ includes also the empty word $\lambda$. For a word $u \in A^*$, let $|u|$ denote the length of $u$. For words $u, v, w \in A^*$ satisfying $w = uv$, let $w \setminus u$ denote the factor $v$.

2.1.1. Semidirect products of pseudovarieties

Let $M$ and $N$ be monoids. It is convenient to write $M$ additively, without however assuming that $M$ is commutative. We denote by $0$ (respectively $1$) the unit element of $M$ (respectively $N$). A left action of $N$ on $M$ is a morphism $\varphi$ from $N$ into the monoid of monoid endomorphisms of $M$, where endomorphisms of $M$ are written on the left.

Given a left action $\varphi$ of $N$ on $M$, we define the semidirect product $M \rtimes N$ as follows. The elements of $M \rtimes N$ are pairs $(m, n)$ with $m \in M$, $n \in N$. Multiplication is given by the formula

$$(m, n)(m', n') = (m + nm', nn')$$

where $nm'$ represents $\varphi(n)(m')$. (This is what Eilenberg [8] calls a “unitary” semidirect product.) The multiplication in $M \rtimes N$ is associative. Thus $M \rtimes N$ is a monoid with $(0, 1)$ as unit element.

We now relate the notion of pseudovariety with that of a semidirect product. Given pseudovarieties of monoids $V$ and $W$, we denote by $V \rtimes W$ the pseudovariety generated by all semidirect products $M \rtimes N$ with $M \in V$, $N \in W$ and with any left action of $N$ on $M$. The semidirect product of pseudovarieties of monoids is associative.
PROPOSITION 2.1: (Almeida [1]) Let \( V \) and \( W \) be pseudovarieties of monoids such that \( F_V(A) \) and \( F_W(A) \) are finite for all finite \( A \). Then so is \( V \ast W \). Moreover, for a finite set \( A \), let \( N = F_W(A) \) and \( M = F_V(N \times A) \). Consider the left action of \( N \) on \( M \) defined by \( n(n', a) = (nn', a) \) and the associated semidirect product \( M \ast N \). Then, there is an embedding from \( F_{V \ast W}(A) \) into \( M \ast N \) that maps \( a \) into \( ((1, a), a) \).

2.1.2. Pseudovarieties and sequences of identities

Let \( A \) be a set. A monoid identity on \( A \) is an expression of the form \( u = v \) where \( u, v \in A^* \). A monoid \( M \) satisfies an identity \( u = v \) (or the identity is true in \( M \), or holds in \( M \)), abbreviated by \( M \models u = v \), if for every morphism \( \varphi : A^* \to M \) we have \( \varphi(u) = \varphi(v) \).

A class \( C \) of monoids satisfies \( u = v \), written \( C \models u = v \), if each member of \( C \) satisfies \( u = v \). If \( \Sigma \) is a set of identities, we say \( C \) satisfies \( \Sigma \), written \( C \models \Sigma \), if \( C \models u = v \) for each \( u = v \in \Sigma \). An identity \( u = v \) is deducible from a set of identities \( \Sigma \), abbreviated by \( \Sigma \vdash u = v \), if for every monoid \( M \) we have \( M \models \Sigma \) implies \( M \models u = v \). Here, letters can be erased in monoid identities.

Let \( u_i = v_i, i \geq 1 \) be a sequence of identities. Put \( \Sigma = \{ u_i = v_i \mid i \geq 1 \} \), and define \( V(\Sigma) \) to be the class of finite monoids satisfying \( \Sigma \) or all the identities \( u_i = v_i \). A class \( C \) of finite monoids is said to be defined by \( \Sigma \) (or by the identities \( u_i = v_i, i \geq 1 \)) if \( C = V(\Sigma) \); \( \Sigma \) is said to be a basis for \( C \).

Eilenberg and Schützenberger [9] show that every pseudovariety generated by a single monoid is of the form \( V(\Sigma) \) for some such \( \Sigma \).

2.2. Varieties of sets

Let \( L \) be a subset of \( A^* \). We define a congruence \( \sim_L \) on \( A^* \) as follows: \( u \sim_L v \) holds if \( xuvy \in L \) if and only if \( xvy \in L \) for all \( x, y \in A^* \). The congruence \( \sim_L \) is called the syntactic congruence of \( L \), and the quotient monoid \( A^*/\sim_L \), which we denote by \( M(L) \), is called the syntactic monoid of \( L \). The subset \( L \) of \( A^* \) is saturated for the congruence \( \sim_L \), that is \( u \sim_L v \) and \( u \in L \) imply \( v \in L \). Each pseudovariety of monoids is generated by the syntactic monoids that it contains. The set \( L \) is recognizable if and only if \( M(L) \) is a finite monoid.

