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ONE-RULE SEMI-THUE SYSTEMS WITH LOOPS OF LENGTH ONE, TWO OR THREE (*)

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Abstract. – *A loop of a semi-Thue system is a reduction chain where the start word reappears as a factor of the obtained word after a finite number of reduction steps. A rewriting system admitting a loop is clearly nonterminating. The semi-Thue systems consisting of only one rule which admit loops of length 1, 2 or 3 are characterized. Length-1-loops are trivial. 2-loops turn out to have a unique structure, whereas one-rule systems admitting a 3-loop (but no 2-loop) can belong to three structurally different types.*

Résumé. – *Une boucle d'un système semi-Thue est une chaîne de réduction où le mot de départ réapparaît comme facteur du mot obtenu après un nombre fini de pas de réduction. Un système de réécriture qui admet une boucle a clairement la propriété de ne pas se terminer. Les systèmes semi-Thue consistant en une seule règle qui permettent des boucles de longueur 1, 2 ou 3 sont caractérisés. Les boucles de longueur 1 sont triviales. Les boucles de longueur 2 se révèlent comme possédant une structure unique, tandis que les systèmes semi-Thue consistant en une règle qui admet une boucle de longueur 3 (mais pas de boucle de longueur 2) peuvent faire partie de trois types qui sont différents en structure.*

1. INTRODUCTION

Proving *termination* of rewriting systems is an important problem with a lot of applications in computer science (*see*, e.g., [5]). Here we restrict ourselves to a very special case, to rewriting systems operating on finite *words* (strings), *i.e.* semi-Thue systems. Furthermore, the systems considered here consist of *only one* rewrite rule $u \longrightarrow v$ (where u and v are words). Many one-rule systems are *terminating* (also called *noetherian*), *i.e.* they admit no infinite reduction chain. The problem to decide whether a given one-rule system $u \longrightarrow v$ is terminating or not seems to be nontrivial. McNaughton [8] and Kurth [6] gave several sufficient criteria for termination, and the

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question of termination was settled for all $u \longrightarrow v$ with length of $v \leq 6$ [6], but the general question remains open.

Here a special case of nontermination is considered, namely, the existence of *loops*. E.g., the one-rule system $ba \longrightarrow aabb$ over the alphabet $\{a, b\}$ admits the infinite reduction chain

$$baa \longrightarrow aabba \longrightarrow aa\underline{baa}bb \longrightarrow aaaabbabb \longrightarrow \dots ,$$

which is characterized by the reappearance of the start word baa as a factor (underlined) of the word generated after the second reduction step. Clearly, this self-inclusion can be iterated *ad infinitum*. We speak of a *loop of length* 2 or, for short, 2-*loop*. In this paper we characterize all one-rule semi-Thue systems admitting loops of length 1, 2 or 3. This establishes at the same time some sufficient criteria for nontermination.

A *word* is a finite string with elements (called *letters*) from a finite alphabet Σ. Throughout this paper, lower case letters can represent words or single letters, depending on the context. As usual, the set of all words over Σ, including the *empty word* $\square$, is denoted by Σ^*, and $\Sigma^+ = \Sigma^* - \{\square\}$. $\ell(w)$ is the *length* of the word $w \in \Sigma^*$; we have $\ell(\square) = 0$. The *mirror image* $\tilde{w}$ of a word w is obtained by writing down its letters in reverse direction. Word reflection is obviously an anti-automorphism, *i.e.* $\widetilde{uv} = \tilde{v}\,\tilde{u}$ for all u, $v \in \Sigma^*$. The word u is a *factor* of the word v, written $u \preceq v$, iff there exist words x, $y \in \Sigma^*$ such that $v = xuy$. E.g., $aa \preceq baa$, but $aa \not\preceq aba$. u is a *prefix* or *left factor* of v iff $v = uy$ holds for some $y \in \Sigma^*$.

A *one-rule semi-Thue system* is simply a pair (u, v) of words over some fixed alphabet Σ. We write $u \longrightarrow v$ for short. By $p \longrightarrow q$ we also denote the *reduction step* which results when an occurrence of u as factor of the word $p \in \Sigma^*$ is replaced by v, *i.e.* when the rule $u \longrightarrow v$ is *applied to* p:

$$p = xuy \longrightarrow xvy = q.$$

It will be clear from the context which meaning of the arrow symbol "$\longrightarrow$" has to be assumed. — A *reduction chain* is a sequence of successive reduction steps

$$p_0 \longrightarrow p_1 \longrightarrow p_2 \longrightarrow \dots ,$$

where in each step the same one-rule system $u \longrightarrow v$ is applied. We call p_0 the *start word* of the chain.

