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ON THE SEMIDIRECT PRODUCT OF THE PSEUDOVARIETY OF SEMILATTICES BY A LOCALLY FINITE PSEUDOVARIETY OF GROUPS (*)

by F. BLANCHET-SADRI ([1]) ([2])

Abstract – *In this paper, we give a sequence of identities defining the product pseudovariety* $\mathbf{J}_1 * \mathbf{H}$ *generated by all semidirect products of the form* $M * N$ *with* $M \in \mathbf{J}_1$ *and* $N \in \mathbf{H}$ *(here* $\mathbf{J}_1$ *is the pseudovariety of semilattice monoids and* $\mathbf{H}$ *is a locally finite pseudovariety of groups) A sequence of sets of identities ultimately defining* $\mathbf{J}_1 * \mathbf{G}_p$ *results (here* $\mathbf{G}_p$ *is the pseudovariety of p-groups)*

Résumé – *Dans cet article, nous donnons une suite d'identités définissant la pseudovariéte* $\mathbf{J}_1 * \mathbf{H}$ *engendrée par les produits semidirects de la forme* $M * N$ *où* $M \in \mathbf{J}_1$ *et* $N \in \mathbf{H}$ *(ici* $\mathbf{J}_1$ *est la pseudovariété des demi-treillis et* $\mathbf{H}$ *une pseudovariété de groupes localement finie) Une suite d'ensembles d'identités définissant ultimement* $\mathbf{J}_1 * \mathbf{G}_p$ *en résulte (ici* $\mathbf{G}_p$ *est la pseudovariété des p-groupes)*

1. INTRODUCTION

In this paper, we discuss a technique to produce identities for the semidirect product pseudovariety $\mathbf{J}_1 * \mathbf{H}$ generated by all semidirect products of the form $M * N$ with $M \in \mathbf{J}_1$ and $N \in \mathbf{H}$, where $\mathbf{J}_1$ is the pseudovariety of all semilattice monoids and $\mathbf{H}$ is a locally finite pseudovariety of groups.

The notion of congruence plays a central role in our approach. For any finite alphabet A denote by A^* the free monoid generated by A. We say that a monoid M is *A-generated* if there exists a congruence β on A^* such that M is isomorphic to A^*/β. A pseudovariety of monoids $\mathbf{V}$ is *locally finite* if

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for any A there are finitely many A-generated monoids in $\mathbf{V}$. Equivalently, there exists for each A a congruence β_A such that an A-generated monoid M is in $\mathbf{V}$ if and only if M is a morphic image of A^*/β_A.

Let $\mathbf{H}$ be a locally finite pseudovariety of groups. Let γ be the congruence generating $\mathbf{H}$ for the finite alphabet A. The idea is to associate with $\mathbf{J}_1 * \mathbf{H}$ a congruence $\sim_\gamma$ on A^*. Section 3 gives a criterion to determine when an identity on A is satisfied in $\mathbf{J}_1 * \mathbf{H}$ with the help of $\sim_\gamma$. This leads to a proof that such $\mathbf{J}_1 * \mathbf{H}$ are locally finite and hence decidable. This criterion follows from Almeida's semidirect product representation of the free objects in $\mathbf{V} * \mathbf{W}$ in case both $\mathbf{V}$ and $\mathbf{W}$ have finite free objects [1] (Almeida's representation is stated in Section 2.1). In Section 5, we give a basis of identities for $\mathbf{J}_1 * \mathbf{H}$ which follows mainly from a result on graphs due to Simon [8] (Simon's result is stated in Section 4) and the identity criterion of Section 3. In Section 6, we give a sequence of sets of identities ultimately defining the pseudovariety $\mathbf{J}_1 * \mathbf{G}_p$, where p is a prime number and $\mathbf{G}_p$ is the pseudovariety of all p-groups, that is the pseudovariety of all groups of order p^k for some nonnegative integer k.

Related known results include the following. The product $\mathbf{J}_1 * \mathbf{G}$ is generated by the inverse monoids (Margolis and Pin [11]) and is the class of finite monoids in which the idempotents commute (Ash [4]) (here $\mathbf{G}$ is the pseudovariety of groups). Blanchet-Sadri and Zhang [6] give identities ultimately defining the product $\mathbf{J}_1 * \mathbf{G}_{com}$ where $\mathbf{G}_{com}$ denotes the pseudovariety of commutative groups. Irastorza [10] shows that if the pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ are finitely based, their product may not be.

The techniques in this paper were used in particular by Pin [13] to give a basis of identities for $\mathbf{J}_1 * \mathbf{J}_1$, by Almeida [2] to generalize Pin's result to iterated semidirect products of finite semilattices, and by Blanchet-Sadri [5] to give a basis of identities for $\mathbf{J}_1 * \mathbf{J}_k$ where $\mathbf{J}_k$ denotes the pseudovariety of $\mathcal{J}$-trivial monoids of height k.

2. PRELIMINARIES

We refer the reader to [3, 7, 8, 12] for terms not explicitly defined here.

2.1. Pseudovarieties of monoids

A nonempty class of finite monoids is called a *pseudovariety* if it is closed under submonoids, morphic images, and finitary direct products. A nonempty class of monoids is called a *variety* if it is closed under submonoids, morphic images, and direct products.

