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ON FREE INVERSE MONOID LANGUAGES (*) (**)

by Pedro V. SILVA [1]

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Abstract. – *This is a study on the class of FIM(X)-languages and its important subfamily consisting of inverse automata languages (i-languages). Both algebraic and combinatorial approaches are used to obtain several results concerning closure operators on* $(X \cup X^{-1})^*$*-languages, including a classification of FIM(X)-languages by i-languages. In particular, it is proved that the i-closure of a recognizable* $(X \cup X^{-1})^*$*-language is at most deterministic context-free. Infinite trees are an essential tool in this process, and they are also helpful in producing counterexamples for other closure problems. Applications to X*-languages are also produced, involving particular classes of codes.*

Résumé. – *Nous étudions la classe des langages dans* $FIM(X)$ *et la sous-famille importante des langages à automates inverses (i-langages). Les approches algébrique et combinatoire sont utilisées pour obtenir plusieurs résultats concernant la fermeture par certains opérateurs des langages de* $(X \cup X^{-1})^*$ *et entre autre une classification des langages des* $FIM(X)$ *par les i-langages. En particulier, il est prouvé que la i-fermeture des langages reconnaissables de* $(X \cup X^{-1})^*$ *est au plus algébrique déterministe. Les arbres infinis sont un outil essentiel dans cette démarche et ils sont aussi utiles pour produire des contre—exemples pour les autres propriétés de fermeture. Des applications aux langages de X* sont aussi exhibées dont des clases particulières de codes.*

1. INTRODUCTION

The first connections between inverse semigroup theory and automata theory are due to the work of W. D. Munn [11], who developed a description of the free inverse semigroup in terms of finite labelled trees which turned out to be finite automata. Unfortunately, this innovative approach had no immediate followers and purely algebraic methods dominated the theory of inverse semigroups for years to come.

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However, the work of J. B. Stephen [15] in the late eighties revived the spirit of Munn's work and boosted combinatorial inverse semigroup theory as one of the fashionable subjects in algebra, attracting the interest of computer scientists, group theorists, logicians and others. His work related the study of presentations of inverse monoids to a certain class of automata called inverse. These are trim deterministic automata on a dual alphabet of the form $X \cup X^{-1}$, and must satisfy a duality condition on their edges. Inverse graphs (underlying graphs of inverse automata) were already a major tool in other areas of mathematics such as combinatorial group theory [13], and inverse automata have now acquired great relevance in several other domains as well.

Stephen's techniques produced great developments in recent years and many of them are due to the work of S. W. Margolis and J. C. Meakin ([6] to [10]). Their methods and results brought together semigroup theory, automata theory, combinatorics and logic, and created a new interest for inverse monoids inside computer science itself.

This new emphasis on combinatorial methods in the study of inverse semigroups forced consideration of free inverse monoid languages, since the language of an inverse automaton can be viewed as a free inverse monoid language. This study examines free inverse monoid languages from an automata theoretic point of view. We answer several standard questions concerning this class of languages, and consider various decidability questions. Some of the methods and results are related to previous work by the author [14]. It is expected that free inverse monoid languages will have applications to classical language theory, and we provide some evidence for this assertion.

2. PRELIMINARIES

The reader is assumed to be familiar with elementary language and automata theories, [1], [2] and [3] being standard references. In particular, we assume some knowledge about $RecM$, the class of all recognizable languages $L \subseteq M$, where M is an arbitrary monoid [1].

Let M be a monoid. A subset $L \subseteq M$ is said to be an *M-language*. Given an M-language L, the *syntactic congruence* of L is defined as follows: for all $u, v \in M$, $u \sim_L v$ if and only if

$$\forall x, y \in M, \qquad xuy \in L \quad \Leftrightarrow \quad xvy \in L.$$

We say that $L \in RecM$ is and only if $M/\sim_L$ is finite. Alternatively, $L \in RecM$ if and only if there exists a homomorphism $\phi : M \to N$ into a finite monoid N such that $L = L\phi\phi^{-1}$. It is a well-known fact that $RecM$ is closed for the Boolean operators (union, intersection, complement).

The reader is also expected to know elementary concepts and results regarding finite Σ-automata and languages in $Rec\Sigma^*$ [2], where Σ denotes a finite alphabet and Σ^* denotes the free monoid on Σ. Such concepts and results should include determinism, trimness, the subset construction, the construction of the minimal automaton, Kleene's Theorem, etc. We use the following notation for automata: a Σ-automaton is a quadruple $A = (Q, i, T, E)$, where Q is a nonempty set, $i \in Q$, $T \subseteq Q$ and $E \subseteq Q \times \Sigma \times Q$; the Σ-language recognized by A is denoted by $L(A)$; for every $p \in Q$, we denote the Σ-language $L(Q, p, T, E)$ by $p^{-1} T$.

Now we introduce the basic definitions concerning inverse monoids and free inverse monoids. For further details, see [12], or [4] for general semigroup theory.

A monoid M is said to be *inverse* if

$$\forall u \in M, \quad \exists ! v \in M : uvu = u \text{ and } vuv = v.$$

We say then that v is the *inverse* of u and denote it by u^{-1}. Alternatively, M is inverse if and only if

$$\forall u \in M, \quad \exists v \in M : uvu = u$$
$$\forall e, f \in E(M), \quad ef = fe$$

both hold, where $E(M)$ denotes the set of idempotents of M. It follows easily that, in an inverse monoid M, $(uv)^{-1} = v^{-1} u^{-1}$ and $(u^{-1})^{-1} = u$ for all $u, v \in M$.

Let M be an inverse monoid. A subset $N \subseteq M$ is said to be an *inverse submonoid* of M if N is a submonoid of M and $u^{-1} \in N$ for every $u \in N$. Now let L be an arbitrary subset of M and let $L^{-1} = \{u^{-1} : u \in L\}$. It is easy to see that $(L \cup L^{-1})^*$ is the smallest inverse submonoid of M containing L. We say that $(L \cup L^{-1})^*$ is the inverse submonoid of M generated by L and denote it by $\langle L \rangle$.

Two relations play an important role throughout this paper. The equivalence relation $\mathcal{R}$ on M is defined by

$$u \mathcal{R} v \quad \Leftrightarrow \quad uu^{-1} = vv^{-1}.$$

The *natural partial order* of M is defined by

$$u \leq v \quad \Leftrightarrow \quad u = uu^{-1}v.$$

In fact, this can be shown to be equivalent to having $u = e_0 v_1 e_1 \dots v_n e_n$ for some $e_0, \dots, e_n \in E(M)$ and $v_1, \dots, v_n \in M$ such that $v_1 \dots v_n = v$. It follows easily that

$$u \leq v,\ u' \leq v' \quad \Rightarrow \quad uu' \leq vv',$$
$$u \leq v \quad \Rightarrow \quad u^{-1} \leq v^{-1}.$$

Such facts will be used later with no further comment.

Now let X denote a finite alphabet. We associate to X a set of formal inverses $X^{-1} = \{x^{-1} : x \in X\}$ disjoint from X. We extend the operator $^{-1}$ to $(X \cup X^{-1})^*$ inductively by defining $(x^{-1})^{-1} = x$ for every $x \in X$ and by using the rule $(uv)^{-1} = v^{-1}u^{-1}$. The *free inverse monoid* on X [12] is defined as the quotient $(X \cup X^{-1})^*/\rho$, where ρ denotes the congruence on $(X \cup X^{-1})^*$ generated by the relation

$$\{(uu^{-1}u,\ u) : u \in (X \cup X^{-1})^*\}$$
$$\cup \{(uu^{-1}vv^{-1},\ vv^{-1}uu^{-1}) : u,\ v \in (X \cup X^{-1})^*\}.$$

The congruence ρ is known as the *Vagner* congruence on $(X \cup X^{-1})^*$ and we denote the free inverse monoid on X by $FIM(X)$.

The projection homomorphism $(X \cup X^{-1})^* \to FIM(X) : u \mapsto u\rho$ is denoted by θ. For technical reasons, we shall favour the use of $u\theta$ instead of $u\rho$.

3. $FIM(X)$-LANGUAGES

Some results in this section are probably well-known but, since no appropriate references could be found, we include full proofs for the sake of completeness. It is essential to relate $FIM(X)$-languages to $(X \cup X^{-1})^*$-languages, and the next results establish the basic connections.

LEMMA 3.1: *Let $L \subseteq FIM(X)$. Then*

$$L \in RecFIM(X) \Leftrightarrow L\theta^{-1} \in Rec(X \cup X^{-1})^*.$$

Proof: Suppose that $L \in RecFIM(X)$. Since recognizable languages are closed under inverse homomorphism, it follows that $L\theta^{-1} \in Rec(X \cup X^{-1})^*$.

Conversely, suppose that $L\theta^{-1} \in Rec(X \cup X^{-1})^*$. Suppose that u, $v \in (X \cup X^{-1})^*$ are such that $u \sim_{L\theta^{-1}} v$. Then, for all $a, b \in (X \cup X^{-1})^*$, we have $(a\theta)(u\theta)(b\theta) \in L \Leftrightarrow (aub)\theta \in L \Leftrightarrow aub \in L\theta^{-1} \Leftrightarrow avb \in L\theta^{-1} \Leftrightarrow (avb)\theta \in L \Leftrightarrow (a\theta)(v\theta)(b\theta) \in L$. Hence $u\theta \sim_L v\theta$ and since $\sim_{L\theta^{-1}}$ has finite index, it follows that $\sim_L$ has finite index as well. Thus $L \in RecFIM(X)$.

Given $P \subseteq (X \cup X^{-1})^*$, we say that P is *ρ-closed* if P is a union of ρ-classes (equivalently, $P = P\theta\theta^{-1}$).

THEOREM 3.2: *Let $P \subseteq (X \cup X^{-1})^*$. The following conditions are equivalent :*

(i) $P = L\theta^{-1}$ *for some* $L \subseteq FIM(X)$;

(ii) $\rho \subseteq \sim_P$;

(iii) $(X \cup X^{-1})^* / \sim_P$ is inverse and $(x \sim_P)^{-1} = x^{-1} \sim_P$ for every $x \in X$;

(iv) P is ρ-closed.

Moreover, if $P \in Rec(X \cup X^{-1})^*$, then $L \in RecFIM(X)$ in (i).

Proof: (i)⇒(ii). Suppose that $P = L\theta^{-1}$ for some $L \subseteq FIM(X)$. Let $(u, v) \in \rho$. Then, for all $a, b \in (X \cup X^{-1})^*$, we have $aub \in P \Leftrightarrow aub \in L\theta^{-1} \Leftrightarrow (aub)\theta \in L \Leftrightarrow (a\theta)(u\theta)(b\theta) \in L \Leftrightarrow (a\theta)(v\theta)(b\theta) \in L \Leftrightarrow avb \in P$. Hence $u \sim_P v$ and $\rho \subseteq \sim_P$.

(ii)⇒(iii). Suppose that $\rho \subseteq \sim_P$. Then we can define a surjective homomorphism $\phi : FIM(X) \to (X \cup X^{-1})^* / \sim_P$ by $(u\rho)\phi = u \sim_P$ for $u \in (X \cup X^{-1})^*$. Since $(X \cup X^{-1})^* / \sim_P$ is a homomorphic image of an inverse monoid, it must be inverse as well [4], and for every $x \in X$, we have $(x \sim_P)^{-1} = [(x\rho)\phi]^{-1} = [(x\rho)^{-1}]\phi = (x^{-1}\rho)\phi = x^{-1} \sim_P$. Thus (iii) holds.