A reduction chain $p_0 \longrightarrow \ldots \longrightarrow p_n$ of length n is called a *cycle of length* n, iff $p_n = p_0$, and a *loop of length* n (short: an n-*loop*), iff p_0 is a factor of p_n. We say that the one-rule system $u \longrightarrow v$ *has an* n-*cycle* (resp., *an* n-*loop*), iff there exists some start word p_0 from which a cycle (resp., a loop) of length n can be obtained. (This terminology coincides with that used in the context of term rewriting, *see* [4]. The notion "cycle" was also used in our sense in the case of strings in [7].) A semi-Thue system having a loop is sometimes also called "self-embedding" [12].

A cycle is a special case of loop. It can easily be seen that a one-rule semi-Thue system $u \longrightarrow v$ has a cycle only in the trivial case $u = v$. For loops of length 1, the situation is trivial as well:

PROPOSITION: *Let* $u, v \in \Sigma^*$. *The one-rule semi-Thue system* $u \longrightarrow v$ *has a 1-loop iff* $u \preceq v$.

Proof: From $u \preceq v$ there follows clearly the existence of a 1-loop with start word u. Let $u \longrightarrow v$ have a 1-loop $y \longrightarrow xyz$. As y undergoes a reduction step, we must have $y = sut$, and therefore $svt = xyz = xsutz$. One can conclude $sut \preceq svt$, and hence $u \preceq v$ (the final deduction step representing a type of argumentation which will appear again in the proofs of our theorems below). □

2. LOOPS OF LENGTH 2

The following theorem characterizes the one-rule semi-Thue systems admitting 2-loops.

THEOREM 1: *Let* $u, v \in \Sigma^*$, $u \not\preceq v$. *The following statements are equivalent:*

(1) $u \longrightarrow v$ *has* a *2-loop.*

(2) There exist words $c, d, e, f, g, h \in \Sigma^+$, *such that*

$$u = ef = gh$$

and

$$v = cge = hfd.$$

(3) There exist words $g, h, i \in \Sigma^*$ *such that* $u = gh$, $v = hi$ *and* $ggh \preceq hii$.

(4) There exist words $e, f, j \in \Sigma^*$ *such that* $u = ef$, $v = je$ *and* $eff \preceq jje$.

Furthermore, if (2) is fulfilled, the inequality

$$|\ell(f) - \ell(g)| \leq \ell(v) - \ell(u)$$

holds. 2-loops can be started from the words uf and gu.

Proof: We show (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\wedge$ (4), then (3) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (1). Let (1) be fulfilled, that is, $u \longrightarrow v$ has a 2-loop $y \longrightarrow \ldots \longrightarrow xyz$ (x, y, $z \in \Sigma^*$). Then there exist words p, q, r, $s \in \Sigma^*$ such that

$$y = puq \longrightarrow pvq \overset{(A)}{=} rus \longrightarrow rvs \overset{(B)}{=} xpuqz = xyz.$$

The factors of the left side of equation (A) can be arranged in four different ways with respect to the factors of the right side (if $u \not\preceq v$ is already taken into account) as shown by the four schemes (*cf.* [1] for the notion of a *scheme* of a word equation) in Figure 1.

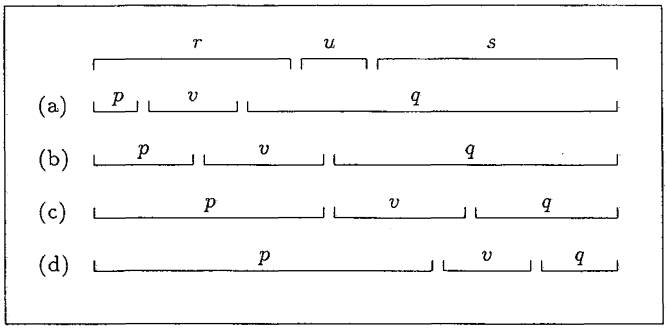

Figure 1.

Here, scheme (d) is the mirror image of (a), and (c) that of (b). If (a) occurs, we have $r = pvt$ and $q = tus$ with some $t \in \Sigma^*$. From equation (B) one obtains $puq \preceq rvs$, that is, $putus \preceq pvtvs$. One can conclude $utu \preceq vtv$, and hence $u \preceq v$, contradicting the preliminaries. Hence, scheme (a) can be excluded, and (d) as well.