As the intersection of a class of pseudovarieties of monoids is again a pseudovariety, and as all finite monoids form a pseudovariety, we can conclude that for every class C of finite monoids there is a smallest pseudovariety containing C, called *the pseudovariety generated by* C. Now, if C is a class of monoids, the smallest variety containing C is called *the variety generated by* C.

For a pseudovariety $\mathbf{V}$ and a set A, $F_{\mathbf{V}}(A)$ denotes the *free object* on A (or generated by A) in the variety generated by $\mathbf{V}$. If A is finite, say $A = \{a_1, \ldots, a_r\}$, we often write $F_{\mathbf{V}}(a_1, \ldots, a_r)$ for $F_{\mathbf{V}}(A)$. In case $\mathbf{V}$ is the pseudovariety of all finite semigroups (respectively all finite monoids), the semigroup (respectively monoid) $F_{\mathbf{V}}(A)$ is usually denoted by A^+ (respectively A^*). Elements of A^+ are viewed as nonempty words of elements of A, and the multiplication is given by concatenation of words. The monoid A^* includes also the empty word 1. For a word $u \in A^*$, let $|u|$ denote the length of u. For words $u, v, w \in A^*$ satisfying $w = uv$, let $w \setminus u$ denote the factor v.

2.1.1. *Semidirect products of pseudovarieties*

Let M and N be monoids. It is convenient to write M additively, without however assuming that M is commutative. We denote by 0 (respectively 1) the unit element of M (respectively N). A *left action* of N on M is a morphism φ from N into the monoid of monoid endomorphisms of M, where endomorphisms of M are written on the left.

Given a left action φ of N on M, we define the *semidirect product* $M * N$ as follows. The elements of $M * N$ are pairs (m, n) with $m \in M$, $n \in N$. Multiplication is given by the formula

$$(m, n)(m', n') = (m + nm', nn')$$

where nm' represents $\varphi(n)(m')$. (This is what Eilenberg [8] calls a "unitary" semidirect product.) The multiplication in $M * N$ is associative. Thus $M * N$ is a monoid with $(0, 1)$ as unit element.

We now relate the notion of pseudovariety with that of a semidirect product. Given pseudovarieties of monoids $\mathbf{V}$ and $\mathbf{W}$, we denote by $\mathbf{V} * \mathbf{W}$ the pseudovariety generated by all semidirect products $M * N$ with $M \in \mathbf{V}$, $N \in \mathbf{W}$ and with any left action of N on M. The semidirect product of pseudovarieties of monoids is associative.

PROPOSITION 2.1: (Almeida [1]) *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of monoids such that* $F_{\mathbf{V}}(A)$ *and* $F_{\mathbf{W}}(A)$ *are finite for all finite* A. *Then so is* $\mathbf{V} * \mathbf{W}$. *Moreover, for a finite set* A, *let* $N = F_{\mathbf{W}}(A)$ *and* $M = F_{\mathbf{V}}(N \times A)$. *Consider the left action of* N *on* M *defined by* $n(n', a) = (nn', a)$ *and the associated semidirect product* $M * N$. *Then, there is an embedding from* $F_{\mathbf{V}*\mathbf{W}}(A)$ *into* $M * N$ *that maps* a *into* $((1, a), a)$.

2.1.2. *Pseudovarieties and sequences of identities*

Let A be a set. A monoid *identity* on A is an expression of the form $u = v$ where $u, v \in A^*$. A monoid M *satisfies* an identity $u = v$ (or the identity *is true in* M, or *holds in* M), abbreviated by $M \models u = v$, if for every morphism $\varphi : A^* \to M$ we have $\varphi(u) = \varphi(v)$.

A class C of monoids satisfies $u = v$, written $C \models u = v$, if each member of C satisfies $u = v$. If Σ is a set of identities, we say C satisfies Σ, written $C \models \Sigma$, if $C \models u = v$ for each $u = v \in \Sigma$. An identity $u = v$ is *deducible* from a set of identities Σ, abbreviated by $\Sigma \vdash u = v$, if for every monoid M we have $M \models \Sigma$ implies $M \models u = v$. Here, letters can be erased in monoid identities.

Let $u_i = v_i, i \geq 1$ be a sequence of identities. Put $\Sigma = \{u_i = v_i \mid i \geq 1\}$, and define $\mathbf{V}(\Sigma)$ to be the class of finite monoids satisfying Σ or all the identities $u_i = v_i$. A class C of finite monoids is said to be *defined* by Σ (or by the identities $u_i = v_i, i \geq 1$) if $C = \mathbf{V}(\Sigma)$; Σ is said to be a *basis* for C. Eilenberg and Schützenberger [9] show that every pseudovariety generated by a single monoid is of the form $\mathbf{V}(\Sigma)$ for some such Σ.

2.2. Varieties of sets

Let L be a subset of A^*. We define a congruence $\sim_L$ on A^* as follows: $u \sim_L v$ holds if $xuy \in L$ if and only if $xvy \in L$ for all $x, y \in A^*$. The congruence $\sim_L$ is called the *syntactic congruence* of L, and the quotient monoid $A^*/\sim_L$, which we denote by $M(L)$, is called the *syntactic monoid* of L. The subset L of A^* is saturated for the congruence $\sim_L$, that is $u \sim_L v$ and $u \in L$ imply $v \in L$. Each pseudovariety of monoids is generated by the syntactic monoids that it contains. The set L is recognizable if and only if $M(L)$ is a finite monoid.