Suppose that for each finite alphabet \( A \), a family \( A^*V \) of recognizable sets of \( A^* \) is given. We then say that \( V = \{ A^*V \} \) is a \( * \)-variety of sets if it satisfies the following conditions:

- \( A^*V \) is closed under boolean operations;
• If \( L \in A^*\mathcal{V} \) and \( a \in A \), then the sets \( a^{-1}L = \{w \in A^* \mid aw \in L\} \) and \( La^{-1} = \{w \in A^* \mid wa \in L\} \) are in \( A^*\mathcal{V} \);

• If \( \varphi : B^* \to A^* \) is a monoid morphism and if \( L \in A^*\mathcal{V} \), then \( \varphi^{-1}(L) \in B^*\mathcal{V} \).

Pseudovarieties of monoids and \(*\)-varieties of sets are in 1–1 correspondence. If \( \mathcal{V} \) is a \(*\)-variety of sets, then the pseudovariety of monoids generated by \( \{M(L) \mid L \in A^*\mathcal{V} \text{ for some } A\} \) defines the corresponding pseudovariety of monoids \( \mathcal{V} \). If \( \mathcal{V} \) is a pseudovariety of monoids, then \( A^*\mathcal{V} = \{L \subseteq A^* \mid M(L) \in \mathcal{V}\} \) defines the corresponding \(*\)-variety of sets \( \mathcal{V} \).

3. CONGRUENCES FOR \( J_1 \ast H \)

In this section, we give a criterion to determine when an identity is satisfied in the semidirect product \( J_1 \ast H \) where \( H \) is a locally finite pseudovariety of groups. This criterion is used in Section 5 to obtain a basis of identities for \( J_1 \ast H \).

Let \( A \) be a finite set. For a word \( u \in A^* \), let \( \alpha(u) \) denote the set of elements of \( A \) that occur in \( u \). Then the free object of \( J_1 \) on \( A \) is isomorphic to the quotient \( A^*/\alpha \) where the congruence \( \alpha \) on \( A^* \) is defined by \( u\alpha v \) if and only if \( \alpha(u) = \alpha(v) \). Now, let \( \gamma \) be the congruence of finite index on \( A^* \) such that an \( A \)-generated monoid \( M \) belongs to \( H \) if and only if \( M \) is a morphic image of \( A^*/\gamma \). The free object \( F_H(A) \) is isomorphic to the quotient \( A^*/\gamma \). The pseudovarieties \( J_1 \) and \( H \) have hence finite finitely generated free objects. We denote by \( \pi_\gamma \) the canonical projection from \( A^* \) into \( F_H(A) \) that maps \( a \) onto the generator \( a \) of \( F_H(A) \). If \( u, v \in A^* \), then \( \pi_\gamma(u) = \pi_\gamma(v) \) if and only if \( uv = vu \).

**Definition 3.1**: Let \( w \in A^* \).

- Let \( \sigma_\gamma : A^* \to (F_H(A) \times A)^* \) be the function defined by

\[
\sigma_\gamma(a_1 \ldots a_i) = (1, a_1)(\pi_\gamma(a_1), a_2) \ldots (\pi_\gamma(a_1 \ldots a_{i-1}), a_i)
\]

if \( i > 0 \), \( 1 \) otherwise.

- Let \( \sigma^w_\gamma : A^* \to (F_H(A) \times A)^* \) be the function defined by

\[
\sigma^w_\gamma(a_1 \ldots a_i) = (\pi_\gamma(w), a_1)(\pi_\gamma(wa_1), a_2) \ldots (\pi_\gamma(wa_1 \ldots a_{i-1}), a_i)
\]

if \( i > 0 \), \( 1 \) otherwise.
The sequential function $\sigma_\gamma$ is realized by the transducer whose states are the elements of $F_H(A)$ (1 being the initial state) and whose transitions are given by

$$ n \xrightarrow{a/(n,a)} na $$

where $n \in F_H(A)$ and $a \in A$.

We define an equivalence relation on $A^*$ by requesting that

$$ u \sim_\gamma v \text{ if and only if } \alpha(\sigma_\gamma(u)) = \alpha(\sigma_\gamma(v)) \text{ and } u \gamma v. $$

**Lemma 3.1:** The equivalence relation $\sim_\gamma$ is a congruence of finite index on $A^*$.

**Proof:** Assume $u \sim_\gamma v$ and $u' \sim_\gamma v'$. We have

$$ \alpha(\sigma_\gamma(u)) = \alpha(\sigma_\gamma(v)) \text{ and } u \gamma v $$

and similarly with $u$ and $v$ replaced by $u'$ and $v'$. Since $\gamma$ is a congruence we have $uw' \gamma vv'$. The above and the fact that $\pi_\gamma(u) = \pi_\gamma(v)$ imply that

$$ \alpha(\sigma_\gamma(uw')) = \alpha(\sigma_\gamma(u)\sigma_\gamma(u')) = \alpha(\sigma_\gamma(v)\sigma_\gamma(v')) = \alpha(\sigma_\gamma(vv')). $$