We take scheme (b) for given. Let t be the overlap of r with v, that is, $r = pt$, and furthermore e the overlap of v with u, that is, $v = te$, and f the overlap of u with q, that is, $u = ef$ and $q = fs$ (*Fig.* 2).

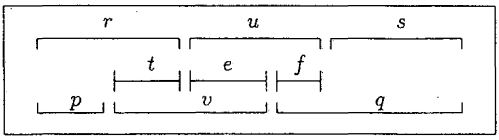

Figure 2.

Here, we have $f \neq \square$ (otherwise $u \preceq v$). Equation (B) yields $ptvs = xpufsz$. From the trivial relations $\ell(xp) \geq \ell(p)$ and $\ell(sz) \geq \ell(s)$ we obtain for the central factors in this word equation: $uf \preceq tv$, that is, there exist c, $d \in \Sigma^*$ with

$$tv = cufd. \tag{$*$}$$

If the reduction is started with uf instead of puq, one obtains with $uf \longrightarrow vf = tu \longrightarrow tv = cufd$ a simpler 2-loop than the one previously considered.

Let us now consider $(*)$. The factor u in $c\underline{u}fd$ can neither lie completely in v, nor (because of $t \preceq v$) completely in t. Therefore, there exist g, $h \neq \square$ such that $u = gh$, $v = hfd$ and $t = cg$ (*Fig.* 3).

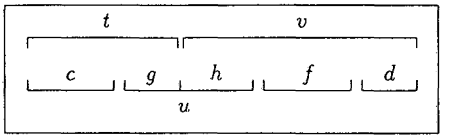

Figure 3.

Especially, $f \preceq v$. From this, one can conclude $e \neq \square$, because otherwise $f = u \preceq v$. Summarizing, we have $u = ef = gh$, $v = cge = hfd$, and e, f, g, $h \neq \square$. Under reflection, this system of equations reveals itself as self-equivalent, such that scheme (c) must yield the same system. Now let us assume $c = \square$, *i.e.* $v = ge = hfd$. By induction on n, one obtains easily $h(fdf)^n = (gg)^n h$ for all $n \geq 1$, which has as a consequence that h (and gh as well) is a prefix of the infinite word $ggg\ldots$. This implies that h is a prefix of gh. On the other hand, gh has the prefix e, and from $\ell(u) \leq \ell(v)$ one obtains $\ell(h) \leq \ell(e)$, such that h turns out to be a prefix of e, giving the contradiction $u = gh \preceq ge = v$. Analogously, one can exclude the case $d = \square$. Hence, (2) is obtained.

Starting with gu, the 2-loop $gu \longrightarrow gv = ufd \longrightarrow vfd = cgud$ is developed. We proceed with the length inequality of the theorem: One has $0 \leq \ell(c) = \ell(v) - \big(\ell(g) + \ell(e)\big) = \ell(v) - \big(\ell(g) + \ell(u) - \ell(f)\big) = \ell(v) - \ell(u) + \ell(f) - \ell(g)$, and therefore $\ell(g) - \ell(f) \leq \ell(v) - \ell(u)$. Analogously, from $0 \leq \ell(d)$ one obtains $\ell(f) - \ell(g) \leq \ell(v) - \ell(u)$, and altogether

$$|\ell(f) - \ell(g)| \leq \ell(v) - \ell(u).$$

Now let us assume that (2) is fulfilled. We set $i = fd$ and $j = cg$. Then we obtain $u = ef = gh$, $v = hi = je$, $ggh \preceq cgghd = cgefd = vfd =$

$hfdfd = hii$, $eff \preceq ceffd = cghfd = cgv = cgcge = jje$, that is, (3) and (4).

On the other hand, if (3) is fulfilled, one has with $ggh = gu \longrightarrow gv = ghi = ui \longrightarrow vi = hii$ because of $ggh \preceq hii$ a 2-loop. Analogously in the case of (4): $eff = uf \longrightarrow vf = jef = ju \longrightarrow jv = jje$, $eff \preceq jje$. □

The conditions in part (2) have the consequence that u and v overlap each other from both sides, as it is demonstrated in Figure 4.