Suppose that for each finite alphabet A, a family $A^*\mathcal{V}$ of recognizable sets of A^* is given. We then say that $\mathcal{V} = \{A^*\mathcal{V}\}$ is a $*$-*variety* of sets if it satisfies the following conditions:

- $A^*\mathcal{V}$ is closed under boolean operations;

• If $L \in A^*\mathcal{V}$ and $a \in A$, then the sets $a^{-1}L = \{w \in A^* \mid aw \in L\}$ and $La^{-1} = \{w \in A^* \mid wa \in L\}$ are in $A^*\mathcal{V}$;

• If $\varphi : B^* \to A^*$ is a monoid morphism and if $L \in A^*\mathcal{V}$, then $\varphi^{-1}(L) \in B^*\mathcal{V}$.

Pseudovarieties of monoids and $*$-varieties of sets are in 1–1 correspondence. If $\mathcal{V}$ is a $*$-variety of sets, then the pseudovariety of monoids generated by $\{M(L) \mid L \in A^*\mathcal{V}$ for some $A\}$ defines the corresponding pseudovariety of monoids $\mathbf{V}$. If $\mathbf{V}$ is a pseudovariety of monoids, then $A^*\mathcal{V} = \{L \subseteq A^* \mid M(L) \in \mathbf{V}\}$ defines the corresponding $*$-variety of sets $\mathcal{V}$.

3. CONGRUENCES FOR $\mathbf{J}_1 * \mathbf{H}$

In this section, we give a criterion to determine when an identity is satisfied in the semidirect product $\mathbf{J}_1 * \mathbf{H}$ where $\mathbf{H}$ is a locally finite pseudovariety of groups. This criterion is used in Section 5 to obtain a basis of identities for $\mathbf{J}_1 * \mathbf{H}$.

Let A be a finite set. For a word $u \in A^*$, let $\alpha(u)$ denote the set of elements of A that occur in u. Then the free object of $\mathbf{J}_1$ on A is isomorphic to the quotient A^*/α where the congruence α on A^* is defined by $u\alpha v$ if and only if $\alpha(u) = \alpha(v)$. Now, let γ be the congruence of finite index on A^* such that an A-generated monoid M belongs to $\mathbf{H}$ if and only if M is a morphic image of A^*/γ. The free object $F_{\mathbf{H}}(A)$ is isomorphic to the quotient A^*/γ. The pseudovarieties $\mathbf{J}_1$ and $\mathbf{H}$ have hence finite finitely generated free objects. We denote by π_γ the canonical projection from A^* into $F_{\mathbf{H}}(A)$ that maps a onto the generator a of $F_{\mathbf{H}}(A)$. If $u, v \in A^*$, then $\pi_\gamma(u) = \pi_\gamma(v)$ if and only if $u\gamma v$.

DEFINITION 3.1: *Let* $w \in A^*$.

• *Let* $\sigma_\gamma : A^* \to (F_{\mathbf{H}}(A) \times A)^*$ *be the function defined by*

$$\sigma_\gamma(a_1 \ldots a_i) = (1, a_1)(\pi_\gamma(a_1), a_2) \ldots (\pi_\gamma(a_1 \ldots a_{i-1}), a_i)$$

if $i > 0$, 1 *otherwise.*

• *Let* $\sigma_\gamma^w : A^* \to (F_{\mathbf{H}}(A) \times A)^*$ *be the function defined by*

$$\sigma_\gamma^w(a_1 \ldots a_i) = (\pi_\gamma(w), a_1)(\pi_\gamma(wa_1), a_2) \ldots (\pi_\gamma(wa_1 \ldots a_{i-1}), a_i)$$

if $i > 0$, 1 *otherwise.*

The sequential function σ_γ is realized by the transducer whose states are the elements of $F_{\mathbf{H}}(A)$ (1 being the initial state) and whose transitions are given by

$$n \overset{a/(n,a)}{\longrightarrow} na$$

where $n \in F_{\mathbf{H}}(A)$ and $a \in A$.

We define an equivalence relation on A^* by requesting that

$$u \sim_\gamma v \text{ if and only if } \alpha(\sigma_\gamma(u)) = \alpha(\sigma_\gamma(v)) \text{ and } u\gamma v.$$

LEMMA 3.1: *The equivalence relation $\sim_\gamma$ is a congruence of finite index on A^*.*

Proof: Assume $u \sim_\gamma v$ and $u' \sim_\gamma v'$. We have

$$\alpha(\sigma_\gamma(u)) = \alpha(\sigma_\gamma(v)) \text{ and } u\gamma v$$

and similarly with u and v replaced by u' and v'. Since γ is a congruence we have $uu'\gamma vv'$. The above and the fact that $\pi_\gamma(u) = \pi_\gamma(v)$ imply that $\alpha(\sigma_\gamma(uu')) = \alpha(\sigma_\gamma(u)\sigma_\gamma^u(u')) = \alpha(\sigma_\gamma(u)\sigma_\gamma^v(u')) = \alpha(\sigma_\gamma(v)\sigma_\gamma^v(v')) = \alpha(\sigma_\gamma(vv'))$. Thus $uu' \sim_\gamma vv'$ showing that $\sim_\gamma$ is a congruence. This obviously is a finite congruence since α and γ are finite. □

LEMMA 3.2: *If $u = v$ is an identity on A, then the following conditions are equivalent:*

- $\mathbf{J}_1 * \mathbf{H} \models u = v$;
- $u \sim_\gamma v$.