Thus $uw' \sim_\gamma vv'$ showing that $\sim_\gamma$ is a congruence. This obviously is a finite congruence since $\alpha$ and $\gamma$ are finite. □

**Lemma 3.2:** If $u = v$ is an identity on $A$, then the following conditions are equivalent:

- $J_1 \ast H \models u = v$;
- $u \sim_\gamma v$.

Consequently, an $A$-generated monoid $M$ belongs to $J_1 \ast H$ if and only if $M$ is a morphic image of $A^*/\sim_\gamma$.

**Proof:** Let $u = v$ be an identity on $A$, say $u = a_1 \ldots a_i$ and $v = b_1 \ldots b_j$. Let $N = F_H(A)$ and $M = F_{J_1}(N \times A)$. Consider the left action of $N$ on $M$ defined by $n(n', a) = (nn', a)$ and the associated semidirect product $M \ast N$. The embedding of Proposition 2.1 from $F_{J_1,H}(A)$ into $M \ast N$ that maps $a$ into $((1, a), a)$ maps $u$ into

$$ (1) \quad ((1, a_1) + (a_1, a_2) + \cdots + (a_1 \ldots a_{i-1}, a_i), a_1 \ldots a_i), $$

and $v$ into

$$ (2) \quad ((1, b_1) + (b_1, b_2) + \cdots + (b_1 \ldots b_{j-1}, b_j), b_1 \ldots b_j). $$

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Denote by \( u' \) (respectively \( v' \)) the first component of (1) (respectively (2)). Then, we have \( J_1 \ast H \models u = v \) if and only if \( F_{J_1}(F_H(A) \times A) \models u' = v' \) and \( F_H(A) \models u = v, \) or \( \alpha(\sigma_G(u)) = \alpha(\sigma_G(v)) \) and \( u' = v'. \)

4. A RESULT ON GRAPHS

In the next section, we give a basis of identities for \( J_1 \ast H \). In order to do this, we use a result on graphs due to Simon which we state in this section.

A (directed) graph \( G \) consists in a set \( V \) of vertices, a set \( E \) of edges and two mappings \( f, g : E \to V \) which to each edge \( e \) assigns the start vertex \( f(e) \) and the end vertex \( g(e) \) of that edge. Two edges \( e_1, e_2 \) are consecutive if \( g(e_1) = f(e_2) \). A path of length \( i, i > 0, \) is a sequence \( e_1 \ldots e_i \) of \( i \) consecutive edges. The mappings \( f \) and \( g \) are extended to mappings \( f, g : P \to V \) by letting \( f(e_1 \ldots e_i) = f(e_1) \) and \( g(e_1 \ldots e_i) = g(e_i) \) (\( P \) denotes the set of all paths in \( G \)). For each vertex \( v \) we allow an empty path \( 1_v \) of length 0 for which \( f(1_v) = g(1_v) = v \). A loop about \( v \) is a path \( x \) such that \( f(x) = g(x) = v \).

An equivalence relation \( \cong \) on \( P \) is called a congruence if it satisfies the following two conditions:

- If \( x \cong y \), then \( x \) and \( y \) are coterminal (that is \( f(x) = f(y) \) and \( g(x) = g(y) \));
- If \( x \cong x', y \cong y' \) and \( g(x) = f(y) \), then \( xy \cong x'y' \).

We agree that each path \( 1_v \) is congruent only to itself.

**Proposition 4.1** (Simon [8]): Let \( \cong \) be the smallest congruence relation on \( P \) satisfying

\[
xx \cong x, \\
xy \cong yx,
\]

for any two loops \( x, y \) about the same vertex. Then any two coterminal paths traversing the same set of edges (without regard to order and multiplicity) are \( \cong \)-equivalent.

The graph \( G_\gamma \) of the transducer of the preceding section is useful in the proof of our main result. The set of vertices of \( G_\gamma \) is \( F_H(A) \), and its set of edges is \( F_H(A) \times A \). The start vertex of the edge \( (n, a) \) is \( n \) and its end vertex is \( na \). We use the notation \( P_\gamma \) for the set of all paths in \( G_\gamma \). To any path

\[
x = (n_1, a_1) \ldots (n_i, a_i)
\]

in \( P_\gamma \), we associate the word \( \bar{x} = a_1 \ldots a_i \) in \( A^* \).