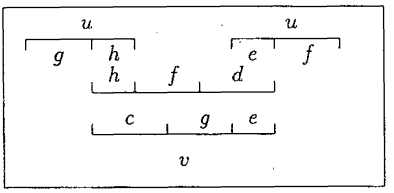

Figure 4.

For example, for the one-rule semi-Thue system $bab \longrightarrow abba$ (already mentioned by Metivier, [10]), one has $c = d = a$, $e = ba$, $f = g = b$, $h = ab$, and 2-loops can be obtained from the shortest start words $uf = babb$ or $gu = bbab$.

In a special case, the structure of a 2-loop can be described even simpler than in Theorem 1.

COROLLARY: *Let $u, v \in \Sigma^*$, $u \not\preceq v$, and $\ell(v) \geq 2\ell(u)$. Then $u \longrightarrow v$ has a 2-loop iff one of the following two conditions is satisfied:*

(i) *There exist words $e, h, s, t \in \Sigma^*$ with $e, h \neq \square$ such that $u = esh$ and $v = hshtese$.*

(ii) *There exist words $f, g, s, t \in \Sigma^*$ with $f, g \neq \square$ such that $u = gsf$ and $v = sfftggs$.*

Proof: We use (2) from Theorem 1. $2\ell(u) = \ell(h)+\ell(f)+\ell(g)+\ell(e) \leq \ell(v)$ has as a consequence that f lies inside v strictly left from g and does not overlap with g (in contradiction to the situation indicated in Figure 4). That is, between f and g there is a middle factor $t \in \Sigma^*$ (see *Fig.* 5).

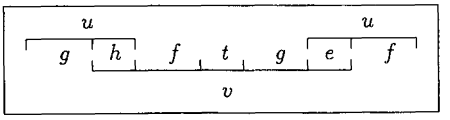

Figure 5.

From $u = gh = ef$ one obtains the existence of an $s \in \Sigma^*$ with either $g = es$, $f = sh$, or $e = gs$, $h = sf$. In combination with the conditions from (2), the conditions (i) or (ii) result. □

3. LOOPS OF LENGTH 3

We now restrict our attention to one-rule semi-Thue systems admitting neither 1-loops nor 2-loops.

THEOREM 2: *Let $u, v \in \Sigma^*$, $u \not\preceq v$, and let $u \longrightarrow v$ not have any 2-loop. Then $u \longrightarrow v$ has a 3-loop iff one of the following five conditions is fulfilled:*

(A) There exist words $e, f, h, i, j, \ell, m, n \in \Sigma^+$ and $k \in \Sigma^*$, such that

$$u = ef = ij = mn$$

and

$$v = kiem = jnh = f\ell.$$

($\tilde{A}$) Condition (A) holds for the mirror images $\tilde{u}$, $\tilde{v}$ instead of u and v.

(B) There exist words $i, j, m, n, p, q \in \Sigma^+$ and $f, k, \ell \in \Sigma^*$, such that

$$u = pqf = ij = mn$$

and

$$v = qm = kpi = fjn\ell.$$

($\tilde{B}$) Condition (B) holds for the mirror images $\tilde{u}$, $\tilde{v}$ instead of u and v.

(C) There exist words $f, i, j, m, n, p \in \Sigma^+$ and $k, \ell, q \in \Sigma^*$, such that

$$u = pqf = ij = mn$$

and

$$v = jqm = kip = fn\ell.$$

In case (A), the words iu, un and ieu can serve as start words for 3-loops, in case (B) the words pqu, ujn and pun, and in case (C) the words iun, uqu, $uqfn$ and $ipqu$.

Remark: We note that condition (C) is invariant under word reflection (substitute $\tilde{f}$ by p, $\tilde{i}$ by n, $\tilde{j}$ by m, $\tilde{k}$ by ℓ, $\tilde{\ell}$ by k, $\tilde{m}$ by j, $\tilde{n}$ by i, $\tilde{p}$ by

f, $\tilde{q}$ by q). Thus Theorem 2 is completely symmetric under the operation of word reflection.

Proof: It is easy to verify that, given u and v fulfilling one of the five conditions, in fact 3-loops can be obtained from the start words specified above. We have to show that each one-rule system $u \longrightarrow v$ having a 3-loop, but no 1- or 2-loop, fulfills one of the five conditions.