Consequently, an A-generated monoid M belongs to $\mathbf{J}_1 * \mathbf{H}$ if and only if M is a morphic image of $A^*/\sim_\gamma$.

Proof: Let $u = v$ be an identity on A, say $u = a_1 \ldots a_i$ and $v = b_1 \ldots b_j$. Let $N = F_{\mathbf{H}}(A)$ and $M = F_{\mathbf{J}_1}(N \times A)$. Consider the left action of N on M defined by $n(n', a) = (nn', a)$ and the associated semidirect product $M * N$. The embedding of Proposition 2.1 from $F_{\mathbf{J}_1 * \mathbf{H}}(A)$ into $M * N$ that maps a into $((1, a), a)$ maps u into

$$((1, a_1) + (a_1, a_2) + \cdots + (a_1 \ldots a_{i-1}, a_i), a_1 \ldots a_i), \tag{1}$$

and v into

$$((1, b_1) + (b_1, b_2) + \cdots + (b_1 \ldots b_{j-1}, b_j), b_1 \ldots b_j). \tag{2}$$

Denote by u' (respectively v') the first component of (1) (respectively (2)). Then, we have $\mathbf{J}_1 * \mathbf{H} \models u = v$ if and only if $F_{\mathbf{J}_1 * \mathbf{H}}(A) \models u = v$. This is equivalent to the two conditions $F_{\mathbf{J}_1}(F_{\mathbf{H}}(A) \times A) \models u' = v'$ and $F_{\mathbf{H}}(A) \models u = v$, or $\alpha(\sigma_\gamma(u)) = \alpha(\sigma_\gamma(v))$ and $u\gamma v$. □

4. A RESULT ON GRAPHS

In the next section, we give a basis of identities for $\mathbf{J}_1 * \mathbf{H}$. In order to do this, we use a result on graphs due to Simon which we state in this section.

A (directed) graph G consists in a set V of *vertices*, a set E of *edges* and two mappings $f, g : E \to V$ which to each edge e assigns the *start* vertex $f(e)$ and the *end* vertex $g(e)$ of that edge. Two edges e_1, e_2 are *consecutive* if $g(e_1) = f(e_2)$. A *path* of *length* $i, i > 0$, is a sequence $e_1 \ldots e_i$ of i consecutive edges. The mappings f and g are extended to mappings $f, g : P \to V$ by letting $f(e_1 \ldots e_i) = f(e_1)$ and $g(e_1 \ldots e_i) = g(e_i)$ (P denotes the set of all paths in G). For each vertex v we allow an empty path 1_v of length 0 for which $f(1_v) = g(1_v) = v$. A *loop* about v is a path x such that $f(x) = g(x) = v$.

An equivalence relation $\cong$ on P is called a *congruence* if it satisfies the following two conditions:

- If $x \cong y$, then x and y are coterminal (that is $f(x) = f(y)$ and $g(x) = g(y)$);
- If $x \cong x'$, $y \cong y'$ and $g(x) = f(y)$, then $xy \cong x'y'$.

We agree that each path 1_v is congruent only to itself.

PROPOSITION 4.1 (Simon [8]): *Let $\cong$ be the smallest congruence relation on P satisfying*

$$xx \cong x,$$

$$xy \cong yx,$$

for any two loops x, y about the same vertex. Then any two coterminal paths traversing the same set of edges (without regard to order and multiplicity) are $\cong$-equivalent.

The graph G_γ of the transducer of the preceding section is useful in the proof of our main result. The set of vertices of G_γ is $F_{\mathbf{H}}(A)$, and its set of edges is $F_{\mathbf{H}}(A) \times A$. The start vertex of the edge (n, a) is n and its end vertex is na. We use the notation P_γ for the set of all paths in G_γ. To any path

$$x = (n_1, a_1) \ldots (n_i, a_i)$$

in P_γ, we associate the word $\overline{x} = a_1 \ldots a_i$ in A^*.

If $u \sim_\gamma v$, then $\sigma_\gamma(u)$ and $\sigma_\gamma(v)$ are coterminal paths (with start vertex 1 and end vertex $\pi_\gamma(u) = \pi_\gamma(v)$) traversing the same set of edges.

Given a morphism $\varphi : A^* \to M$ where M denotes a finite monoid, we can define a congruence $\cong_\gamma$ on P_γ by $x \cong_\gamma y$ if x and y are coterminal, and if for all paths z from the vertex 1 to the start vertex of x and y we have $\varphi(\overline{z}\,\overline{x}) = \varphi(\overline{z}\,\overline{y})$.

5. IDENTITIES FOR $\mathbf{J}_1 * \mathbf{H}$

In this section, we give a basis of identities for $\mathbf{J}_1 * \mathbf{H}$.

Let A be a finite alphabet. Let γ be the congruence generating $\mathbf{H}$ for A and let q be a positive integer such that $u^q \gamma 1$ for all words u on A.