(iii)⇒(iv). Suppose that $(X \cup X^{-1})^* / \sim_P$ is inverse and $(x \sim_P)^{-1} = x^{-1} \sim_P$ for every $x \in X$. Then $(u \sim_P)^{-1} = u^{-1} \sim_P$ for every $u \in (X \cup X^{-1})^*$. It follows that, for all $u, v \in (X \cup X^{-1})^*$, we have $(uu^{-1}u) \sim_P = u \sim_P$ and $(uu^{-1}vv^{-1}) \sim_P = (vv^{-1}uu^{-1}) \sim_P$ and so every $\sim_P$-class is a union of ρ-classes. Since P is a union of $\sim_P$-classes, it follows that P itself is a union of p-classes. Thus $P = P\theta\theta^{-1}$ and (iv) holds.

(iv)⇒(i). Suppose that $P = P\theta\theta^{-1}$. Let $L = P\theta$. Then $L\theta^{-1} = P\theta\theta^{-1} = P$.

The final remark follows from Lemma 3.1.

COROLLARY 3.3: *Given $P \in Rec(X \cup X^{-1})^*$, it is decidable whether or not $P = L\theta^{-1}$ for some $L \in RecFIM(X)$.*

Proof: By Theorem 3.2, we only need to decide if condition (iii) in the theorem holds for P. Since $(X \cup X^{-1})^* / \sim_P$ is finite and effectively constructible, we can certainly decide whether or not $(X \cup X^{-1})^* / \sim_P$ is inverse and $(x \sim_P)^{-1} = x^{-1} \sim_P$ for every $x \in X$.

4. *i*-LANGUAGES

In the last few years, inverse automata have become a very useful tool in inverse semigroup theory [15]. They are naturally related to $FIM(X)$ by the Munn description [11], they play a major role in the study of presentations [9], [14], and so languages recognized by these automata induce a subclass of $FIM(X)$-languages important in its own right.

A trim deterministic $(X \cup X^{-1})^*$-automaton $A = (Q, i, T, E)$ is said to be *inverse* if

$$(p, x, q) \in E \Leftrightarrow (q, x^{-1}, p) \in E$$

holds for all $p, q \in Q$ and $x \in X$ (duality of edges). If A is inverse, it follows easily that $L(A)$ is ρ-closed [15].

A language $L \subseteq FIM(X)$ is said to be an *i-language* if $L\theta^{-1} = L(A)$ for some inverse $(X \cup X^{-1})^*$-automaton A. Sometimes we will refer to the language of an inverse $(X \cup X^{-1})^*$-automaton as an i-language too.

Given a language $L \subseteq FIM(X)$, we say that L is *closed* if

$$\forall u \in L, \quad \forall v \in FIM(X), \qquad v \geq u \quad \Rightarrow \quad v \in L,$$

and we say that L is *elastic* if

$$\forall a, b \in L, \quad aa^{-1}b \in L.$$

THEOREM 4.1: *Let $L \subseteq FIM(X)$. Then L is an i-language if and only if L is closed and elastic.*

Proof: Suppose that L is an i-language. Then $L\theta^{-1} = L(A)$ for some inverse $(X \cup X^{-1})^*$-automaton A. Let $u \in L$, $v \in FIM(X)$ be such that $v \geq u$. Then $u = uu^{-1}v$. Let $v' \in v\theta^{-1}$ and $u' \in u\theta^{-1}$. Then $(u'u'^{-1}v')\theta = uu^{-1}v = u$ and so $u'u'^{-1}v'$ labels a successful path in A. Since A is deterministic, $u'u'^{-1}$ must label exactly one path starting at the initial vertex. Since A has duality of edges, that path must necessarily be a loop. Therefore v' also labels a successful path in A, so $v' \in L(A) = L\theta^{-1}$ and $v = v'\theta \in L$. Thus L is closed.

Now suppose that $a\rho$, $b\rho \in L$ for some $a, b \in (X \cup X^{-1})^*$. Then $a, b \in L(A)$. Since A has duality of edges, aa^{-1} must label a loop at the initial vertex in A and so $aa^{-1}b$ labels a successful path in A. Thus $(a\theta)(a\theta)^{-1}(b\theta) \in L$ and L is elastic.

Conversely, suppose that L is closed and elastic. Let $P = L\theta^{-1}$. We intend to show that the minimal automaton of P, denoted by $A = (Q, i, T, E)$, is inverse. Since A is minimal, A is certainly trim and deterministic and so we only need to prove duality of edges. Let $(p, x, q) \in E$, with $p, q \in Q$ and $x \in X \cup X^{-1}$. Since A is trim, there is a path in A of the form

$$i \xrightarrow{u} p \xrightarrow{x} q \xrightarrow{v} t \in T.$$

Hence $uxv \in P$, and since $P = P\theta\theta^{-1}$, we have $uxx^{-1}xv \in P$. Since A is deterministic, it follows that A has an edge (q, x^{-1}, r) for some $r \in Q$. We want to show that $r = p$. Since A is minimal, this is equivalent to $p^{-1}T = r^{-1}T$ [2].

Let $w \in r^{-1}T$. Then $uxx^{-1}w \in P$ and so $(uxx^{-1}w)\rho \in L$. Since $(uxx^{-1}w)\rho = (uxx^{-1}u^{-1})\rho(uw)\rho$ and L is closed, it follows that $(uw)\rho \in L$ and so $uw \in P$. Since A is deterministic, it follows that $w \in p^{-1}T$ and so $r^{-1}T \subseteq p^{-1}T$.

Conversely, let $z \in p^{-1}T$. Then $uz \in P$. Since $uxv \in P$ and L is elastic, it follows that $(uxvv^{-1}x^{-1}u^{-1}uz)\rho \in L$. Now $(uxx^{-1}z)\rho \geq (uxvv^{-1}x^{-1}u^{-1}uz)\rho$ and so $(uxx^{-1}z)\rho \in L$, since L is closed. Therefore $uxx^{-1}z \in P$ and we must have $z \in r^{-1}T$. Hence $p^{-1}T \subseteq r^{-1}T$ and so $r^{-1}T = p^{-1}T$. Thus A is inverse and L is an i-language.

It follows from the previous proof that $L \subseteq FIM(X)$ is a recognizable i-language if and only if $L\theta^{-1}$ is recognized by a finite inverse $(X \cup X^{-1})^*$-automaton. We denote by $iRecFIM(X)$ the class of all recognizable i-languages of $FIM(X)$.

For $L \subseteq FIM(X)$, let

$$L^{\omega} = \{v \in FIM(X) : v \geq u \text{ for some } u \in L\}.$$

It is immediate that L^{ω} is the smallest closed $FIM(X)$-language containing L.

THEOREM 4.2: *$RecFIM(X)$ is the Boolean closure of $iRecFIM(X)$.*

Proof: Since $iRecFIM\,(X) \subseteq RecFIM\,(X)$ and $RecFIM\,(X)$ is closed for the Boolean operations, it follows that $RecFIM\,(X)$ contains the Boolean closure of $iRecFIM\,(X)$.

Conversely, let $L \in RecFIM\,(X)$. Then there exists a finite inverse monoid M and a homomorphism $\phi : FIM\,(X) \to M$ such that $L = L\phi\phi^{-1}$.

Let $u \in M$. Obviously, $u^\omega\,\phi^{-1} \in RecFIM\,(X)$. We are going to show that $u^\omega\,\phi^{-1}$ is an i-language.

Let $v \in u^\omega\,\phi^{-1}$ and let $w \in FIM\,(X)$ be such that $w \geq v$. Then $w\phi \geq v\phi \geq u$ and so $w \in u^\omega\,\phi^{-1}$. Hence $u^\omega\,\phi^{-1}$ is closed. Now suppose that $a, b \in u^\omega\,\phi^{-1}$. Then $a\phi \geq u$ and $b\phi \geq u$. It follows that $(aa^{-1}\,b)\,\phi \geq uu^{-1}\,u = u$ and so $aa^{-1}\,b \in u^\omega\,\phi^{-1}$. Therefore $u^\omega\,\phi^{-1}$ is elastic and, by Theorem 4.1, an i-language.

Thus $u^\omega\,\phi^{-1} \in iRecFIM\,(X)$ for all $u \in M$.

Now $L = L\phi\phi^{-1} = \bigcup_{u \in L\phi} u\phi^{-1} = \bigcup_{u \in L\phi} [u^\omega \phi^{-1} \backslash \bigcup_{v > u} (v^\omega\,\phi^{-1})]$. Thus L belongs to the Boolean closure of $iRecFIM\,(X)$ and the result follows.

Now we are able to define the *i-closure* of a language $L \subseteq FIM\,(X)$, which we prove to be the smallest i-language containing L. Let

$$\overline{L} = [(\bigcup_{u \in L} uu^{-1})^*\,L]^\omega.$$

Now we have

THEOREM 4.3: *Let $L \subseteq FIM\,(X)$. Then $\overline{L}$ is the smallest i-language containing L.*

Proof: By definition, $\overline{L}$ is closed. Now let $a, b \in \overline{L}$. Then there exist $u_1, \ldots, u_{n+1}, v_1, \ldots, v_{m+1} \in L$ with

$$a \geq (u_1\,u_1^{-1}) \ldots (u_n\,u_n^{-1})\,u_{n+1}$$

and

$$b \geq (v_1\,v_1^{-1}) \ldots (v_m\,v_m^{-1})\,v_{m+1}.$$

Thus

$$aa^{-1}\,b \geq (u_1\,u_1^{-1}) \ldots (u_n\,u_n^{-1})\,u_{n+1}\,u_{n+1}^{-1}\,(u_n\,u_n^{-1}) \times \ldots \times (u_1\,u_1^{-1})(v_1\,v_1^{-1}) \ldots (v_m\,v_m^{-1})\,v_{m+1} \in \overline{L}.$$

Since $\overline{L}$ is closed, we have $aa^{-1}b \in \overline{L}$ and so $\overline{L}$ is elastic. By Theorem 3.1, it follows that $\overline{L}$ is an i-language.

It is immediate that $L \subseteq \overline{L}$.

Finally, let L' be an i-language containing L. Since $L \subseteq L'$, and L' is elastic, it follows that

$$(\bigcup_{u \in L} uu^{-1})^* L \subseteq L'.$$

Since L' is closed, it follows that $\overline{L} \subseteq L'$ and so $\overline{L}$ is in fact the smallest i-language containing L.

Let R_X denote the subset of all reduced words of $(X \cup X^{-1})^*$. We denote bu $\iota : (X \cup X^{-1})^* \to R_X$ the reduction map which assigns to every word u in $(X \cup X^{-1})^*$ the corresponding reduced word $u\iota$. It is well-known that R_X under the binary operation defined by $(u, v) \to (uv)\iota$ constitutes a description of $FG(X)$, the free group on X. This perspective of R_X should be kept in mind. Note in particular that every (free) subgroup of $FG(X)$ corresponds to a subset $P \subseteq R_X$ which is closed under this binary operation and formal inversion. This is the correspondence that should be kept in mind when we refer to such a set P as a subgroup of R_X.