First we consider the inequations of the type "$w \neq \square$" which are implied by the notation $w \in \Sigma^+$. If one of them is violated while the equations of the corresponding condition are fulfilled, one obtains either $u \preceq v$, or $v \preceq u$, or a 2-loop, in contradiction to our assumptions. (A 2-loop appears under (B) for $p = \square$ and under (C) for $f = \square$ or $p = \square$, as can be seen by comparison with Theorem 1.) That $h = \square$ under (A) implies $u \preceq v$ can be shown in the same manner as $u \preceq v$ from $c = \square$ in the proof of Theorem 1. The other inequations are straightforward. Thus it remains only the task to prove the equations.

A 3-loop has generally the form

$$y = aub \longrightarrow avb \overset{(1)}{=} rus \longrightarrow rvs \overset{(2)}{=} tuw \longrightarrow tvw \overset{(3)}{=} xaubz = xyz$$

with a, b, r, s, t, w, x, y, $z \in \Sigma^*$. For equation (1), we obtain the same four schemes as in the case of equation (A) in the 2-loop theorem, (see *Fig.* 1) (replace p by a, q by b). Schemes (d) and (c) are the mirror images of (a) and (b), thus we can restrict ourselves to (a) and (b).

Case (a): There is a $q \in \Sigma^*$ with $r = avq$ and $b = qus$. That is, $rvs = avqvs$ and $y = aub = auqus$. For equation (2), taking $u \not\preceq v$ into account, we get 8 possible schemes, (see *Fig.* 6.)

In case (a_1), we have $a = tuf$, $w = fvqvs$ with $f \in \Sigma^*$, and hence equation (3) says $tvfvqvs = xtufuqusz$, entailing $u \preceq v$ and thus giving a contradiction. Case (a_8) is analogous.

In case (a_2), we have $u = gh$, $a = tg$, $v = hf$ and $w = fqvs$ with f, g, $h \in \Sigma^*$, and from equation (3) we get $tvfqvs = xtguqusz$. Because of $u \not\preceq v$, one can conclude $gu \preceq vf$. Starting with gu, we obtain the 2-loop $gu \longrightarrow gv = ghf \longrightarrow vf$ and thus again a contradiction. Case (a_7) is analogous.

In case (a_3), one obtains $u = gh$, $v = eg$, $q = hf$, $t = ae$, $w = fvs$ with e, f, g, $h \in \Sigma^*$, and equation (3) delivers $aevfvs = xauhfusz$, that is, $uhfu \preceq evfv$. Because of $u \not\preceq v$, one can conclude $uh \preceq ev$. Starting with uh, we obtain again a 2-loop: $uh \longrightarrow vh = egh \longrightarrow ev$, and thus a contradiction. Case (a_6) is analogous.

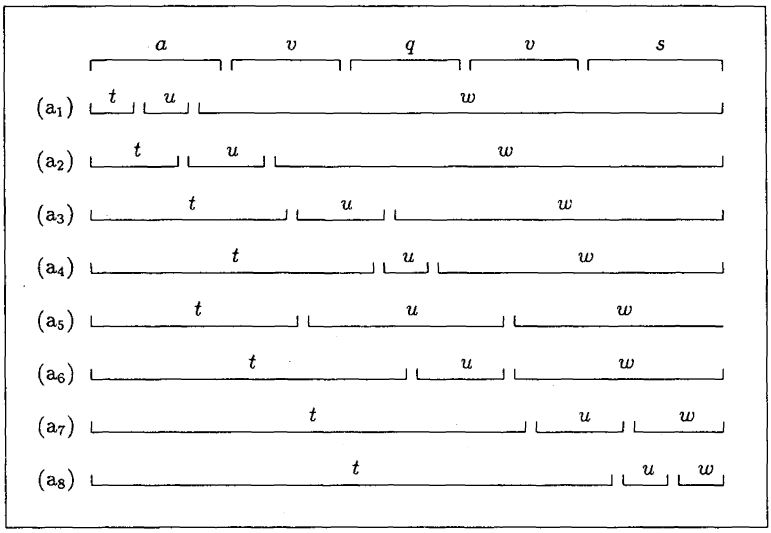

Figure 6.

In case (a_4), the situation is $q = euf$, $t = ave$, $w = fvs$ with e, $f \in \Sigma^*$, and equation (3) yields $avevfvs = xaueufusz$, leading to $u \preceq v$, a contradiction.

Only case (a_5) remains. Here, we have $t = ah$, $v = hp$, $u = pqf$, $v = fe$, and $w = es$ with e, f, h, $p \in \Sigma^*$. Equation (3) yields $ahves = xauqusz$, that is, $uqu \preceq hve$. As we have $h \preceq v$, $e \preceq v$ and $u \not\preceq v$, the factor uqu can only be positioned according to the scheme indicated in Figure 7.