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If \( u \sim_\gamma v \), then \( \sigma_\gamma(u) \) and \( \sigma_\gamma(v) \) are coterminal paths (with start vertex 1 and end vertex \( \pi_\gamma(u) = \pi_\gamma(v) \)) traversing the same set of edges.

Given a morphism \( \varphi : A^* \to M \) where \( M \) denotes a finite monoid, we can define a congruence \( \cong_\gamma \) on \( P_\gamma \) by \( x \cong_\gamma y \) if \( x \) and \( y \) are coterminal, and if for all paths \( z \) from the vertex 1 to the start vertex of \( x \) and \( y \) we have \( \varphi(z \overline{x}) = \varphi(z \overline{y}) \).

5. IDENTITIES FOR \( J_1 * H \)

In this section, we give a basis of identities for \( J_1 * H \).

Let \( A \) be a finite alphabet. Let \( \gamma \) be the congruence generating \( H \) for \( A \) and let \( q \) be a positive integer such that \( u^q \gamma 1 \) for all words \( u \) on \( A \).

**Définition 5.1:** We call a list \( a_1, \ldots, a_i \) of elements of \( A \) \( \gamma \)-circular on \( A \) if \( a_1 \ldots a_i \gamma 1 \) but no nonempty proper prefix of \( a_1 \ldots a_i \) is \( \gamma \)-equivalent to 1. We write \( A_\gamma \) for the set of such \( \gamma \)-circular lists on \( A \).

**Définition 5.2:** We write \( \Sigma_{A_\gamma} \) for the set consisting of the identities

(3) \[ x^{2q} = x^q, \]

(4) \[ x^q y^q = y^q x^q, \]

together with all the identities of the form

(5) \[ (y_1 z_1^q \ldots y_{i-1} z_{i-1}^q y_i)^2 = y_1 z_1^q \ldots y_{i-1} z_{i-1}^q y_i, \]

where \( y_1, \ldots, y_i \) is a list in \( A_\gamma \).

The following definition and lemmas will be useful in the proof of Theorem 5.1.

Let us define recursively what we mean by “a \( \gamma \)-word \( w \) on \( A \).”

**Définition 5.3:** Basis. The empty word 1 is a \( \gamma \)-word on \( A \).

Recursive step. If there exists a list \( a_1, \ldots, a_i \) in \( A_\gamma \), and there exist \( v_1, \ldots, v_{i-1} \) which are finite concatenations of \( \gamma \)-words on \( A \) satisfying \( w = a_1 v_1 \ldots a_i v_{i-1} a_i \), then we say that \( w \) is a \( \gamma \)-word on \( A \).

Closure. A word \( w \) is a \( \gamma \)-word on \( A \) only if it can be obtained from the basis by a finite number of applications of the recursive step.
Note that if a word $w$ is a $\gamma$-word on $A$, it is built only from elements of $A$ which build the lists in $A_\gamma$.

**Lemma 5.1:** We have $\Sigma_{A,\gamma,q} \vdash (u_1^q \ldots u_i^q)^2 = u_1^q \ldots u_i^q$ and so $\Sigma_{A,\gamma,q} \vdash (u_1^q \ldots u_i^q)^q = u_1^q \ldots u_i^q$.

**Proof:** We have $\Sigma_{A,\gamma,q} \vdash u_1^q \ldots u_i^q = u_1^q \ldots u_i^q$ since the identity $x^{2q} = x^q$ belongs to $\Sigma_{A,\gamma,q}$, and so $\Sigma_{A,\gamma,q} \vdash u_1^q \ldots u_i^q = (u_1^q \ldots u_i^q)^2$ by using Identity (4) repeatedly.

**Lemma 5.2:** 1. If $w$ is a $\gamma$-word on $A$, then $\Sigma_{A,\gamma,q} \vdash w^2 = w$ and so $\Sigma_{A,\gamma,q} \vdash w^q = w$;
2. If $w$ and $w'$ are $\gamma$-words on $A$, then $\Sigma_{A,\gamma,q} \vdash ww' = w'w$.