We denote by $\Gamma(X)$ the Cayley graph of $FG(X)$ with respect to the generators $X \cup X^{-1}$. Therefore R_X is the set of vertices of $\Gamma(X)$ and (p, x, q) is an edge of $\Gamma(X)$, with $p, q \in R_X$, $x \in X \cup X^{-1}$, if and only if $q = (px)\iota$. We fix 1 as the initial vertex. Since $\Gamma(X)$ has duality of edges, we can turn $\Gamma(X)$ into an inverse automaton by assigning to it a set T of terminal vertices: provided T is nonempty, our automaton will be trim, and determinism follows in any case. Moreover, if we replace $\Gamma(X)$ by one of its nonempty connected subtress containing 1, say Γ, the result is still an inverse automaton. This automaton will be denoted by (Γ, T), T denoting a nonempty subset of vertices of Γ.

For every $u \in (X \cup X^{-1})^*$, we denote by $MT(u)$ (the *Munn tree* of u), the finite connected subtree of $\Gamma(X)$ defined by the path beginning at vertex 1 and having u as its label. It is well-known that $\{(MT(u), u\iota) : u \in (X \cup X^{-1})^*\}$, under the product defined by $(MT(u), u\iota)(MT(v), v\iota) = (MT(uv), (uv)\iota)$, constitutes a description of $FIM(X)$ [11]; in fact, it solves the word problem for ρ, since

$$u\rho = v\rho \quad \Leftrightarrow \quad (MT(u), u\iota) = (MT(v), v\iota).$$

Now we can define $MT(w)$ and $w\iota$ for $w \in FIM(X)$ as being respectively $MT(u)$ and $u\iota$ for some $u \in w\theta^{-1}$. It is easy to see that, for all u, $v \in FIM(X)$,

(i) $u\mathcal{R}v \Leftrightarrow MT(u) = MT(v)$;

(ii) $u \geq v \Leftrightarrow MT(u)$ is a subtree of $MT(v)$ and $u\iota = v\iota$;

(iii) $L((MT(u), u\iota)) = u^{\omega}\theta^{-1}$.

A word $u \in (X \cup X^{-1})^*$ is said to be a *Dyck* word if $u\iota = 1$. We denote the language consisting of all Dyck words of $(X \cup X^{-1})^*$ by D_X. The language D_X is a well-known example of a context-free $(X \cup X^{-1})^*$-language (CFL) [1]. It is easy to see that, given $u \in (X \cup X^{-1})^*$, $u\rho \in E[FIM(X)]$ if and only if $u \in D_X$.

Now for $P \subseteq (X \cup X^{-1})^*$, we define a connected subtree $MT(P)$ of $\Gamma(X)$ to be the union of all subtrees of the form $MT(u)$, with $u \in P$. Finally, for $L \subseteq FIM(X)$, we define $MT(L)$ to be the union of all subtrees of the form $MT(w)$, with $w \in L$.

THEOREM 4.4: *Let $L \subseteq FIM(X)$ and let $A = (MT(L), L_\iota)$. Then A is an inverse automaton and $L(A) = \overline{L}\theta^{-1}$.*

Proof: It follows from previous remarks that A is inverse. Suppose that $w \in L\theta^{-1}$. Then $w \in L((MT(w), w\iota))$ and so $w \in L(A)$. Thus $L\theta^{-1} \subseteq L(A)$ and so $L \subseteq [L(A)]\theta$. Since $[L(A)]\theta$ is an i-language containing L, it follows from the previous result that $\overline{L} \subseteq [L(A)]\theta$ and so $\overline{L}\theta^{-1} \subseteq [L(A)]\theta\theta^{-1} = L(A)$.

Conversely, suppose that $v \in L(A)$. Since $A = (MT(L), L\iota)$, there exist $u_1, \ldots, u_n \in L$ such that v is in the language of $(MT(u_1) \cup \ldots \cup MT(u_n), u_n\iota)$. But now v is in the language of $(MT(u_1 u_1^{-1} \ldots u_{n-1} u_{n-1}^{-1} u_n), (u_1 u_1^{-1} \ldots u_{n-1} u_{n-1}^{-1} u_n)\iota)$ and so if follows that $v\theta \geq u_1 u_1^{-1} \ldots u_{n-1} u_{n-1}^{-1} u_n$. But $u_1 u_1^{-1} \ldots u_{n-1} u_{n-1}^{-1} u_n \in \overline{L}$ and so, since $\overline{L}$ is closed, we obtain $v\rho \in \overline{L}$. Thus $v \in \overline{L}\theta^{-1}$ and the theorem holds.

We note that, despite being inverse, the automaton $A = (MT(L), L\iota)$ is not necessarily minimal, any infinite recognizable $FIM(X)$-language being a counterexample.

THEOREM 4.5: *Let L be a finite $FIM(X)$-language. Then*

(i) *L is recognizable;*

(ii) *$\overline{L}$ is finite;*

(iii) $\min_{\overline{L}\theta^{-1}} = (MT(L), L\iota)$;

(iv) *L is an i-language if and only if there exist an $\mathcal{R}$-class R of $FIM(X)$ and $F \subseteq R$ such that $L = F^{\omega}$;*

(v) *if $L = \{u\}$, then* L is an i-language if and only if u is reduced.

Proof: (i) For every $n \geq 1$, the subset $I_n = \{u \in FIM(X) : MT(u)$ has at least n vertices$\}$ is an ideal of $FIM(X)$. The corresponding Rees congruence τ_n on $FIM(X)$ has I_n as one of its congruence classes, all the others being singular, and so $FIM(X)/\tau_n$ is finite. Since L is finite, we have $L \subseteq FIM(X) \backslash I_m$ for some $m \geq 1$. Let $\phi : FIM(X) \to FIM(X)/\tau_m$ denote the projection homomorphism. It follows that $L = L\phi\phi^{-1}$ and so $L \in RecFIM(M)$.

(ii) We have $\overline{L} = [L(A)]\theta$, where $A = (MT(L), L\iota)$. It follows that $\overline{L}$ is contained in the subset F consisting of all $u \in FIM(X)$ such that $MT(u)$ is a subtree of $MT(L)$. Since L is finite, then F is finite and so $\overline{L}$ is finite.

(iii) We know that $\overline{L}\theta^{-1} = L(A)$, where $A = (MT(L), L\iota)$, therefore we only need to prove that A is a minimal automaton. Since A is inverse, it is certainly trim and deterministic. By a well-known algorithm [2], we only need to show that $p^{-1}T \neq q^{-1}T$ for all vertices p, q of A with $p \neq q$. Let p, q be vertices of A.

Suppose first that there exists $u \in (X \cup X^{-1})^*$ such that u labels a path, say, from p, but u labels no path from q. Then $uu^{-1}v \in p^{-1}T$ for some $v \in (X \cup X^{-1})^*$ but $uu^{-1}v \notin q^{-1}T$, since $uu^{-1}v$ cannot label a path from q in A. Thus $p^{-1}T \neq q^{-1}T$.

Now suppose that paths from p and paths from q produce exactly the same labels. Let w be a reduced word labeling a path from p to q. We show by induction that w^n labels a path from p for every $n \geq 1$. It is true for $n = 1$. Now suppose that w^n labels a path from p. Then w^n labels a path from q and so w^{n+1} labels a path from p. It follows by induction that w^n labels a path from p for every $n \geq 1$. We can write $w = aca^{-1}$ for some words a, c with c cyclically reduced, therefore $ac^n a^{-1}$ is a reduced word labeling a path from p for every $n \geq 1$. Since $MT(L)$ is a finite tree, the length of a reduced word labeling a path in A is clearly bounded. It follows that $c = 1$ and so $w = 1$ and $p = q$. Thus A is minimal.

(iv) Suppose first that L is an i-language. Then $L = \overline{L} = [L(A)]\theta$, where $A = (MT(L), L\iota)$. Since L is finite, there exist $u, v_1, \ldots, v_m \in L$ such that $MT(L) = MT(u)$ and $L\iota = \{v_1\iota, \ldots, v_m\iota\}$. It follows that $L(A) = \bigcup_{i=1}^m L((MT(L), v_i\iota)) = \bigcup_{i=1}^m L((MT(u), v_i\iota)) = \bigcup_{i=1}^m L((MT(uu^{-1}v_i), (uu^{-1}v_i)\iota))$. Therefore $L = [L(A)]\theta =$

$\bigcup_{i=1}^{m} (uu^{-1} v_i)^\omega = (\bigcup_{i=1}^{m} uu^{-1} v_i)^\omega$. Since $MT(uu^{-1} v_i) = MT(u)$ for $i = 1, \ldots, m$, it follows that the $uu^{-1} v_i$ are all $\mathcal{R}$-related.

Conversely, suppose that there exist an $\mathcal{R}$-class R of $FIM(X)$ and $F \subseteq R$ such that $L = F^\omega$. Since L is closed by hypothesis, we only need to show that L is elastic. Let $a, b \in L$. Then there exist $c, d \in F$ such that $a \geq c$ and $b \geq d$. It follows that $aa^{-1} b \geq cc^{-1} d = dd^{-1} d = d \in F$. Thus $aa^{-1} b \in F^\omega = L$ and L is an i-language.

(v) Suppose that $L = \{u\}$. By (iv), $\{u\}$ is an i-language if and only if $\{u\} = F^\omega$ for some subset F of an R-class of $FIM(X)$. But then we must have $|F| = 1$ and this yields $F = \{u\}$ and $\{u\} = u^\omega$. It is easy to see that $\{u\} = u^\omega$ if and only if u is reduced.

In general, $L \in RecFIM(X)$ does not imply $\overline{L} \in RecFIM(X)$, as the next example shows. We remark that $L\iota = \overline{L}\iota$ for every $L \subseteq FIM(X)$.

EXAMPLE 4.6: Let $X = \{x, y\}$ and let $L = \langle x\rho \rangle \cup \langle y\rho \rangle$. Obviously, we have $\langle x\rho \rangle \theta^{-1} = (x \cup x^{-1})^* \in Rec(X \cup X^{-1})^*$ and so $\langle x\rho \rangle \in RecFIM(X)$. Similarly, $\langle y\rho \rangle \in RecFIM(X)$. Since $RecFIM(X)$ is closed for union, it follows that $L \in RecFIM(X)$. We prove that $\overline{L} \notin RecFIM(X)$ by showing that $\sim_{\overline{L}}$ does not have finite index.

Let $m, k \geq 1$ with $m \neq k$. Since $\overline{L}$ is elastic, we have $(x^m \rho)(x^{-m} y)\rho \in \overline{L}$. On the other hand, $(x^k x^{-m} y)\iota \notin L\iota = \overline{L}\iota$ and so $(x^k \rho)(x^{-m} y)\rho \notin \overline{L}$. Hence $(x^m \rho) \sim_{\overline{L}} \neq (x^k \rho) \sim_{\overline{L}}$. It follows that $FIM(X)/\sim_{\overline{L}}$ is infinite and so $\overline{L} \notin RecFIM(X)$.