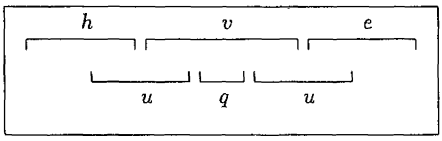

Figure 7.

That means, we have factorizations $h = ki$, $u = ij$, $v = jqm$, $u = mn$, $e = n\ell$ with i, j, k, ℓ, m, $n \in \Sigma^*$. Together with $v = hp = kip$ and $v = fe = fn\ell$, this yields condition (C).

Case (b): There are m, n, $q \in \Sigma^*$ with $r = aq$, $v = qm$, $u = mn$ and $b = ns$. We have then $rvs = aqvs$ and $y = aub = auns$. For equation (2), taking $u \not\preceq v$ into account, we get 6 possible schemes (see *Fig.* 8).

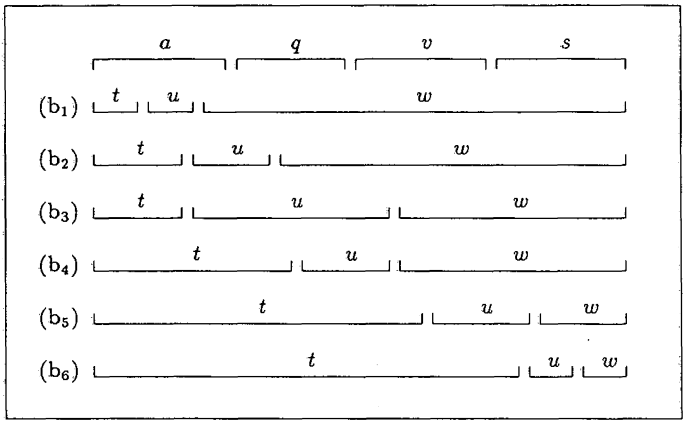

Figure 8.

In case (b_1), we have $a = tuf$, $w = fqvs$ with $f \in \Sigma^*$, and equation (3) gives $tvfqvs = xtufunsz$. Because of $u \not\preceq v$, one can conclude $un \preceq qv$, but this yields the 2-loop $un \longrightarrow vn = qmn \longrightarrow qv$ and hence a contradiction. Case (b_6) is analogous.

Case (b_2) is characterized by $a = ti$, $u = ij$, $q = jg$, $w = gvs$ with some g, i, $j \in \Sigma^*$, and from eq. (3) we get $tiuns \preceq tvgvs$, that is, $iun \preceq vgv$. If we had $iu \preceq vg$, there would be a 2-loop $iu \longrightarrow iv = ijgm \longrightarrow vgm$. Analogously, if $un \preceq gv$, there would be the 2-loop $un \longrightarrow vn = jgmn \longrightarrow jgv$. Thus, for $iun \preceq vgv$ only the scheme indicated in Figure 9 remains, and that means, there exist f, k, ℓ, $p \in \Sigma^*$ with $v = kip$, $u = pgf$ and $v = fn\ell$.

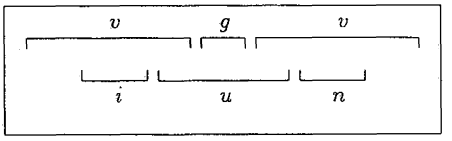

Figure 9.

When we rename g into q, we arrive exactly at the equations of condition (C).

In case (b_3), there are $f, g, p \in \Sigma^*$ such that $a = tp$, $u = pqf$, $v = fg$ and $w = gs$. Equation (3) yields $tpuns \preceq tvgs$, that is, $pun \preceq vg$. As we have $u \not\preceq g \preceq v$, the factor u from pun must intersect both the v and the g in vg (*Fig.* 10), that is, there are i, j, k, $\ell \in \Sigma^*$ with $v = kpi$, $u = ij$ and $g = jn\ell$.

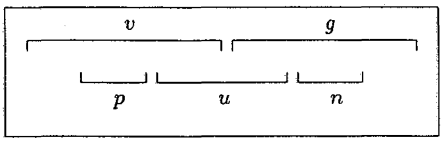

Figure 10.

From $g = jn\ell$ we arrive at $v = fg = fjn\ell$, thus receiving the last equation which remained to be deduced to establish condition (B).