**Proof:** Assertion 1 follows by induction on $w$. Trivially, $\Sigma_{A,\gamma,q} \vdash 1^2 = 1$ and so $\Sigma_{A,\gamma,q} \vdash 1^q = 1$. If $v$ is a finite concatenation of $\gamma$-words on $A$, say $v = u_1 \ldots u_j$, then by using the inductive assumption on $u_1, \ldots, u_j$ as well as Lemma 5.1 we get $\Sigma_{A,\gamma,q} \vdash v^2 = (u_1 \ldots u_j)^2 = (u_1^q \ldots u_j^q)^2 = u_1^q \ldots u_j^q = v$, and so $\Sigma_{A,\gamma,q} \vdash v^q = v$. Now, if there exists a list $a_1, \ldots, a_i$ in $A_\gamma$, and there exist $v_1, \ldots, v_{i-1}$ which are finite concatenations of $\gamma$-words on $A$ satisfying $w = a_1v_1 \ldots a_{i-1}v_{i-1}v_i$, then by using an identity of the form (5) we get $\Sigma_{A,\gamma,q} \vdash w^2 = (a_1v_1 \ldots a_{i-1}v_{i-1}v_i)^2 = (a_1v_1^q \ldots a_{i-1}v_{i-1}^q v_i^q)^2 = a_1v_1^q \ldots a_{i-1}v_{i-1}^q v_i^q = w$ and so $\Sigma_{A,\gamma,q} \vdash w^q = w$.

Assertion 2 follows from $\Sigma_{A,\gamma,q} \vdash ww' = w^q(w')^q = (w')^qw^q = w'w$.

**Lemma 5.3:** If $w\gamma_1$, then $\alpha(\sigma_\gamma(u^2)) = \alpha(\sigma_\gamma(u))$. As consequences, $u^{2q} \sim_\gamma u^2$ and $u^qv^q \sim_\gamma v^qu^q$.

**Proof:** If $w\gamma_1$, then $\sigma_\gamma(u^2) = \sigma_\gamma(u)\sigma_\gamma(u) = \sigma_\gamma(u)\sigma_\gamma(u)$ since $\pi_\gamma(u) = 1$. We have $u^qv\gamma_1$ and $u^q\gamma_1$, and so $u^q, u^{2q}, u^qv^q$ and $v^qu^q$ are $\gamma$-equivalent to 1. The equalities $\alpha(\sigma_\gamma(u^{2q})) = \alpha(\sigma_\gamma(u^2))$ and $\alpha(\sigma_\gamma(u^qv^q)) = \alpha(\sigma_\gamma(v^qu^q))$ are easy to check.

Now, let $r$ be a positive integer and put $A_r = \{x_1, \ldots, x_r\}$. Let $\gamma_r$ be the congruence generating $H$ for $A_r$ and let $q_r$ be a positive integer such that $u^{q_r} \gamma_r 1$ for all words $u$ on $A_r$.

**Theorem 5.1:** We have $\mathbf{J}_1 \ast \mathbf{H} = \mathbf{V}(\bigcup_{r \geq 1} \Sigma_{A_r,\gamma_r,q_r})$.

**Proof:** We will show that an $A$-generated monoid $M$ is in $\mathbf{J}_1 \ast \mathbf{H}$ if and only if $M \vdash \Sigma_{A,\gamma,q}$ where $A$ abbreviates $A_r$, $\gamma$ abbreviates $\gamma_r$ and $q$ abbreviates $q_r$. By Lemma 3.2, $A$-generated monoids in $\mathbf{J}_1 \ast \mathbf{H}$ satisfy identities $u = v$
where \( u \sim_\gamma v \) (that is \( \alpha(\sigma_\gamma(u)) = \alpha(\sigma_\gamma(v)) \) and \( u_\gamma v \)). Lemma 5.3 implies that \( x^{2q} \sim_\gamma x^q \) and \( x^q y^q \sim_\gamma y^q x^q \). We also have \( x^2 \sim_\gamma x \) for all the identities \( x^2 = x \) of the form (5). To see this, put \( x = y_1 x_1^q \ldots y_{i-1} x_{i-1}^q y_i \) with \( y_1, \ldots, y_i \) a list in \( A_\gamma \). Since \( x \) is \( \gamma \)-equivalent to 1, we get \( x^2 \gamma x \). The equality \( \alpha(\sigma_\gamma(x^2)) = \alpha(\sigma_\gamma(x)) \) follows from Lemma 5.3.

Conversely, let \( \varphi : A^* \to M \) be a surjective morphism satisfying \( \varphi(u) = \varphi(v) \) for every identity \( u = v \) in \( \Sigma_{A, \gamma, q} \). We also denote by \( \varphi \) the (nuclear) congruence on \( A^* \) associated with \( \varphi \) and defined by \( w \varphi v \) if and only if \( \varphi(u) = \varphi(v) \). We show the inclusion \( \sim_\gamma \subseteq \varphi \) which yields \( M = A^*/\varphi \) is a morphic image of \( A^*/\sim_\gamma \). The membership of \( M \) to \( J_1 * H \) follows by Lemma 3.2.