5. CLOSURES FOR $(X \cup X^{-1})^*$-LANGUAGES

In this section we discuss the $FIM(X)$-languages $P\theta$ and $\overline{P\theta}$ for $P \in Rec(X \cup X^{-1})^*$. We shall be forced to consider far more general classes of languages than recognizable. We say that $L \subseteq FIM(X)$ is *context-free* (deterministic context-free, context-sensitive) if $L\theta^{-1}$ is context-free (deterministic context-free, context-sensitive) [3]. Note that these definitions are compatible with the concept of recognizability.

It is easy to see that $P\theta$ is recursive for every $P \in Rec(X \cup X^{-1})^*$. In fact, given $u \in FIM(X)$, we have $u \in P\theta$ if and only if $P \cap (u\theta^{-1}) \neq \emptyset$. Since $\{u\} \in RecFIM(X)$ by Theorem 4.5, we have $u\theta^{-1} \in Rec(X \cup X^{-1})^*$. But $Rec(X \cup X^{-1})^*$ is closed for intersection, therefore $P \cap (u\theta^{-1}) \in Rec(X \cup X^{-1})^*$ and we can certainly decide whether or not this is empty. It follows that we can decide whether or not $u \in P\theta$ and so $P\theta$ is recursive.

However, $P\theta$ does not have to be context-free, as the next example shows.

EXAMPLE 5.1: Let $X = \{x\}$ and let $P = x^*$. We show that $P\theta$ is not a context-free $FIM(X)$-language, that is, $P\theta\theta^{-1}$ is not a context-free $(X \cup X^{-1})^*$-language.

Suppose that $P\theta\theta^{-1}$ is a CFL. Since $x^* (x^{-1})^* x^* \in Rec(X \cup X^{-1})^*$ and the intersection of a CFL with a recognizable language is still a CFL [3], it follows that $Q = (P\theta\theta^{-1}) \cap [x^* (x^{-1})^* x^*]$ is a context-free $(X \cup X^{-1})^*$-language. Since $P\theta\theta^{-1}$ is the union of the languages of all inverse automata of the form

$$I \ni c_1 \xrightarrow{x} c_2 \dots c_{d-1} \xrightarrow{x} c_d \in T$$

for $d \geq 1$ (the dual edges are ommited), it is easy to see that $Q = \{x^m x^{-n} x^k : m \geq n \text{ and } k \geq n\}$. By the Pumping Lemma for CFLs [3], there exists $N \geq 1$ such that, for every $u \in Q$ with $|u| \geq N$, there exist $a, v, w, z, b \in (X \cup X^{-1})^*$ satisfying

(i) $vz \neq 1$;

(ii) $|vwz| \leq N$;

(iii) $avwzb = u$;

(iv) $av^n wz^n b \in Q$ for every $n \geq 0$.

Take $u = x^N x^{-N} x^N \in Q$ and let a, v, w, z, b satisfy the Pumping Lemma conditions. If x^{-1} occurs in v or z, it is easy to see that $av^2 wz^2 b \notin Q$, contradiction. On the other hand, if x^{-1} does not occur in either v or z, then $awb \notin Q$, contradiction again. It follows that Q is not a CFL and so $P\theta\theta^{-1}$ is not a CFL either.

In this particular example, it is not dificult to show that $P\theta$ is context-sensitive, since $P\theta\theta^{-1}$ is the intersection of two context-free $(X \cup X^{-1})^*$-languages. Whether or not $P\theta$ is always context-sensitive is an open question.

The remaining part of this section is devoted to the study of $\overline{P\theta}$, the i-closure of P. We can provide positive answers but some technical work is required.

For every $P \subseteq (X \cup X^{-1})^*$, we define P^{pr} to be the $(X \cup X^{-1})^*$-language consisting of all prefixes of words in P. If $P \in Rec(X \cup X^{-1})^*$, then it is easy to see that $P^{pr} \in Rec(X \cup X^{-1})^*$ as well, just by allowing all vertices in the minimal automaton of P to be terminal.

Given an $(X \cup X^{-1})^*$-automaton $A = (Q, I, T, E)$, we define $L_i(A)$ to be the language recognized by the $(X \cup X^{-1})^*$-automaton (Q, I, Q, E).

The words in $L_i(A)$ are said to label initial paths in A. Obviously, it is always true that $[L(A)]^{pr} \subseteq L_i(A)$. If A is trim, then the reverse inclusion holds as well.

LEMMA 5.2: *Let P, N be $(X \cup X^{-1})^*$-languages. Then $\overline{P\theta} = \overline{N\theta}$ if and only if $P^{pr}\iota = N^{pr}\iota$ and $P\iota = N\iota$.*

Proof: Let Q be an $(X \cup X^{-1})^*$-language. It follows from Theorem 4.4 that $\overline{Q\theta}$ is fully determined by $MT(Q)$ and $Q\iota$. Since $MT(Q)$ is a tree, it is determined by its geodesics, where a geodesic means the shortest path connecting the initial vertex 1 to a certain vertex. Since $MT(Q)$ has duality of edges, it is immediate that the set of labels of these geodesics is precisely $Q^{pr}\iota$. Thus $P^{pr}\iota = N^{pr}\iota$ and $P\iota = N\iota$ together imply $\overline{P\theta} = \overline{N\theta}$.

Conversely, suppose that $\overline{P\theta} = \overline{N\theta}$. Let $u \in P^{pr}\iota$. Then $u = v\iota$ for some $v \in P^{pr}$. Hence v labels an initial path in $MT(P)$ and so in $MT(\overline{P\theta})$ as well. Since $\overline{P\theta} = \overline{N\theta}$ and $MT(\overline{N\theta})$ has duality of edges, it follows that $u = v\iota$ labels an initial path in $MT(\overline{N\theta})$. Hence u labels an initial path in $MT(w_1 w_1^{-1} \dots w_n w_n^{-1} w_{n+1})$ for some $w_1, \dots, w_{n+1} \in N$, $n \geq 0$, and we can assume that such n is minimal. Since u is reduced and n is minimal, it follows that u must label an initial path in $MT(w_{n+1})$ and so $u \in (w_{n+1})^{pr}\iota \subseteq N^{pr}\iota$. Thus $P^{pr}\iota \subseteq N^{pr}\iota$. Similarly, we show that $N^{pr}\iota \subseteq P^{pr}\iota$ and so $P^{pr}\iota = N^{pr}\iota$.

On the other hand, it follows from the definition that $(\overline{Q\theta})\iota = Q\iota$ for every $(X \cup X^{-1})^*$-language Q, hence $P\iota = (\overline{P\theta})\iota = (\overline{N\theta})\iota = N\iota$ and the lemma is proved.

Next we present a slightly stronger version of Lemma 2.4 of [9], usually related to Benois' Theorem [1].

LEMMA 5.3: *Let $P \in Rec(X \cup X^{-1})^*$ be nonempty. Then we can effectively construct a finite deterministic $(X \cup X^{-1})^*$-automaton A such that $L(A) = P\iota$ and $L_i(A) = P^{pr}\iota$.*

Proof: Let $B = \min_P = (Q, i, T, E)$. For all $q, q' \in Q$, we denote by $L_{q,q'}$ the set of labels of all paths in B from q to q'. We define

$$A_0 = (Q_0, i_0, T_0, E_0),$$

where $Q_0 = [Q \times (X \cup X^{-1})] \cup \{i_0\}$, $T_0 = T \times (X \cup X^{-1})$ (we add i_0 if $D_X \cap P \neq \emptyset$) and $E_0 = \{((q, y), x, (q', x)) : x, y \in X \cup X^{-1}, y \neq x^{-1}, q, q' \in Q, L_{q,q'} \cap (x\iota^{-1}) \neq \emptyset\} \cup \{(i_0, x, (q', x)) : x \in X \cup X^{-1}, q' \in Q, L_{i,q'} \cap (x\iota^{-1}) \neq \emptyset\}$.

It is a well-known fact that the Dyck language $D_X = 1\iota^{-1}$ is a CFL [1], and so is $x\iota^{-1} = D_X x D_X$ for every $x \in X \cup X^{-1}$. Therefore it is always decidable whether or not these languages intersect recognizable languages such as P or $L_{q,q'}$, with $q, q' \in Q$ [3]. It follows that A_0 is an effectively constructible finite $(X \cup X^{-1})^*$-automaton.

Let $u \in P^{pr}\iota$. Since $1 \in L_i(A_0)$ and $1 \in P^{pr}\iota$ trivially, we can assume that $u \neq 1$. Then there exist $x_1, \ldots, x_n \in X \cup X^{-1}$ and $e_1, \ldots, e_n, e'_n \in D_X$ such that $u = x_1 \ldots x_n$ and $e_1 x_1 e_2 \ldots e_n x_n e'_n$ labels an initial path in B. Let

$$i \xrightarrow{e_1 x_1} q_1 \xrightarrow{e_2 x_2} q_2 \ldots \xrightarrow{e_{n-1} x_{n-1}} q_{n-1} \xrightarrow{e_n x_n e'_n} q_n$$

be an initial path in B. We show that

$$i_0 \xrightarrow{x_1} (q_1, x_1) \xrightarrow{x_2} (q_2, x_2) \ldots \xrightarrow{x_{n-1}} (q_{n-1}, x_{n-1}) \xrightarrow{x_n} (q_n, x_n)$$

is a path in A_0.

In fact, $e_1 x_1 \in L_{i,q_1}$ yields $(i_0, x_1, (q_1, x_1)) \in E_0$. For every $j \in \{2, \ldots, n-1\}$, we have that $x_j \neq x_{j-1}^{-1}$ (since $x_1 \ldots x_n$ is reduced) and $e_j x_j \in L_{q_{j-1}, q_j}$. It follows that $((q_{j-1}, x_{j-1}), x_j, (q_j, x_j)) \in E_0$ for every $j \in \{2, \ldots, n-1\}$. Similarly, we prove that $((q_{n-1}, x_{n-1}), x_n, (q_n, x_n)) \in E_0$ and so $u = x_1 \ldots x_n$ labels an initial path in A_0.

Therefore $P^{pr}\iota \subseteq L_i(A_0)$. Now, if $u \in P\iota$, with $u = x_1 \ldots x_n$ just as before, then we can assume that $q_n \in T$. But then $(q_n, x_n) \in T_0$ and so $u \in L(A_0)$. Finally, if $1 \in P\iota$, then $D_X \cap P \neq \emptyset$ and so $i_0 \in T_0$ and $1 \in L(A_0)$. It follows that $P\iota \subseteq L(A_0)$.