Case (b_4) can be described by $t = ag$, $q = ge$, $u = ef$, $v = f\ell$ and $w = \ell s$ with e, f, g, $\ell \in \Sigma^*$. Equation (3) says $auns \preceq agv\ell s$, that is, $un \preceq gv\ell$. Three schemes are possible for the positioning of the factor un in $gv\ell$, (see *Fig.* 11). (u cannot be a factor of g, v or ℓ alone because of g, $\ell \preceq v$ and $u \not\preceq v$.)

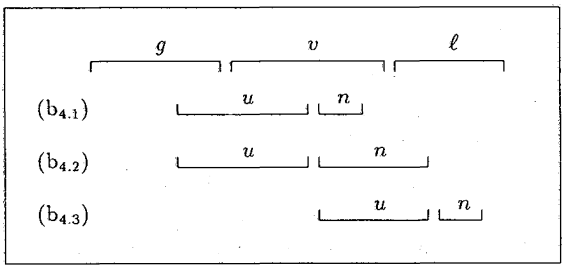

Figure 11.

In case ($b_{4.1}$), there exist h, i, j, $k \in \Sigma^*$ such that $g = ki$, $u = ij$, $v = jnh$. From $v = qm$ and $q = ge$ we obtain $v = kiem$. Thus we have obtained all the equations of condition (A).

In case ($b_{4.2}$), there exist c, d, h, i, j, $k \in \Sigma^*$ such that $g = ki$, $u = ij$, $v = jc$, $n = cd$, and $\ell = dh$. We obtain $u = mcd = ef = ij$, $v = jc = fdh = kiem$. If we turn to the mirror images $\tilde{u}$, $\tilde{v}$ and apply the renaming

$$\begin{pmatrix} \tilde{c} & \tilde{d} & \tilde{e} & \tilde{f} & \tilde{h} & \tilde{i} & \tilde{j} & \tilde{k} & \tilde{m} \\ q & p & j & i & k & n & m & \ell & f \end{pmatrix},$$

these equations reveal themselves as the characterization of condition (B), *i.e.* u and v satisfy condition ($\tilde{\text{B}}$).

In case ($b_{4.3}$), there are $h, i, j, k \in \Sigma^*$ such that $v = hi$, $u = ij$, $\ell = jnk$. We obtain $u = ij = mn = ef$, $v = fjnk = gem = hi$. Reflection and application of the renaming

$$\begin{pmatrix} \tilde{e} & \tilde{f} & \tilde{g} & \tilde{h} & \tilde{i} & \tilde{j} & \tilde{k} & \tilde{m} & \tilde{n} \\ n & m & h & \ell & f & e & k & j & i \end{pmatrix}$$

leads to the equations characterizing condition (A), *i.e.* u and v fulfill condition ($\tilde{\text{A}}$).

Case (b_5) can be described by $t = aqg$, $v = gc$, $u = cd$ and $s = dw$ with $c, d, g \in \Sigma^*$. Equation (3) leads to $aundw \preceq aqgvw$, *i.e.* $und \preceq qgv$. From $q, g \preceq v$ and $u \not\preceq v$ we deduce that u cannot be a factor of q, g or v alone. Furthermore, the assumption $un \preceq qg$ leads to the 2-loop $un \longrightarrow vn = qmn \longrightarrow qgc$. Therefore, $und \preceq qgv$ can be realized only in three possible schemes, (see *Fig.* 12).

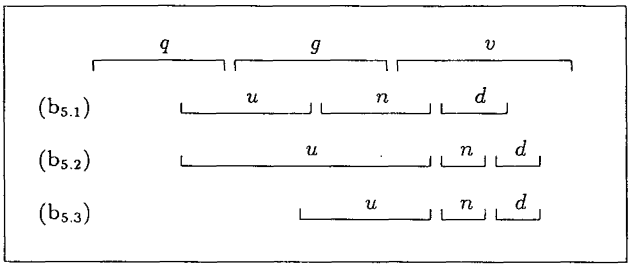

Figure 12.

In case ($b_{5.1}$), there exist $e, f, i, j, k, \ell \in \Sigma^*$ such that $q = ki$, $u = ij$, $g = je$, $n = ef$, $v = fd\ell$. Summarizing, we obtain $u = mef = ij = cd$, $v = jec = kim = fd\ell$. With the renaming

$$\begin{pmatrix} c & d & e & m \\ m & n & q & p \end{pmatrix},$$

we can identify condition (C).