DEFINITION 5.1: *We call a list $a_1, \ldots, a_i$ of elements of A γ-*circular *on A if $a_1 \ldots a_i \gamma 1$ but no nonempty proper prefix of $a_1 \ldots a_i$ is γ-equivalent to* 1. *We write A_γ for the set of such γ-circular lists on A.*

DEFINITION 5.2: *We write $\Sigma_{A,\gamma,q}$ for the set consisting of the identities*

$$x^{2q} = x^q, \tag{3}$$

$$x^q y^q = y^q x^q, \tag{4}$$

together with all the identities of the form

$$(y_1 z_1^q \ldots y_{i-1} z_{i-1}^q y_i)^2 = y_1 z_1^q \ldots y_{i-1} z_{i-1}^q y_i, \tag{5}$$

where $y_1, \ldots, y_i$ is a list in A_γ.

The following definition and lemmas will be useful in the proof of Theorem 5.1.

Let us define recursively what we mean by "a γ-word w on A".

DEFINITION 5.3: Basis. *The empty word 1 is a γ-word on A.*

Recursive step. *If there exists a list $a_1, \ldots, a_i$ in A_γ, and there exist $v_1, \ldots, v_{i-1}$ which are finite concatenations of γ-words on A satisfying $w = a_1 v_1 \ldots a_{i-1} v_{i-1} a_i$, then we say that w is a γ-word on A.*

Closure. *A word w is a γ-word on A only if it can be obtained from the basis by a finite number of applications of the recursive step.*

Note that if a word w is a γ-word on A, it is built only from elements of A which build the lists in A_γ.

LEMMA 5.1: *We have* $\Sigma_{A,\gamma,q} \vdash (u_1^q \dots u_i^q)^2 = u_1^q \dots u_i^q$ *and so* $\Sigma_{A,\gamma,q} \vdash (u_1^q \dots u_i^q)^q = u_1^q \dots u_i^q$.

Proof: We have $\Sigma_{A,\gamma,q} \vdash u_1^q \dots u_i^q = u_1^{2q} \dots u_i^{2q}$ since the identity $x^{2q} = x^q$ belongs to $\Sigma_{A,\gamma,q}$, and so $\Sigma_{A,\gamma,q} \vdash u_1^q \dots u_i^q = (u_1^q \dots u_i^q)^2$ by using Identity (4) repeatedly. □

LEMMA 5.2 : 1. *If w is a γ-word on A, then* $\Sigma_{A,\gamma,q} \vdash w^2 = w$ *and so* $\Sigma_{A,\gamma,q} \vdash w^q = w$;

2. *If w and w' are γ-words on A, then* $\Sigma_{A,\gamma,q} \vdash ww' = w'w$.

Proof: Assertion 1 follows by induction on w. Trivially, $\Sigma_{A,\gamma,q} \vdash 1^2 = 1$ and so $\Sigma_{A,\gamma,q} \vdash 1^q = 1$. If v is a finite concatenation of γ-words on A, say $v = u_1 \dots u_j$, then by using the inductive assumption on $u_1, \dots, u_j$ as well as Lemma 5.1 we get $\Sigma_{A,\gamma,q} \vdash v^2 = (u_1 \dots u_j)^2 = (u_1^q \dots u_j^q)^2 = u_1^q \dots u_j^q = v$, and so $\Sigma_{A,\gamma,q} \vdash v^q = v$. Now, if there exists a list $a_1, \dots, a_i$ in A_γ, and there exist $v_1, \dots, v_{i-1}$ which are finite concatenations of γ-words on A satisfying $w = a_1 v_1 \dots a_{i-1} v_{i-1} a_i$, then by using an identity of the form (5) we get $\Sigma_{A,\gamma,q} \vdash w^2 = (a_1 v_1 \dots a_{i-1} v_{i-1} a_i)^2 = (a_1 v_1^q \dots a_{i-1} v_{i-1}^q a_i)^2 = a_1 v_1^q \dots a_{i-1} v_{i-1}^q a_i = w$ and so $\Sigma_{A,\gamma,q} \vdash w^q = w$.

Assertion 2 follows from $\Sigma_{A,\gamma,q} \vdash ww' = w^q (w')^q = (w')^q w^q = w'w$. □

LEMMA 5.3: *If $u\gamma 1$, then* $\alpha(\sigma_\gamma(u^2)) = \alpha(\sigma_\gamma(u))$. *As consequences,* $u^{2q} \sim_\gamma u^q$ and $u^q v^q \sim_\gamma v^q u^q$.

Proof: If $u\gamma 1$, then $\sigma_\gamma(u^2) = \sigma_\gamma(u)\sigma_\gamma^u(u) = \sigma_\gamma(u)\sigma_\gamma(u)$ since $\pi_\gamma(u) = 1$. We have $u^q\gamma 1$ and $v^q\gamma 1$, and so $u^q, u^{2q}, u^q v^q$ and $v^q u^q$ are γ-equivalent to 1. The equalities $\alpha(\sigma_\gamma(u^{2q})) = \alpha(\sigma_\gamma(u^q))$ and $\alpha(\sigma_\gamma(u^q v^q)) = \alpha(\sigma_\gamma(v^q u^q))$ are easy to check. □

Now, let r be a positive integer and put $A_r = \{x_1, \dots, x_r\}$. Let γ_r be the congruence generating $\mathbf{H}$ for A_r and let q_r be a positive integer such that $u^{q_r}\gamma_r 1$ for all words u on A_r.

THEOREM 5.1: *We have* $\mathbf{J}_1 * \mathbf{H} = \mathbf{V}(\bigcup_{r\geq 1} \Sigma_{A_r,\gamma_r,q_r})$.