Conversely, let $u \in L_i(A_0)$. We can assume that $u \neq 1$, say $u = x_1 \ldots x_n$, with $x_1, \ldots, x_n \in X \cup X^{-1}$. Then u must label a path in A_0 of the form

$$i_0 \xrightarrow{x_1} (q_1, x_1) \xrightarrow{x_2} (q_2, x_2) \ldots \xrightarrow{x_{n-1}} (q_{n-1}, x_{n-1}) \xrightarrow{x_n} (q_n, x_n).$$

It follows from the definition of E_0 that $x_1 \in L_{i,q_1}\iota$ and that, for all $j \in \{2, \ldots, n\}$,$x_j \neq x_{j-1}^{-1}$ and $x_j \in L_{q_{j-1},q_j}\iota$. Denoting i by q_0, we can say that there exist $v_1, \ldots, v_n \in (X \cup X^{-1})^*$ such that, for all $j \in \{1, \ldots, n\}$, $v_j \in L_{q_{j-1}, q_j}$ and $v_j\iota = x_j$. Now $v = v_1 \ldots v_n$ labels a path in B from $i = q_0$ to q_n and $v\iota = (v_1 \ldots v_n)\iota = (v_1\iota \ldots v_n\iota)\iota = (x_1 \ldots x_n)\iota = x_1 \ldots x_n = u$. Hence $u = v\iota \in P^{pr}\iota$ and $L_i(A_0) \subseteq P^{pr}\iota$. Thus $L_i(A_0) = P^{pr}\iota$.

Now if $u \in L(A_0)$, with $u = x_1 \ldots x_n$ just as before, we can assume that $(q_n, x_n) \in T_0$, that is, $q_n \in T$. It follows that v labels a successful path in B and so $u = v\iota \in P\iota$. Finally, if $1 \in L(A_0)$, then $i_0 \in T_0$ and so $D_X \cap P \neq \varnothing$. Therefore $1 = v\iota$ for some $v \in P$, that is, $1 \in P\iota$. It follows that $L(A_0) \subseteq P\iota$ and so $L(A_0) = P\iota$.

Now let A denote the $(X \cup X^{-1})^*$-automaton obtained by applying the subset construction [2] to A_0 and deleting the state corresponding to the empty subset of Q_0. Of course A is deterministic and $L(A) = P\iota$.

Suppose that $u \in L_i(A_0)$. Then u labels a path in A_0 from i_0 to some state $k \in Q_0$. If follows that u labels a path in A from $\{i_0\}$ to some state K which, as a subset, contains k. Thus $u \in L_i(A)$ and $L_i(A_0) \subseteq L_i(A)$.

Conversely, let $u \in L_i(A)$, and we can assume that $u \neq 1$. Then there is a path in A of the form

$$\{i_0\} = K_0 \xrightarrow{x_1} K_1 \xrightarrow{x_2} K_2 \dots \xrightarrow{x_n} K_n$$

with $x_1, \dots, x_n \in X \cup X^{-1}$ and $x_1 \dots x_n = u$. Let $k_n \in K_n$. For $j = n, \dots, 1$, we can successively choose $k_{j-1} \in K_{j-1}$ such that $(k_{j-1}, x_j, k_j) \in E_0$. Since k_0 must necessarily be i_0, we obtain a path

$$i_0 = k_0 \xrightarrow{x_1} k_1 \xrightarrow{x_2} k_2 \dots \xrightarrow{x_n} k_n$$

in A_0 and so $u \in L_i(A_0)$. Thus $L_i(A) \subseteq L_i(A_0)$ and so $L_i(A) = L_i(A_0) = P^{pr}\iota$.

In a similar spirit to Corollary 4.3 of [9], we can now obtain the next result.

THEOREM 5.4: *Let $P \in Rec(X \cup X^{-1})^*$. Then $\overline{P\theta}$ is deterministic context-free and $(\overline{P\theta})\,\theta^{-1}$ is effectively constructible.*

Proof: We assume that P is nonempty. By our previous lemma, we can effectively construct a finite deterministic $(X \cup X^{-1})^*$-automaton $A = (Q, i, T, E)$ such that $L(A) = P\iota$ and $L_i(A) = P^{pr}\iota$. For every $\lambda \in E$, we denote by $|\lambda|$ the label of λ.

We define a pushdown $(X \cup X^{-1})^*$-automaton [3] $A' = (Q, i, T, \Gamma, s, \delta)$, where $\Gamma = E \cup \{s\}$, $s \notin E$ and δ is described by the following transitions, with $x \in X \cup X^{-1}$; $q, q' \in Q$; $\lambda, \mu \in E$:

$$\begin{array}{ll} x : (q, s) \vdash (q', \lambda s) & \text{if } \lambda = (q, x, q') \\ x : (q, \mu) \vdash (q', \lambda\mu) & \text{if } \lambda = (q, x, q') \text{ and } |\mu| \neq x^{-1} \\ x : (q, \mu) \vdash (q', 1) & \text{if } \mu = (q', x^{-1}, q). \end{array}$$

Obviously, A' is an effectively constructible pushdown $(X \cup X^{-1})^*$-automaton. Since A is deterministic, it follows that A' is deterministic as well, and we shall prove that $L(A') = (\overline{P\theta})\,\theta^{-1}$.

We are going to successively prove a few remarks, where we assume that $x \in X \cup X^{-1}$; $u,\ v \in (X \cup X^{-1})^*$, $e \in D_X$; $q,\ q' \in Q$; $t \in T$; $\lambda_1, \ldots, \lambda_n \in E$; $\gamma,\ \gamma',\ \gamma'' \in \Gamma^*$.

(i) If $u : (i,\ s) \overset{*}{\vdash} (q,\ \lambda_n \ldots \lambda_1 s)$, then $|\lambda_n| \ldots |\lambda_1|$ is a reduced word.

This follows from the definition of δ, namely the constraint $|\mu| \neq x^{-1}$ for transitions of the form $x : (q,\ \mu) \vdash (q',\ \lambda\mu)$.

(ii) If $(i, s) \overset{*}{\vdash} (q, \gamma)$ and $u : (q, \gamma) \overset{*}{\vdash} (q', \gamma')$, then $uu^{-1} : (q, \gamma) \overset{*}{\vdash} (q, \gamma)$.

By a simple induction, it is enough to consider $u \in X \cup X^{-1}$. Suppose then that $(i, s) \overset{*}{\vdash} (q, \gamma)$ and $x : (q, \gamma) \vdash (q', \gamma')$, with $x \in X \cup X^{-1}$. Suppose first that $\gamma' = \lambda\gamma$ with $\lambda \in E$. Then $\lambda = (q, x, q')$ and so $x^{-1} : (q',\ \lambda) \vdash (q,\ 1)$. Thus $xx^{-1} : (q,\ \gamma) \overset{*}{\vdash} (q,\ \gamma)$. Now suppose that $\gamma = \mu\gamma'$, with $\mu \in E$. Then $\mu = (q',\ x^{-1},\ q)$. Let $\gamma' = \lambda\gamma''$, with $\lambda \in \Gamma$. By (i), it follows that either $\lambda = s$ or $\lambda \in E$ with $|\lambda| \neq |\mu^{-1}| = x$. Therefore we have $x^{-1} : (q',\ \lambda) \vdash (q,\ \mu\lambda)$ and so $xx^{-1} : (q,\ \gamma) \overset{*}{\vdash} (q,\ \gamma)$. Thus (ii) holds.

(iii) $L(A')$ is ρ-closed.

Let $a,\ b,\ u,\ v \in (X \cup X^{-1})^*$. Suppose that $aub \in L(A')$. Then we have $a : (i, s) \overset{*}{\vdash} (q, \gamma)$, $u : (q, \gamma) \overset{*}{\vdash} (q', \gamma')$ and $b : (q', \gamma') \overset{*}{\vdash} (t, \gamma'')$ for some q, $q' \in Q$; $t \in T$; $\gamma, \gamma', \gamma'' \in \Gamma^+$. It follows from (ii) that $uu^{-1} : (q, \gamma) \overset{*}{\vdash} (q, \gamma)$ and so $uu^{-1}u : (q,\ \gamma) \overset{*}{\vdash} (q',\ \gamma')$. Therefore $auu^{-1}ub \in L(A')$. Suppose now that $auu^{-1}ub \in L(A')$. Since A' is deterministic, it follows from (ii) that $aub \in L(A')$. Similarly, we show that $auu^{-1}vv^{-1}b \in L(A')$ if and only if $avv^{-1}uu^{-1}b \in L(A')$ and so $L(A')$ is ρ-closed.

(iv) If $u,\ v \in L(A')$, then $uu^{-1}v \in L(A')$.

By (ii), we have $uu^{-1} : (i,\ s) \overset{*}{\vdash} (i,\ s)$ and so $uu^{-1}v \in L(A')$.

(v) If $v\rho \geq u\rho$ and $u \in L(A')$, then $v \in L(A')$.

If $v\rho \geq u\rho$, then $u\rho = (uu^{-1}v)\,\rho$. Since $u \in L(A')$, then $uu^{-1}v \in L(A')$ by (iii).

Similarly to previous cases, it follows from (ii) that $v \in L(A')$.

(vi) If u is reduced, then $u \in [L(A')]^{pr}$ if and only if $u \in L_i(A)$.

Let $u = x_1 \dots x_n$, with $x_1, \dots, x_n \in X \cup X^{-1}$. We have $u \in L_i(A)$ if and only if there is a path in A of the form

$$i = q_0 \xrightarrow{x_1} q_1 \xrightarrow{x_2} q_2 \dots \xrightarrow{x_n} q_n.$$

Since u is a reduced word, it follows from the definitions that this is equivalent to $u : (i, s) \overset{*}{\vdash} (q_n, (q_{n-1}, x_n, q_n) \dots (q_0, x_1, q_1)\, s)$, which is equivalent to have $u : (i, s) \overset{*}{\vdash} (q, \gamma)$ for some q and γ. Finally, this is equivalent to $u \in [L(A')]^{pr}$, because if $u : (i, s) \overset{*}{\vdash} (q, \gamma)$, then $uu^{-1} v \in L(A')$ for any $v \in L(A')$.

(vii) If u is reduced, then $u \in L(A')$ if and only if $u \in L(A)$.

Similar to the previous proof, considering $q_n \in T$.

Now we know from (iii), (iv) and (v) that $L(A')$ is an i-language. It follows that $[L(A')]^{pr}\iota = [L(A')]^{pr} \cap R_X$ and $[L(A')]\,\iota = L(A') \cap R_X$. On the other hand, since $L(A')$ is an i-language, $L(A') = (\overline{P\theta})\,\theta^{-1}$ is equivalent to $[L(A')]\,\theta = \overline{P\theta}$. By Lemma 5.2, we must prove that $[L(A')]^{pr} \cap R_X = P^{pr}\iota$ and $L(A') \cap R_X = P\iota$. Since $P^{pr}\iota = L_i(A)$, it follows from (vi) that $[L(A')]^{pr} \cap R_X = P^{pr}\iota$, and since $P\iota = L(A)$, it follows from (vii) that $L(A') \cap R_X = P\iota$. Thus $[L(A')]\,\theta = \overline{P\theta}$, $L(A') = (\overline{P\theta})\,\theta^{-1}$ and $(\overline{P\theta})$ is deterministic context-free.

Naturally, one can raise the question of determining exactly which (deterministic context-free) i-languages can be obtained as i-closures of recognizable $(X \cup X^{-1})^*$-languages. It is useful to consider the following lemma, though quite obvious.

LEMMA 5.5: *Let $a, b, c \in R_X$, with b prefix of c. Then $(ab)\,\iota$ is a prefix either of a or of $(ac)\,\iota$.*

Proof: If $a = a' b^{-1}$ for some $a' \in R_X$, then $(ab)\iota = a'$ is a prefix of a. Otherwise, we have $(ac)\,\iota = (ab)\iota \cdot c'$. where $c' \in R_X$ is such that $c = bc'$.