We consider the graph \( G_\gamma \) and the congruence relation \( \cong_\gamma \) on its set of paths \( P_\gamma \) defined at the end of Section 4. Let \( x \) and \( y \) be two loops about the same vertex \( \pi_\gamma(w) \), or

\[
x = (\pi_\gamma(w), a_1) \ldots (\pi_\gamma(wa_1 \ldots a_{i-1}), a_i),
\]

\[
y = (\pi_\gamma(w), b_1) \ldots (\pi_\gamma(wb_1 \ldots b_{j-1}), b_j),
\]

where \( wa_1 \ldots a_i \gamma w \gamma wb_1 \ldots b_j \). We show the following two claims: Claim 1 or \( xx \cong_\gamma x \), and Claim 2 or \( xy \cong_\gamma yx \). Now if \( u \sim_\gamma v \), then \( \sigma_\gamma(u) \) and \( \sigma_\gamma(v) \) are two coterminally paths traversing the same set of edges (the start vertex of \( \sigma_\gamma(u) \) and \( \sigma_\gamma(v) \) is 1 and their end vertex is \( \pi_\gamma(u) = \pi_\gamma(v) \)). Hence, by Proposition 4.1, \( \sigma_\gamma(u) \cong_\gamma \sigma_\gamma(v) \). Therefore, \( \varphi(\sigma_\gamma(u)) = \varphi(\sigma_\gamma(v)) \) or \( \varphi(u) = \varphi(v) \) and the inclusion \( \sim_\gamma \subseteq \varphi \) follows.

Let us now prove Claim 1 and Claim 2. Since \( wa_1 \ldots a_i \gamma w \) and \( wb_1 \ldots b_j \gamma w \), we have \( \bar{x} = a_1 \ldots a_i \gamma 1 \) and \( \bar{y} = b_1 \ldots b_j \gamma 1 \) since \( H \) is a pseudovariety of groups.

**Proof of Claim 1:** The condition \( xx \cong_\gamma x \) follows by showing that \( \varphi(\bar{x} \bar{x}) = \varphi(\bar{x} \bar{x}) \) for all paths \( z \) from the vertex 1 to the start vertex of \( x \). Here we can show that \( \varphi(\bar{x} \bar{x}) = \varphi(\bar{x}) \) (and therefore \( \varphi(\bar{x}^q) = \varphi(\bar{x}) \)). The word \( \bar{x} \) has the property \( \mathcal{P} \) that "it is \( \gamma \)-equivalent to 1". The word \( \bar{x} \) can be factorized as follows: let \( u_1 \) be the smallest nonempty prefix of \( \bar{x} \) with Property \( \mathcal{P} \); let \( u_2 \) be the smallest nonempty prefix of \( \bar{x} \setminus u_1 \) with Property \( \mathcal{P} \); ... So \( \bar{x} \) is a concatenation of factors \( u_1 \ldots u_n \) with Property \( \mathcal{P} \). Since no nonempty proper prefix of \( u_1 \) has Property \( \mathcal{P} \), let \( c_1 v_1 \) be the shortest prefix of \( u_1 \) such that \( \pi_\gamma(c_1 v_1) = \pi_\gamma(c_1) \); ... let \( c_{\ell-1} v_{\ell-1} \) be the shortest prefix of \( u_1 \setminus c_1 v_1 \ldots c_{\ell-2} v_{\ell-2} \) such that \( \pi_\gamma(c_1 v_1 \ldots c_{\ell-2} v_{\ell-2} c_{\ell-1} v_{\ell-1}) = \pi_\gamma(c_1 v_1 \ldots c_{\ell-2} v_{\ell-2} c_{\ell-1}) \); and
let $c_\ell = u_1 \setminus c_1 v_1 \ldots c_{\ell-1} v_{\ell-1}$ satisfying $\pi_\gamma(c_1 v_1 \ldots c_{\ell-1} v_{\ell-1} c_\ell) = \pi_\gamma(1)$. 
So $u_1 = c_1 v_1 \ldots c_{\ell-1} v_{\ell-1} c_\ell$ where $c_1, \ldots, c_\ell \in A_\gamma$ and where the $v$-factors have Property $P$ (similar statements hold for $u_2, \ldots, u_n$). Since the $v$-factors have Property $P$, they can be factorized as above and the process can be repeated. Factors in $\bar{x}$ are hence $\gamma$-words on $A$. We have $\varphi(u_1) = \varphi(u_1^q)$, $\ldots$, $\varphi(u_n) = \varphi(u_n^q)$ (as in Lemma 5.2). Therefore $\varphi(\bar{x}) = \varphi(u_1 \ldots u_n) = \varphi(u_1^q \ldots u_n^q) = \varphi((u_1^q \ldots u_n^q)^2)$ (as in Lemma 5.1) $= \varphi(x^2) = \varphi(x \bar{x})$.