Proof: We will show that an A-generated monoid M is in $\mathbf{J}_1 * \mathbf{H}$ if and only if $M \models \Sigma_{A,\gamma,q}$ where A abbreviates A_r, γ abbreviates γ_r and q abbreviates q_r. By Lemma 3.2, A-generated monoids in $\mathbf{J}_1 * \mathbf{H}$ satisfy identities $u = v$

where $u \sim_\gamma v$ (that is $\alpha(\sigma_\gamma(u)) = \alpha(\sigma_\gamma(v))$ and $u\gamma v$). Lemma 5.3 implies that $x^{2q} \sim_\gamma x^q$ and $x^q y^q \sim_\gamma y^q x^q$. We also have $x^2 \sim_\gamma x$ for all the identities $x^2 = x$ of the form (5). To see this, put $x = y_1 z_1^q \dots y_{i-1} z_{i-1}^q y_i$ with $y_1, \dots, y_i$ a list in A_γ. Since x is γ-equivalent to 1, we get $x^2 \gamma x$. The equality $\alpha(\sigma_\gamma(x^2)) = \alpha(\sigma_\gamma(x))$ follows from Lemma 5.3.

Conversely, let $\varphi : A^* \to M$ be a surjective morphism satisfying $\varphi(u) = \varphi(v)$ for every identity $u = v$ in $\Sigma_{A,\gamma,q}$. We also denote by φ the (nuclear) congruence on A^* associated with φ and defined by $u\varphi v$ if and only if $\varphi(u) = \varphi(v)$. We show the inclusion $\sim_\gamma \subseteq \varphi$ which yields $M = A^*/\varphi$ is a morphic image of $A^*/\sim_\gamma$. The membership of M to $\mathbf{J}_1 * \mathbf{H}$ follows by Lemma 3.2.

We consider the graph G_γ and the congruence relation $\cong_\gamma$ on its set of paths P_γ defined at the end of Section 4. Let x and y be two loops about the same vertex $\pi_\gamma(w)$, or

$$x = (\pi_\gamma(w), a_1) \dots (\pi_\gamma(wa_1 \dots a_{i-1}), a_i),$$

$$y = (\pi_\gamma(w), b_1) \dots (\pi_\gamma(wb_1 \dots b_{j-1}), b_j),$$

where $wa_1 \dots a_i \gamma w \gamma w b_1 \dots b_j$. We show the following two claims: Claim 1 or $xx \cong_\gamma x$, and Claim 2 or $xy \cong_\gamma yx$. Now if $u \sim_\gamma v$, then $\sigma_\gamma(u)$ and $\sigma_\gamma(v)$ are two coterminal paths traversing the same set of edges (the start vertex of $\sigma_\gamma(u)$ and $\sigma_\gamma(v)$ is 1 and their end vertex is $\pi_\gamma(u) = \pi_\gamma(v)$). Hence, by Proposition 4.1, $\sigma_\gamma(u) \cong_\gamma \sigma_\gamma(v)$. Therefore, $\varphi(\overline{\sigma_\gamma(u)}) = \varphi(\overline{\sigma_\gamma(v)})$ or $\varphi(u) = \varphi(v)$ and the inclusion $\sim_\gamma \subseteq \varphi$ follows.

Let us now prove Claim 1 and Claim 2. Since $wa_1 \dots a_i \gamma w$ and $wb_1 \dots b_j \gamma w$, we have $\overline{x} = a_1 \dots a_i \gamma 1$ and $\overline{y} = b_1 \dots b_j \gamma 1$ since $\mathbf{H}$ is a pseudovariety of groups.

Proof of Claim 1: The condition $xx \cong_\gamma x$ follows by showing that $\varphi(\overline{z}\,\overline{xx}) = \varphi(\overline{z}\,\overline{x})$ for all paths z from the vertex 1 to the start vertex of x. Here we can show that $\varphi(\overline{xx}) = \varphi(\overline{x})$ (and therefore $\varphi(\overline{x}^q) = \varphi(\overline{x})$). The word $\overline{x}$ has the property $\mathcal{P}$ that "it is γ-equivalent to 1". The word $\overline{x}$ can be factorized as follows: let u_1 be the smallest nonempty prefix of $\overline{x}$ with Property $\mathcal{P}$; let u_2 be the smallest nonempty prefix of $\overline{x} \setminus u_1$ with Property $\mathcal{P}$; .... So $\overline{x}$ is a concatenation of factors $u_1 \dots u_n$ with Property $\mathcal{P}$. Since no nonempty proper prefix of u_1 has Property $\mathcal{P}$, let $c_1 v_1$ be the shortest prefix of u_1 such that $\pi_\gamma(c_1 v_1) = \pi_\gamma(c_1)$; ... let $c_{\ell-1} v_{\ell-1}$ be the shortest prefix of $u_1 \setminus c_1 v_1 \dots c_{\ell-2} v_{\ell-2}$ such that $\pi_\gamma(c_1 v_1 \dots c_{\ell-2} v_{\ell-2} c_{\ell-1} v_{\ell-1}) = \pi_\gamma(c_1 v_1 \dots c_{\ell-2} v_{\ell-2} c_{\ell-1})$; and let $c_\ell = u_1 \setminus c_1 v_1 \dots c_{\ell-1} v_{\ell-1}$ satisfying $\pi_\gamma(c_1 v_1 \dots c_{\ell-1} v_{\ell-1} c_\ell) = \pi_\gamma(1)$. So $u_1 = c_1 v_1 \dots c_{\ell-1} v_{\ell-1} c_\ell$ where $c_1, \dots, c_\ell \in A_\gamma$ and where the v-factors have Property $\mathcal{P}$ (similar statements hold for $u_2, \dots, u_n$). Since the v-factors have Property $\mathcal{P}$, they can be factorized as above and the process can be repeated. Factors in $\overline{x}$ are hence γ-words on A. We have $\varphi(u_1) = \varphi(u_1^q)$, …, $\varphi(u_n) = \varphi(u_n^q)$ (as in Lemma 5.2). Therefore $\varphi(\overline{x}) = \varphi(u_1 \dots u_n) = \varphi(u_1^q \dots u_n^q) = \varphi((u_1^q \dots u_n^q)^2)$ (as in Lemma 5.1) $= \varphi(\overline{x}^2) = \varphi(\overline{x}\overline{x})$.