THEOREM 5.6: *Let $P \subseteq (X \cup X^{-1})^*$ be an i-language. Then $P = (\overline{N\theta})\,\theta^{-1}$ for some $N \in Rec(X \cup X^{-1})^*$ if an only if $P^{pr}\,\iota$, $P\iota \in Rec(X \cup X^{-1})^*$ and $P^{pr}\,\iota \subseteq (PF)\,\iota$ for some finite subset F of R_X.*

Proof: Suppose that $P = (\overline{N\theta})\,\theta^{-1}$ with $N \in Rec(X \cup X^{-1})^*$. Since $P = (\overline{N\theta})\,\theta^{-1}$, we have $P^{pr}\,\iota = N^{pr}\iota$ and $P\iota = N\iota$ by Lemma 5.2. By Lemma 5.3, we have $N^{pr}\iota$, $N\iota \in Rec(X \cup X^{-1})^*$. Therefore $P^{pr}\iota$, $P\iota \in Rec(X \cup X^{-1})^*$. Let $A = (Q, i, T, E)$ denote the minimal automaton of N. Fix $t \in T$. Since A is trim, we can fix a path α_q from q to t for every $q \in Q$, and we denote by w_q the label of α_q. Let $F = \{w_q^{-1} : q \in Q\}$.

Let $u \in P^{pr}\iota$. Then $u \in N^{pr}\iota$ and so $u = v\iota$ for some $v \in N^{pr}$. It follows that v labels a path in A from i to some $q \in Q$. But then vw_q labels a path in A from i to t and so $vw_q \in L(A) = N$. It follows that $(vw_q)\iota \in N\iota = P\iota$ and so $(vw_q)\iota = z\iota$ for some $z \in P$. Now $u = v\iota = (vw_q w_q^{-1})\iota = ((vw_q)\iota w_q^{-1})\iota = (z\iota w_q^{-1})\iota = (zw_q^{-1})\iota \in (PF)\iota$ and so $P^{pr}\iota \subseteq (PF)\iota$.

Conversely, suppose that $P^{pr}\iota$, $P\iota \in Rec(X \cup X^{-1})^*$ and $P^{pr}\iota \subseteq (PF)\iota$ with $F \subseteq R_X$ finite, say $F = \{u_1, \ldots, u_n\}$. We define

$$N = P\iota \cup \left(\bigcup_{i=1}^{n} [P\iota \cap (P^{pr} u_i^{-1})\iota]\, u_i\, u_i^{-1} \right).$$

Since P is an i-language, we have $P = (\overline{N\theta})\,\theta^{-1}$ if and only if $\overline{P\theta} = \overline{N\theta}$. By Lemma 5.2, we must have $P\iota = N\iota$ and $P^{pr}\iota = N^{pr}\iota$. Since $P\iota = N\iota$ is obviously true, we only need to prove the last equality.

Let $v \in P^{pr}\iota$, say $v = v'\iota$, with $v' \in P^{pr}$. Since $P^{pr}\iota \subseteq (PF)\iota = (P\iota \cdot F)\iota$, we have $v = (pu_i)\iota$ for some $p \in P\iota$ and $i \in \{1, \ldots, n\}$. Hence $p = (vu_i^{-1})\iota = (v'u_i^{-1})\iota$ and so $p \in P\iota \cap (P^{pr} u_i^{-1})\iota$. It follows that $pu_i \in N^{pr}$ and so $v = (pu_i)\iota \in N^{pr}\iota$. Thus $P^{pr}\iota = N^{pr}\iota$.

Conversely, let $v \in N^{pr}\iota$. Then $v = w\iota$ for some $w \in N^{pr}$.

Suppose first that $w \in (P\iota)^{pr}$. Then $w \in R_X$ and so $v = w$. Hence $v \in (P\iota)^{pr} \subseteq P^{pr}\iota$.

Now suppose that $w \notin (P\iota)^{pr}$. Then $w = pa$ for some $p \in P\iota \cap (P^{pr} u_i^{-1})\iota$, $i \in \{1, \ldots, n\}$, and some prefix a of $u_i u_i^{-1}$. Moreover, $p = (qu_i^{-1})\iota$ for some $q \in P^{pr}$ and so $v = w\iota = (pa)\iota = (qu_i^{-1}a)\iota = [(q\iota)(u_i^{-1}a)\iota]\iota$. Since a is a prefix of $u_i u_i^{-1}$ and $u_i \in R_X$, then $(u_i^{-1}a)\iota$ is a prefix of u_i^{-1}. By Lemma 5.5, it follows that v is a prefix of either $q\iota$ or $(q\iota.u_i^{-1})\iota = p$. In any case, we obtain $v \in (P^{pr}\iota)^{pr} \subseteq P^{pr}\iota$ and so $N^{pr}\iota \subseteq P^{pr}\iota$. Thus $P^{pr}\iota = N^{pr}\iota$ and so $P = (\overline{N\theta})\,\theta^{-1}$, proving the result.

For $P \subseteq R_X$, let $Stab(P) = \{u \in R_X : (uP)\iota = P\}$. It is immediate that $Stab(P)$ is a subgroup of R_X and $Stab(P) \subseteq P$.

THEOREM 5.7: *Let $P \subseteq (X \cup X^{-1})^*$ be an i-language. Then $P \in Rec(X \cup X^{-1})^*$ if and only if $P^{pr}\iota = (HF)\iota$ for some subgroup H of $Stab(P\iota)$ and some finite subset F of R_X.*

Proof: By Theorem 4.4, the language P is recognized by the inverse $(X \cup X^{-1})^*$-automaton $A = (MT(P), P\iota)$. If we write $A = (Q, i, T, E)$, we know that $P \in Rec(X \cup X^{-1})^*$ if and only if the set $\{q^{-1}T;\ q \in Q\}$ is finite [2]. We can assume that $Q = P^{pr}\iota$. If follows from the definition of A that $q^{-1}T = L((MT(q^{-1}P), (q^{-1}P)\iota)) = (\overline{(q^{-1}P)\theta})\theta^{-1}$ for every $q \in P^{pr}\iota$. Therefore, for all $a,\ b \in P^{pr}\iota$, we have $a^{-1}T = b^{-1}T$ if and only if $\overline{(a^{-1}P)\theta} = \overline{(b^{-1}P)\theta}$ if and only if $(a^{-1}P)^{pr}\iota = (b^{-1}P)^{pr}\iota$ and $(a^{-1}P)\iota = (b^{-1}P)\iota$. Since $(q^{-1}P)^{pr}\iota = (q^{-1}P^{pr})\iota$ for every $q \in P^{pr}\iota$, it follows that $P \in Rec(X \cup X^{-1})^*$ if and only if the sets $\{(a^{-1}P^{pr})\iota\ :\ a \in P^{pr}\iota\}$ and $\{(a^{-1}P)\iota\ :\ a \in P^{pr}\iota\}$ are both finite. We show that this holds if and only if $P^{pr}\iota = (HF)\iota$ for some subgroup H of $Stab(P\iota)$ and some finite subset F of R_X.

Suppose that $P^{pr}\iota = (HF)\iota$ for some subgroup H of $Stab(P\iota)$ and some finite subset F of R_X, say $F = \{f_1, \ldots, f_n\}$. Every $a \in P^{pr}\iota$ is of the form $a = (hf_i)\iota$ for some $h \in H$ and $i \in \{1, \ldots, n\}$, and we have $(a^{-1}P)\iota = (f_i^{-1}h^{-1}P)\iota = (f_i^{-1}P)\iota$ as well as $(a^{-1}P^{pr})\iota = (f_i^{-1}h^{-1}P^{pr})\iota = (f_i^{-1}h^{-1}HF)\iota = (f_i^{-1}HF)\iota = (f_i^{-1}P^{pr})\iota$. It follows that the sets $\{(a^{-1}P^{pr})\iota\ :\ a \in P^{pr}\iota\}$ and $\{(a^{-1}P)\iota\ :\ a \in P^{pr}\iota\}$ have at most n elements each and are therefore finite.

Conversely, suppose that $\{(a^{-1}P^{pr})\iota\ :\ a \in P^{pr}\iota\}$ and $\{(a^{-1}P)\iota\ :\ a \in P^{pr}\iota\}$ are both finite. Suppose that $\{(a^{-1}P^{pr})\iota\ :\ a \in P^{pr}\iota\} = \{(a^{-1}P^{pr})\iota, \ldots, (a_m^{-1}P^{pr})\iota\}$, with $m \geq 1$. Let $G = Stab(P^{pr}\iota) \subseteq P^{pr}\iota$ and let $F_0 = \{a_1, \ldots, a_m\}$. We show that $P^{pr}\iota = (GF_0)\iota$.

Let $a \in P^{pr}\iota$. Then $(a^{-1}P^{pr})\iota = (a_i^{-1}P^{pr})\iota$ for some $i \in \{1, \ldots, m\}$. It follows that $(aa_i^{-1}P^{pr})\iota = P^{pr}\iota$ and so $(aa_i^{-1})\iota \in G$. Hence $a \in (Ga_i)\iota \subseteq (GF_0)\iota$.

Conversely, let $g \in G$ and let $i \in \{1, \ldots, m\}$. We have $(g^{-1}P^{pr})\iota = P^{pr}\iota$ and so $(a_i^{-1}g^{-1}P^{pr})\iota = (a_i^{-1}P^{pr})\iota$. Since $1 \in (a_i^{-1}P^{pr})\iota$, then $1 \in (a_i^{-1}g^{-1}P^{pr})\iota$ and so $(ga_i)\iota \in P^{pr}\iota$. If follows that $(GF_0)\iota \subseteq P^{pr}\iota$. Thus $(GF_0)\iota = P^{pr}\iota$.

Now let $H = G \cap Stab(P\iota)$. Then H is a subgroup of $Stab(P\iota) \subseteq P\iota$. Since $\{(a^{-1}P)\iota\ :\ a \in P^{pr}\iota\}$ is finite, we have $\{(g^{-1}P)\iota\ :\ g \in G\} = \{(b_1^{-1}P)\iota, \ldots, (b_k^{-1}P)\iota\}$ for some $k \geq 1$ and $b_1, \ldots, b_k \in G$. Let $F_1 = \{b_1, \ldots, b_k\}$. We show that $G = (HF_1)\iota$. Since $H,\ F_1 \subseteq G$, it is immediate that $(HF_1)\iota \subseteq G$. Conversely, for every $g \in G$, we have $(g^{-1}P)\iota = (b_j^{-1}P)\iota$ for some $j \in \{1, \ldots, k\}$. It follows that $(gb_j^{-1}P)\iota = P\iota$ and so $(gb_j^{-1})\iota \in Stab(P\iota)$. Since $g,\ b_j \in G$, we have

$(gb_j^{-1})\iota \in H$ and so $g \in (Hb_j)\iota \subseteq (HF_1)\iota$. Thus $G \subseteq (HF_1)\iota$ and so $G = (HF_1)\iota$.

Let $F = (F_1 F_0)\iota$. It follows that $P^{pr}\iota = (GF_0)\iota = (HF_1F_0)\iota = (HF)\iota$. Since F is a finite subset of R_X, the theorem follows.