Proof of Claim 2: The condition $xy \equiv_\gamma yx$ follows from $\varphi(xy) = \varphi(\bar{x} \bar{y}) = \varphi(x) \varphi(y) = \varphi(x^q) \varphi(y^q) = \varphi(\bar{x} \bar{y}^q) = \varphi(\bar{x} \bar{y}^q)$ (using Identity (4)). $\square$

6. IDENTITIES FOR $J_1 \ast G_p$

In this section, we give a sequence of sets of identities ultimately defining $J_1 \ast G_p$.

Let $A$ be a finite alphabet and let $u, w \in A^*$ with $u = a_1 \ldots a_i$. The binomial coefficient $\left(\begin{array}{c} w \\ u \end{array}\right)$ is defined as the number of distinct factorizations of the form

$$w = v_0 a_1 v_1 \ldots a_i v_i$$

with $v_0, \ldots, v_i \in A^*$. Thus the binomial coefficient counts the number of ways in which $u$ is a sub word of $w$. We adopt the convention that $\left(\begin{array}{c} 0 \\ 1 \end{array}\right) = 1$.

Let $a, b \in A$ and $u, w, w' \in A^*$. The following formulas are easily verified:

- $\left(\begin{array}{c} a^j \\ a \end{array}\right) = \left(\begin{array}{c} j \\ 0 \end{array}\right)$ where $i \geq j$;
- $\left(\begin{array}{c} 1 \\ a \end{array}\right) = \begin{cases} 1, & \text{if } u = 1, \\ 0, & \text{otherwise}; \end{cases}$
- $\left(\begin{array}{c} a^j \\ u \end{array}\right) = \begin{cases} 1, & \text{if } u = 1 \text{ or } u = a, \\ 0, & \text{otherwise}; \end{cases}$
- $\left(\begin{array}{c} w^a \\ u \\ b \end{array}\right) = \left(\begin{array}{c} w \\ u \\ b \end{array}\right) + \delta_{a,b} \left(\begin{array}{c} w \\ u \end{array}\right)$ where $\delta_{a,b} = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise}; \end{cases}$
- $\left(\begin{array}{c} w \\ u \\ w' \end{array}\right) = \sum_{u = uv' \in A^*} \left(\begin{array}{c} w \\ u \\ v' \end{array}\right)$.

Given a word $u$ on $A$, we define on $A^*$ the equivalence relation $\gamma_{p,u}$ by $\gamma_{p,u} = u \gamma_{p,u} u$ if and only if $\left(\begin{array}{c} w \\ v \end{array}\right) \equiv \left(\begin{array}{c} w' \\ v' \end{array}\right) \mod p$ whenever $u \in A^* v A^*$.

Now, given an integer $k \geq 0$, we define on $A^*$ the equivalence relation $\gamma_{p,k}$ by $\gamma_{p,k} = \bigcap_{|u| = k} \gamma_{p,u}$. Thus $\gamma_{p,k} = \bigcap_{|u| = k} \gamma_{p,u}$. Thus $\gamma_{p,k} = \bigcap_{|u| = k} \gamma_{p,u}$.

Note that for all $w, w' \in A^*$ we have $w \gamma_{p,0} w'$.
**Lemma 6.1** (Eilenberg [8]): The equivalence relations $\gamma_{p,u}$ and $\gamma_{p,k}$ are congruences of finite index on $A^*$.

**Lemma 6.2** (Eilenberg [8]): Let $k$ be a positive integer and $u \in A^*$. If $w \in A^*$, then $w^{p^{|w|}} \gamma_{p,u}^1$ and $w^{p^k} \gamma_{p,k}^1$.

**Proof:** If $w \in A^*$, then the following conditions are equivalent:

- $w \gamma_{p,k} 1$;
- $(w^i_u) \equiv 0 \mod p$ whenever $0 < |v| \leq k$.

We show the $\gamma_{p,k}$-equivalence of $w^{p^k}$ and $1$. For $k = 1$, the result holds trivially. We proceed by induction and assume $0 < |v| \leq k + 1$. Then

$$
(w^{p^{k+1}}_v) = \sum (w^{p^k}_{v_1}) \cdots (w^{p^k}_{v_p}),
$$

where the summation extends over all factorizations $v = v_1 \ldots v_p$ of $v$. If for some $1 \leq i \leq p$ we have $0 < |v_i| < k + 1$, then by the inductive assumption $(w^{p^k}_{v_i}) \equiv 0 \mod p$ and the summand may be omitted. There remain summands with $v_i = v$, $v_j = 1$ for $j \neq i$. Each such summand yields $(w^{p^k}_v)$ and there are exactly $p$ such summands. Thus $(w^{p^{k+1}}_v) \equiv 0 \mod p$ as required. \(\square\)

The quotients $A^*/\gamma_{p,u}$ and $A^*/\gamma_{p,k}$ are finite monoids by Lemma 6.1. Lemma 6.2 implies that $A^*/\gamma_{p,u}$ satisfies the identity $x^{p^{|v|}} = 1$ and $A^*/\gamma_{p,k}$ the identity $x^{p^k} = 1$. Note that $A^*/\gamma_{p,0}$ is the trivial group. If $A = \{a_1, \ldots, a_r\}$, $A^*/\gamma_{p,1}$ is isomorphic to the set of all words of the form $a_1^{e_1} \ldots a_r^{e_r}$ with $0 \leq e_i < p$ multiplying two such words through the addition of the respective exponents.