Proof of Claim 2: The condition $xy \cong_\gamma yx$ follows from $\varphi(\overline{xy}) = \varphi(\overline{x}\,\overline{y}) = \varphi(\overline{x})\varphi(\overline{y}) = \varphi(\overline{x}^q)\varphi(\overline{y}^q) = \varphi(\overline{x}^q\overline{y}^q) = \varphi(\overline{y}^q\overline{x}^q) = \varphi(\overline{yx})$ (using Identity (4)). □

6. IDENTITIES FOR $\mathbf{J}_1 * \mathbf{G}_p$

In this section, we give a sequence of sets of identities ultimately defining $\mathbf{J}_1 * \mathbf{G}_p$.

Let A be a finite alphabet and let $u, w \in A^*$ with $u = a_1 \dots a_\imath$. The *binomial coefficient* $\binom{w}{u}$ is defined as the number of distinct factorizations of the form

$$w = v_0 a_1 v_1 \dots a_\imath v_\imath$$

with $v_0, \dots, v_\imath \in A^*$. Thus the binomial coefficient counts the number of ways in which u is a subword of w. We adopt the convention that $\binom{w}{1} = 1$.

Let $a, b \in A$ and $u, w, w' \in A^*$. The following formulas are easily verified:

- $\binom{a^\imath}{a^\jmath} = \binom{\imath}{\jmath}$ where $i \geq j$;
- $\binom{1}{u} = \begin{cases} 1, & \text{if } u = 1, \\ 0, & \text{otherwise;} \end{cases}$
- $\binom{a}{u} = \begin{cases} 1, & \text{if } u = 1 \text{ or } u = a, \\ 0, & \text{otherwise;} \end{cases}$
- $\binom{wa}{ub} = \binom{w}{ub} + \delta_{a,b}\binom{w}{u}$ where $\delta_{a,b} = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise;} \end{cases}$
- $\binom{ww'}{u} = \sum_{u=vv'} \binom{w}{v}\binom{w'}{v'}$.

Given a word u on A, we define on A^* the equivalence relation $\gamma_{p,u}$ by

$$w\gamma_{p,u}w' \text{ if and only if } \binom{w}{v} \equiv \binom{w'}{v} \bmod p \text{ whenever } u \in A^*vA^*.$$

Now, given an integer $k \geq 0$, we define on A^* the equivalence relation $\gamma_{p,k}$ by $\gamma_{p.k} = \bigcap_{|u|=k} \gamma_{p,u}$. Thus

$$w\gamma_{p,k}w' \text{ if and only if } \binom{w}{v} \equiv \binom{w'}{v} \bmod p \text{ whenever } |v| \leq k.$$

Note that for all $w, w' \in A^*$ we have $w\gamma_{p.0}w'$.

LEMMA 6.1 (Eilenberg [8]): *The equivalence relations $\gamma_{p,u}$ and $\gamma_{p,k}$ are congruences of finite index on A^*.*

LEMMA 6.2 (Eilenberg [8]): *Let k be a positive integer and $u \in A^*$. If $w \in A^*$, then $w^{p^{|u|}} \gamma_{p,u} 1$ and $w^{p^k} \gamma_{p,k} 1$.*

Proof: If $w \in A^*$, then the following conditions are equivalent:

- $w \gamma_{p,k} 1$;
- $\binom{w}{v} \equiv 0 \bmod p$ whenever $0 < |v| \leq k$.

We show the $\gamma_{p,k}$-equivalence of w^{p^k} and 1. For $k = 1$, the result holds trivially. We proceed by induction and assume $0 < |v| \leq k + 1$. Then

$$\binom{w^{p^{k+1}}}{v} = \sum \binom{w^{p^k}}{v_1} \cdots \binom{w^{p^k}}{v_p},$$

where the summation extends over all factorizations $v = v_1 \ldots v_p$ of v. If for some $1 \leq i \leq p$ we have $0 < |v_i| < k + 1$, then by the inductive assumption $\binom{w^{p^k}}{v_i} \equiv 0 \bmod p$ and the summand may be omitted. There remain summands with $v_i = v$, $v_j = 1$ for $j \neq i$. Each such summand yields $\binom{w^{p^k}}{v}$ and there are exactly p such summands. Thus $\binom{w^{p^{k+1}}}{v} \equiv 0 \bmod p$ as required. $\square$

The quotients $A^*/\gamma_{p,u}$ and $A^*/\gamma_{p,k}$ are finite monoids by Lemma 6.1. Lemma 6.2 implies that $A^*/\gamma_{p,u}$ satisfies the identity $x^{p^{|u|}} = 1$ and $A^*/\gamma_{p,k}$ the identity $x^{p^k} = 1$. Note that $A^*/\gamma_{p,0}$ is the trivial group. If $A = \{a_1, \ldots, a_r\}$, $A^*/\gamma_{p,1}$ is isomorphic to the set of all words of the form $a_1^{e_1} \ldots a_r^{e_r}$ with $0 \leq e_i < p$ multiplying two such words through the addition of the respective exponents.