6. ALGEBRAIC OPERATORS

Unfortunately, $RecFIM(X)$ is not closed for most algebraic operators and so this section consists mainly of counterexamples. By Example 5.1, we know that $RecFIM(X)$ is not closed for the star operator, since $x\rho \in RecFIM(X)$ and $(x\rho)^* \notin RecFIM(X)$. Next, we show that $RecFIM(X)$ is not closed for product either.

EXAMPLE 6.1: Let $X = \{x, y\}$. We saw in Example 4.6 that $\langle x\rho \rangle$, $\langle y\rho \rangle \in RecFIM(X)$, in fact they even belong to $iRecFIM(X)$. Let $L = \langle x\rho \rangle \langle y\rho \rangle$. We prove that $L \notin RecFIM(X)$ by showing that $\sim_L$ does not have finite index.

Let $m,\ k \geq 1$, with $m \neq k$. Then $((yy^{-1}x^m)\rho)(x^{-m}y)\rho = (x^m x^{-m} y)\rho \in L$. It is easy to check that, for every $u \in L$, the edges labelled by y in $MT(u)$ must form a connected subtree. Hence $((yy^{-1}x^k)\rho)(x^{-m}y)\rho \notin L$, and so $((yy^{-1}x^m)\rho) \sim_L \neq ((yy^{-1}x^k)\rho) \sim_L$. It follows that $FIM(X)/\sim_L$ is infinite and so $L \notin RecFIM(X)$. Thus $RecFIM(X)$ is not closed for product.

Next we show that $RecFIM(X)$ is not closed for taking inverse submonoids.

EXAMPLE 6.2: Let $X = \{x, y\}$. Since the set $\{x\rho,\ (xy)\rho,\ (xyy^{-1})\rho\}$ is finite, we have $\{x\rho,\ (xy)\rho,\ (xyy^{-1})\rho\} \in RecFIM(X)$ by Theorem 4.5, and it is not difficult to see that $\{x\rho, (xy)\rho, (xyy^{-1})\rho\}$ is also an i-language. Let $L = \langle x\rho,\ (xy)\rho,\ (xyy^{-1})\rho \rangle$. We prove that $L \notin RecFIM(X)$ by showing that $\sim_L$ does not have finite index.

For all $m,\ k \geq 1$, with $m \neq k$, it follows easily that

$$((yx^m)\rho)(x^{-m}y^{-1}x^{-1})\rho = (x^{-1}xyx^m x^{-m} y^{-1} x^{-1})\rho \in L.$$

It is easy to check that, if $u \in L$ and an edge labelled by y occurs in $MT(u)$, then it must occur in a subtree of the form

$$\xrightarrow{x}\xrightarrow{y}.$$

Hence $((yx^k)\,\rho)(x^{-m}\,y^{-1}\,x^{-1})\rho \notin L$ and so $((yx^m)\,\rho) \sim_L \neq ((yx^k)\,\rho) \sim_L$. It follows that $FIM\,(X)/ \sim_L$ is infinite and so $L \notin RecFIM\,(X)$. Thus $RecFIM\,(X)$ is not closed for taking inverse submonoids.

This example can be used to produce another counterexample concerning homomorphic images.

EXAMPLE 6.3: Let $X = \{x,\, y\}$. Let $\phi\,:\, FIM\,(X) \to FIM\,(X)$ be the homomorphism defined by $(x\rho)\phi = x\rho$ and $(y\rho)\phi = (xy)\rho$. Let $L = FIM\,(X) \in RecFIM\,(X)$. We have $L\phi = \langle\, x\rho,\, (xy)\,\rho\,\rangle = \langle\, x\rho,\, (xy)\,\rho,\, (xyy^{-1})\,\rho\,\rangle$ and so $L\phi \notin RecFIM\,(X)$ by Example 6.2.

Now we present some nontrivial algebraic closure properties from $RecFIM\,(X)$.

THEOREM 6.4: *Let $L \in RecFIM\,(X)$. Then $L^\omega \in RecFIM\,(X)$ and is effectively constructible.*

Proof: Let $P = L\theta^{-1}$ and let $A = (Q,\, i,\, T,\, E)$ be the minimal automaton of P. Considering

$$E' = E \cup \{(q,\, 1,\, q')\,:\, q,\, q' \in Q;\; L_{q,q'} \cap D_X \neq \varnothing\},$$

we can define an $(X \cup X^{-1})^*$-automaton $A' = (Q,\, i,\, T,\, E')$.

Of course, A' is a finite constructible $(X \cup X^{-1})^*$-automaton. We show that $L\,(A') = L^\omega\theta^{-1}$.

Let $u \in L\,(A')$. Then there exist $u_1,\, \ldots,\, u_n \in (X \cup X^{-1})^*$ and $e_0,\, \ldots,\, e_n \in D_X$ such that $u_1 \ldots u_n = u$ and $e_0\, u_1\, e_1 \ldots u_n\, e_n \in P$. It follows that $u\,\theta = (u_1 \ldots u_n)\,\theta \geq (e_0\, u_1\, e_1 \ldots u_n e_n)\,\theta \in P\theta = L$. Hence $u\theta \in L^\omega$. If follows that $u \in L^\omega\theta^{-1}$ and so $L\,(A') \subseteq L^\omega\,\theta^{-1}$.

Conversely, suppose that $u \in L^\omega\theta^{-1}$. Then $u\theta \geq v\theta$ for some $v\theta \in L$ and so there exist $u_1,\, \ldots,\, u_n \in (X \cup X^{-1})^*$ and $e_0,\, \ldots,\, e_n \in D_X$ such that $u_1 \ldots u_n = u$ and $(e_0 u_1 e_1 \ldots u_n e_n)\,\theta = v\theta$. Therefore $e_0 u_1 e_1 \ldots u_n e_n \in v\theta\theta^{-1} \subseteq L\theta^{-1} = P$ and so $u = u_1 \ldots u_n \in L\,(A')$. Hence $L^\omega\,\theta^{-1} \subseteq L\,(A')$ and so $L^\omega\theta^{-1} = L\,(A')$. Thus $L^\omega \in RecFIM\,(X)$.

Given $L \subseteq FIM\,(X)$, it follows easily that there exists a smallest closed inverse submonoid of $FIM\,(X)$ containing L, precisely $\langle\, L\,\rangle^\omega$. Closed inverse submonoids of $FIM\,(X)$ are becoming important for both combinatorial inverse semigroup theory and combinatorial group theory, and the reader is referred to [7] for detailed information.

Given an $(X \cup X^{-1})^*$-automaton $A = (Q, i, T, E)$, $x \in X \cup X^{-1}$ and edges (p, x, q), (p, x, q'), with $q \neq q'$, we can form a new automaton from A by identifying the vertices q and q'. We say that this automaton is obtained by *folding* our two original edges.

The next result can be derived from Lemma 3.6 and Theorem 3.7 in [7], but we give a direct proof for completeness.

THEOREM 6.5: *Let $P \in Rec(X \cup X^{-1})^*$. Then $\langle P\theta \rangle^{\omega} \in iRecFIM(X)$ and is effectively constructible.*

Proof: Let $L = \langle P\theta \rangle^{\omega}$. Let $A = (Q, i, T, E)$ denote the minimal automaton of P, with $T = \{t_1, \ldots, t_m\}$. Let A_0 denote the $(X \cup X^{-1})^*$-automaton obtained from A by identifying the vertices $i, t_1, \ldots, t_m$, and let A_1 denote the $(X \cup X^{-1})^*$-automaton obtained from A_0 by adding to every edge (p, x, q), with $x \in X \cup X^{-1}$, a dual edge (q, x^{-1}, p) (if necessary). Let A_2 denote the $(X \cup X^{-1})^*$-automaton obtained from A_1 by successively folding edges until no more folding can be carried out. In these constructions the new automaton has always fewer vertices than the original and it will follow from this proof that his operation is confluent. It is immediate that A_2 is a finite inverse automaton. We shall prove that $L\theta^{-1} = L(A_2)$.

By construction, A_2 has duality of edges and successive folding forces A_2 to be deterministic as well, therefore A_2 is an inverse $(X \cup X^{-1})^*$-automaton. Since $L(A) \subseteq L(A_0) \subseteq L(A_1) \subseteq L(A_2)$, it is obvious that $P \subseteq L(A_2)$. Since A_2 has a single terminal vertex which is also the single initial vertex, duality of edges yields $P^{-1} \subseteq L(A_2)$, followed by $(P \cup P^{-1})^* \subseteq L(A_2)$. This yields $\langle P\theta \rangle = (P\theta \cup (P\theta)^{-1})^* \subseteq [L(A_2)]\theta$. Since A_2 is inverse, $[L(A_2)]\theta$ is closed and so $L = \langle P\theta \rangle^{\omega} \subseteq [L(A_2)]\theta$. Further, $L(A_2)$ is ρ-closed and so $L\theta^{-1} \subseteq [L(A_2)]\theta\theta^{-1} = L(A_2)$.

To prove the converse inclusion, we shall proceed by steps, considering successively $L(A_0)$, $L(A_1)$ and $L(A_2)$.

Of course, since $P\theta \subseteq L$, we have $L(A) = P \subseteq L\theta^{-1}$. Let $u \in L(A_0)$. Then we can write $u = u_1 \ldots u_n$ where, for every $j \in \{1, \ldots, n\}$, we have $u_j \in L_{p,q}$ in A for some $p, q \in \{i, t_1, \ldots, t_m\}$.

If $u_j \in L_{i,t_k}$ for some $k \in \{1, \ldots, m\}$, then $u_j \in P$ and so $u_j\theta \in L$.

If $u_j \in L_{i,i}$, then for any $v \in P$ we have $u_j v \in P$ and so $(u_j v v^{-1})\theta \in L$. Since L is closed, it follows that $u_j\theta \in L$.

If $u_j \in L_{t_k,i}$ for some $k \in \{1, \ldots, m\}$, then for any $v \in L_{i,t_k}$ we have $v u_j v \in P$ and so $(v^{-1} v u_j v v^{-1})\theta \in L$. Since L is closed, it follows that $u_j\theta \in L$.

If $u_j \in L_{t_k, t_l}$ for some $k, l \in \{1, \ldots, m\}$, then for any $v \in L_{i, t_k}$ we have $vu_j \in P$ and so $(v^{-1} vu_j)\theta \in L$. Since L is closed, it follows that $u_j\theta \in L$.

Therefore $u_1 \theta, \ldots, u_n \theta \in L$ and so $(u_1 \ldots u_n) \theta \in L$. Thus $u \in L\theta^{-1}$ and so $L(A_0) \subseteq L\theta^{-1}$.