We now describe the $*$-variety $G_p$ of sets defined by the pseudovariety $G_p$.

**Lemma 6.3** (Eilenberg [8]): **The pseudovariety $G_p$ is generated by the groups $A^*/\gamma_{p,k}$ for all integers $k \geq 0$ and all finite alphabets $A$, or by the groups $A^*/\gamma_{p,u}$ for all elements $u \in A^*$ and all finite alphabets $A$.

- $A^*G_p$ is the boolean closure of the sets

$$
\{w \in A^* \mid (w^i_u) \equiv i \mod p\}, \ u \in A^*, \ 0 \leq i < p.
$$

Let $k$ be a nonnegative integer and define the pseudovariety $H_{p,k}$ as the locally finite pseudovariety of groups generated by $A^*/\gamma_{p,k}$ for all finite alphabets $A$. The $*$-variety $A^*H_{p,k}$ is then the boolean closure of the sets

$$
\{w \in A^* \mid (w^i_u) \equiv i \mod p\}, \ u \in A^* \text{ with } |u| \leq k, \ 0 \leq i < p.
$$
The pseudovariety $H_{p,0}$ is the trivial pseudovariety $I = V(x = 1)$. Since $I$ is the unit element for the semidirect product operation on pseudovarieties of monoids, we have $J_1 * H_{p,0} = J_1 = V(x^2 = x, xy = yx)$.

Now, let $k$ be a positive integer. A list $a_1, \ldots, a_i$ of elements of $A$ is $\gamma_{p,k}$-circular on $A$ if $(a_1 \ldots a_i)^{\equiv 0 \mod p}$ whenever $0 < |v| \leq k$, but no nonempty proper prefix $w$ of $a_1 \ldots a_i$ satisfies $(w)^{\equiv 0 \mod p}$ for every $0 < |v| \leq k$. For example, $a, b, b, a, a, b, a$ is a list in $\{a, b\}_{\gamma_{2,2}}$.

If $k$ and $r$ are positive integers, we write $\Sigma_{p,k}^r$ for the set consisting of the identities

$$x^{2p^k} = x^{p^k},$$

(6) $$x^{p^k}y^{p^k} = y^{p^k}x^{p^k},$$

(7) together with all the identities of the form

(8) $$(y_1z_1^{p^k} \ldots y_{i-1}z_{i-1}^{p^k} y_i)^2 = y_1z_1^{p^k} \ldots y_{i-1}z_{i-1}^{p^k} y_i,$$

where $y_1, \ldots, y_i$ is a list in $\{x_1, \ldots, x_r\}_{\gamma_{p,k}}$. We write $\Sigma_{p,k}$ for $\bigcup_{r \geq 1} \Sigma_{p,k}^r$.

Continuing with the above example, the identity $x^2 = x$ where

$$x = x_1z_1^{22} x_2z_2^{22} x_2z_3^{22} x_1z_4^{22} x_1z_5^{22} x_2z_6^{22} x_2z_7^{22} x_1,$$

belongs to $\Sigma_{2,2}^2$.

For $r \geq 1$, $\Sigma_{p,k}^r \subseteq \Sigma_{p,k}^{r+1}$. This follows from the fact that if $A \subseteq B$, then $A_{\gamma_{p,k}} \subseteq B_{\gamma_{p,k}}$.

**Corollary 6.1:** The pseudovariety $J_1 * G_p$ is ultimately defined by $\Sigma_{p,k}$, $k \geq 1$ or a monoid is in $J_1 * G_p$ if and only if it satisfies $\Sigma_{p,k}$ for all $k$ sufficiently large.

**Proof:** By Theorem 5.1, the pseudovariety $J_1 * H_{p,k}$ is defined by $\Sigma_{p,k}$. Now, the semidirect product operation on pseudovarieties commutes with directed unions [3]. We get $J_1 * G_p = J_1 * \bigcup_{k \geq 0} H_{p,k} = \bigcup_{k \geq 0} J_1 * H_{p,k} = \bigcup_{k \geq 1} J_1 * H_{p,k}$ and the result follows.

**REFERENCES**


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