We now describe the $*$-variety $\mathcal{G}_p$ of sets defined by the pseudovariety $\mathbf{G}_p$.

LEMMA 6.3 (Eilenberg [8]): • *The pseudovariety $\mathbf{G}_p$ is generated by the groups $A^*/\gamma_{p,k}$ for all integers $k \geq 0$ and all finite alphabets A, or by the groups $A^*/\gamma_{p,u}$ for all elements $u \in A^*$ and all finite alphabets A.*

- *$A^*\mathcal{G}_p$ is the boolean closure of the sets*

$$\{w \in A^* \mid \binom{w}{u} \equiv i \bmod p\},\ u \in A^*,\ 0 \leq i < p.$$

Let k be a nonnegative integer and define the pseudovariety $\mathbf{H}_{p,k}$ as the locally finite pseudovariety of groups generated by $A^*/\gamma_{p,k}$ for all finite alphabets A. The $*$-variety $A^*\mathcal{H}_{p,k}$ is then the boolean closure of the sets

$$\{w \in A^* \mid \binom{w}{u} \equiv i \bmod p\},\ u \in A^* \text{ with } |u| \leq k,\ 0 \leq i < p.$$

The pseudovariety $\mathbf{H}_{p,0}$ is the trivial pseudovariety $\mathbf{I} = \mathbf{V}(x = 1)$. Since $\mathbf{I}$ is the unit element for the semidirect product operation on pseudovarieties of monoids, we have $\mathbf{J}_1 * \mathbf{H}_{p,0} = \mathbf{J}_1 = \mathbf{V}(x^2 = x, xy = yx)$.

Now, let k be a positive integer. A list $a_1, \ldots, a_i$ of elements of A is $\gamma_{p,k}$-circular on A if $\binom{a_1 \ldots a_i}{v} \equiv 0 \bmod p$ whenever $0 < |v| \leq k$, but no nonempty proper prefix w of $a_1 \ldots a_i$ satisfies $\binom{w}{v} \equiv 0 \bmod p$ for every $0 < |v| \leq k$. For example, a, b, b, a, a, b, b, a is a list in $\{a, b\}_{\gamma_{2,2}}$.

If k and r are positive integers, we write $\Sigma_{p,k}^r$ for the set consisting of the identities

$$x^{2p^k} = x^{p^k}, \tag{6}$$

$$x^{p^k} y^{p^k} = y^{p^k} x^{p^k}, \tag{7}$$

together with all the identities of the form

$$(y_1 z_1^{p^k} \ldots y_{i-1} z_{i-1}^{p^k} y_i)^2 = y_1 z_1^{p^k} \ldots y_{i-1} z_{i-1}^{p^k} y_i, \tag{8}$$

where $y_1, \ldots, y_i$ is a list in $\{x_1, \ldots, x_r\}_{\gamma_{p,k}}$. We write $\Sigma_{p,k}$ for $\bigcup_{r \geq 1} \Sigma_{p,k}^r$.

Continuing with the above example, the identity $x^2 = x$ where

$$x = x_1 z_1^{2^2} x_2 z_2^{2^2} x_2 z_3^{2^2} x_1 z_4^{2^2} x_1 z_5^{2^2} x_2 z_6^{2^2} x_2 z_7^{2^2} x_1,$$

belongs to $\Sigma_{2,2}^2$.

For $r \geq 1$, $\Sigma_{p,k}^r \subseteq \Sigma_{p,k}^{r+1}$. This follows from the fact that if $A \subseteq B$, then $A_{\gamma_{p,k}} \subseteq B_{\gamma_{p,k}}$.

Corollary 6.1: *The pseudovariety $\mathbf{J}_1 * \mathbf{G}_p$ is ultimately defined by $\Sigma_{p,k}, k \geq 1$ or a monoid is in $\mathbf{J}_1 * \mathbf{G}_p$ if and only if it satisfies $\Sigma_{p,k}$ for all k sufficiently large.*

Proof: By Theorem 5.1, the pseudovariety $\mathbf{J}_1 * \mathbf{H}_{p,k}$ is defined by $\Sigma_{p,k}$. Now, the semidirect product operation on pseudovarieties commutes with directed unions [3]. We get $\mathbf{J}_1 * \mathbf{G}_p = \mathbf{J}_1 * \bigcup_{k \geq 0} \mathbf{H}_{p,k} = \bigcup_{k \geq 0} \mathbf{J}_1 * \mathbf{H}_{p,k} = \bigcup_{k \geq 1} \mathbf{J}_1 * \mathbf{H}_{p,k}$ and the result follows. □

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