In case ($b_{5.2}$), there exist $i, j, k, \ell \in \Sigma^*$ such that $q = ki$, $u = igj$ and $v = jnd\ell$. We get $u = igj = mn = cd$, $v = gc = kim = jnd\ell$. The renaming

$$\begin{pmatrix} c & d & g & i & j & m & n \\ m & n & q & p & f & i & j \end{pmatrix}$$

leads to condition (B).

In case ($b_{5.3}$), there exist i, j, k, $\ell \in \Sigma^*$ such that $g = ki$, $u = ij$ and $v = jnd\ell$. It follows $u = mn = cd = ij$, $v = jnd\ell = kic = qm$. By reflection and the renaming

$$\begin{pmatrix} \tilde{c} & \tilde{d} & \tilde{i} & \tilde{j} & \tilde{k} & \tilde{\ell} & \tilde{m} & \tilde{n} & \tilde{q} \\ j & i & n & m & h & k & f & e & \ell \end{pmatrix}$$

one obtains condition (A), *i.e.* $u \longrightarrow v$ fulfills condition ($\tilde{\text{A}}$). This case completes the proof of Theorem 2. □

Theorem 2 tells us that each one-rule semi-Thue system with a 3-loop — not admitting a 1- or 2-loop — belongs to one of five structurally different types. The number of types is reduced to three if word reflection (*i.e.* the transformation of $u \longrightarrow v$ into $\tilde{u} \longrightarrow \tilde{v}$) is not considered to create essentially different structures. The following examples show that one-rule systems belonging to the three types do in fact exist.

Example 1: $abb \longrightarrow bbabaab$ fulfills condition (A). Take $e = a$, $f = bb$, $i = ab$, $j = b$, $m = ab$, $n = b$, $h = \ell = abaab$, $k = bb$. A possible start word for the 3-loop is $un = abbb$.

Example 2: $abaaaba \longrightarrow aaababaaab$ fulfills condition (B). Take $f = j = n = a$, $i = m = abaaab$, $k = aa$, $\ell = babaaab$, $p = ab$, $q = aaab$. A possible start word is $ujn = abaaabaaa$.

Example 3: $ababab \longrightarrow babaabab$ fulfills condition (C). Take $f = k = b$, $i = aba$, $j = bab$, $\ell = aabab$, $m = p = abab$, $n = ab$, $q = a$. A possible start word is $uqfn = (ab)^5$.

4. FURTHER EXAMPLES

The one-rule system $baba \longrightarrow ba^n bab$ (n being a positive integer) has a loop of length n, which is obtainable from the start word $(ba)^3$. (A more involved one-rule system with this property was already given by Narendran *et al.* in [11].)

The one-rule system $aaab \longrightarrow b^5a^4$ has a loop of length 125, which suggests that "small" systems can produce relatively "large" loops.

The one-rule system $aaba \longrightarrow ababaaa$ [6] has loops of lengths 2, 4 and 5 (take e.g. the start word $aaaba$ for length 2 and a^4ba for lengths 4 and 5.) This demonstrates that one and the same system can have loops of different lengths, being not necessarily divisible by each other.

At last, a three-rule semi-Thue system is presented which admits an infinite reduction chain (starting with the word *cabbabd*) but which has no loops:

$$\begin{aligned} baab &\longrightarrow abba \\ cabbab &\longrightarrow cbabba \\ babbad &\longrightarrow abbababd \end{aligned}$$

(*see* [6] for the proof). Whether there exists a one-rule system $u \longrightarrow v$ with the same property remains an open question. Also, the problems whether the questions "Does $u \longrightarrow v$ admit an infinite reduction chain?" (*termination problem*) and "Does $u \longrightarrow v$ admit a loop?" are decidable remain open and deserve further attention. For semi-Thue systems with arbitrary finite number of rules, both problems, the termination problem and the existence of loops, are known to be undecidable [12], even in the length-preserving case [2]. For term rewriting systems with one rule, the termination problem is also known to be undecidable [3]. However, the termination problem for one-rule semi-Thue systems becomes decidable if only certain "well-behaved" derivations are permitted [9] or if one restricts the attention to special classes of rules. E.g., Zantema and Geser [13] have completely solved the termination problem (and also the — here equivalent — loop problem) for one-rule systems of the form $a^p b^q \longrightarrow b^r a^s$ over the alphabet $\{a, b\}$.

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