Note that, since A is trim, A_0 is also trim. Now we will show that given a trim $(X \cup X^{-1})^*$-automaton $B' = (Q', i', i', E')$ such that $L(B') \subseteq L\theta^{-1}$, then any automaton B'' obtained from B' by adding a dual edge also satisfies $L(B'') \subseteq L\theta^{-1}$. Since the new automaton B'' is certainly trim, successive application of this fact yields $L(A_1) \subseteq L\theta^{-1}$. Suppose that (q', x^{-1}, p'), with $x \in X \cup X^{-1}$, is the edge added to B' to form B''. We show that $L(B'') \subseteq L\theta^{-1}$ by induction on the number of occurrences of the new edge in a successful path of B''. If u is the label of a successful path α where (q', x^{-1}, p') does not occur, then α is also a successful path in B' and so $u \in L\theta^{-1}$. Now suppose that the labels of successful paths in B'' with no more than k occurrences of (q', x^{-1}, p') all belong to $L\theta^{-1}$. Let α denote a path in B'' with $k+1$ occurrences of (q', x^{-1}, p'), and let u denote the label of α. Then α must be a path of the form

$$i' \xrightarrow{u_1} q' \xrightarrow{x^{-1}} p' \xrightarrow{u_2} i'$$

for some $u_1, u_2 \in (X \cup X^{-1})^*$. Since B' is trim, there exists in B' a path of the form

$$q' \xrightarrow{v} p'.$$

Since $(p', x, q') \in E'$, it follows that $u_1 v u_2$ and $u_1 vxv u_2$ both label successful paths in B'' with no more than k occurrences of (q', x^{-1}, p'). By the induction hypothesis, we now have $u_1 v u_2$, $u_1 vxv u_2 \in L\theta^{-1}$. Hence $[(u_2^{-1} v^{-1} u_1^{-1})(u_1 vxv u_2)(u_2^{-1} v^{-1} u_1^{-1})]\theta \in L$ and so $[(u_2^{-1} v^{-1} u_1^{-1} u_1 v u_2)(u_2^{-1} x u_1^{-1})(u_1 v u_2 u_2^{-1} v^{-1} u_1^{-1})]\theta \in L$. Since L is closed, we must have $(u_2^{-1} x u_1^{-1})\theta \in L$. It follows that $u = u_1 x^{-1} u_2 \in L\theta^{-1}$ and so, by induction, we obtain $L(B'') \subseteq L\theta^{-1}$. Thus $L(A_1) \subseteq L\theta^{-1}$.

Now suppose that $B' = (Q', i', i', E')$ is an $(X \cup X^{-1})^*$-automaton with duality of edges such that $L(B') \subseteq L\theta^{-1}$. We show that any automaton B'' obtained from B' by folding two edges also satisfies $L(B'') \subseteq L\theta^{-1}$. Since the new automaton B'' also has duality of edges, successive application of this fact yields $L(A_2) \subseteq L\theta^{-1}$.

Suppose then that $(p', x, q'), (p', x, r') \in E'$, with $x \in X \cup X^{-1}$ and $q' \neq r'$, and let B'' be obtained from B' by identifying q' and r'. Let

$u \in L(B'')$. Since B' has duality of edges, $x^{-1}x$ labels paths from q' to r' and vice-versa. It follows that there exist $u_0, \ldots, u_n \in (X \cup X^{-1})^*$ such that $u_0 \ldots u_n = u$ and $u_0 x^{-1} x u_1 \ldots x^{-1} x u_n \in L(B')$. Now $u\theta = (u_0 \cdots u_n)\theta \geq (u_0 x^{-1} x u_1 \ldots x^{-1} x u_n)\theta \in L$. Since L is closed, it follows that $u\theta \in L$ and $u \in L\theta^{-1}$. Hence $L(B'') \subseteq L\theta^{-1}$. Successive application of this fact yields $L(A_2) \subseteq L\theta^{-1}$ and so $L(A_2) = L\theta^{-1}$. Thus $\langle P\theta \rangle^{\omega} = L \in i\,RecFIM(X)$ and the theorem is proved.

COROLLARY 6.6: *Let $L \in RecFIM(X)$. Then $\langle L \rangle^{\omega} \in iRecFIM(X)$.*

Note that, even though $RecFIM(X)$ is closed for closed inverse submonoids, it is not so for the closed product $(A, B) \mapsto (AB)^{\omega}$, Example 6.1 being an adequate counterexample.

7. APPLICATIONS TO THE THEORY OF CODES

Some classes of submonoids of X^* are closely related to free inverse monoids, and some of these classes are becoming a popular topic in language theory, particularly in the theory of codes. We intend to give evidence of this and show how $FIM(X)$-languages can play a role on this area.

A languages $P \subseteq X^*$ is said to be a *zigzag* language if

$$uv,\ v,\ vw \in P \quad \Rightarrow \quad uvw \in P$$

for all $u, v, w \in X^*$. The next result can be derived from [5], but we give a direct proof.

THEOREM 7.1: *Let P be a submonoid of X^*. Then the following conditions are equivalent:*

(i) *P is a zigzag language;*

(ii) $P = \langle P\theta \rangle \theta^{-1} \cap X^*$;

(iii) $P = L\theta^{-1} \cap X^*$ *for some inverse submonoid L of $FIM(X)$.*

Proof: (i)⇒(ii). Suppose that P is a zigzag language. Then $P \subseteq \langle P\theta \rangle \theta^{-1} \cap X^*$ is obvious and so we only need to prove the reverse inclusion.

Suppose that $u \in \langle P\theta \rangle \theta^{-1} \cap X^*$. Then $u\theta \in \langle P\theta \rangle$ and so we must have $u\theta = (p_0 q_1^{-1} p_1 \ldots q_n^{-1} p_n)\theta$ for some $p_i, q_j \in P$, $n \geq 0$. We can assume that such n is minimal. Suppose that $n \geq 1$. Let $r \in \{p_0, \ldots, p_n\} \cup \{q_1, \ldots, q_n\}$ have minimal length in this set. Since $u \in X^*$, we must have $|p_0| \geq |q_1|$,

$|p_n| \geq |q_n|$, and so we can assume that one of the following conditions holds:

(1) $r = p_i$ for some $i \in \{1, \ldots, n-1\}$;

(2) $r = q_j$ for some $j \in \{1, \ldots, n\}$.

Suppose that (1) is verified. By minimality of $|p_i|$, we have $|p_i| \leq |q_i|$ and $|p_i| \leq |q_{i+1}|$. Since $(p_0 q_1^{-1} p_1 \ldots q_n^{-1} p_n)\theta = u\theta$ and $u \in X^*$, it follows that $MT(q_i^{-1} p_i q_{i+1}^{-1})$ is linear and so $q_i = p_i a$ and $q_{i+1} = bp_i$ for some $a, b \in R_X$. Thus $bp_i, p_i, p_i a \in P$. Since P is a zigzag language, it follows that $bp_i a \in P$. But $bp_i a = (q_{i+1} p_i^{-1} q_i)\iota$, so

$$(bp_i a)\theta \geq (q_{i+1} p_i^{-1} q_i)\theta \qquad \text{and} \qquad (a^{-1} p_i^{-1} b^{-1})\theta \geq (q_i^{-1} p_i q_{i+1}^{-1})\theta.$$

Therefore we have

$$\begin{aligned} &(p_0 q_1^{-1} \ldots p_{i-1} (a^{-1} p_i^{-1} b^{-1}) p_{i+1} \cdots p_n)\theta \\ &\quad \geq (p_0 q_1^{-1} p_1 \cdots q_n^{-1} p_n)\theta = u\theta. \end{aligned}$$

Since $u \in X^*$, it follows that

$$u\theta = (p_0 q_1^{-1} \ldots p_{i-1} (a^{-1} p_i^{-1} b^{-1}) p_{i+1} \ldots p_n)\theta,$$

contradicting the minimality of n. If (2) is verified, a similar situation arises and the minimality of n is again contradicted. Therefore $n = 0$ and so $u\theta = p_0\theta$. Since $u, p_0 \in X^*$, this yields $u = p_0$ and so $u \in P$. It follows that $\langle P\theta \rangle\theta^{-1} \cap X^* \subseteq P$ and so $P = \langle P\theta \rangle\theta^{-1} \cap X^*$.

(ii)⇒(iii). Immediate.

(iii)⇒(i). Suppose that $P = L\theta^{-1} \cap X^*$ for some inverse submonoid L of $FIM(X)$. Let $u, v, w \in X^*$ be such that $uv, v, vw \in P$. Then $(uv)\theta, v\theta, (vw)\theta \in L$. Since L is an inverse submonoid, we have $v^{-1}\theta \in L$ and $(uv)\theta(v^{-1}\theta)(vw)\theta \in L$, that is, $(uvw)\theta \in L$. Hence $uvw \in L\theta^{-1} \cap X^* = P$ and so P is a zigzag language.

Now we consider a particular class of $FIM(X)$-languages which allow us to characterize all X^*-languages which have an inverse syntactic monoid.

A language $L \subseteq FIM(X)$ is said to be *positive* if

$$\forall x \in X, \quad \exists v \in X^* : (x^{-1}\rho) \sim_L (v\rho).$$

If L is positive and $u \in (X \cup X^{-1})^*$, it follows easily that $(u\rho) \sim_L (v\rho)$ for some $v \in X^*$.

THEOREM 7.2: *Let $P \subseteq X^*$. Then $X^*/\sim_P$ is inverse if and only if $P = L\theta^{-1} \cap X^*$ for some positive $L \subseteq FIM(X)$.*

Proof: Suppose that $X^*/\sim_P$ is inverse. Let $\phi : X^* \to X^*/\sim_P$ denote the projection homomorphism and let $\overline{\theta}$ be the restriction of θ to X^*. Since $X^*/\sim_P$ is inverse, ϕ induces a surjective homomorphism $\psi : FIM(X) \to X^*/\sim_P$ such that $\overline{\theta}\psi = \phi$. Let $L = P\phi\psi^{-1}$ and let $x \in X$. Since $X^*/\sim_P$ is inverse, we have $(x\phi)^{-1} = v\phi$ for some $v \in X^*$. Hence $(x^{-1}\rho)\psi = (x\rho)^{-1}\psi = [(x\rho)\psi]^{-1} = (x\phi)^{-1} = v\phi = (v\rho)\psi$.

Let $a, b \in FIM(X)$. Then we have $a(x^{-1}\rho)b \in L \Leftrightarrow a(x^{-1}\rho)b \in P\phi\psi^{-1} \Leftrightarrow (a\psi)[(x^{-1}\rho)\psi](b\psi) \in P\phi \Leftrightarrow (a\psi)[(v\rho)\psi](b\psi) \in P\phi \Leftrightarrow a(v\rho)b \in P\phi\psi^{-1} \Leftrightarrow a(v\rho)b \in L$ and so $(x^{-1}\rho) \sim_L (v\rho)$. Thus L is positive.

Finally, let $u \in L\theta^{-1} \cap X^*$. Then $u\theta \in L = P\phi\psi^{-1}$ and so $u\theta\psi \in P\phi$, that is, $u\phi \in P\phi$. Since P is $\sim_P$-closed, it follows that $u \in P$ and so $L\theta^{-1} \cap X^* \subseteq P$. Since the converse inclusion is obvious, it follows that $P = L\theta^{-1} \cap X^*$.

Conversely, suppose that $P = L\theta^{-1} \cap X^*$ for some positive $L \subseteq FIM(X)$. Let $\phi : X^* \to FIM(X)/\sim_L$ be the homomorphism defined by $u\phi = (u\rho)\sim_L$. Since L is positive, we know that for every $x \in X$ there exists $v \in X^*$ such that $(x^{-1}\rho)\sim_L = (v\rho)\sim_L$. If follows that $(x\rho)\sim_L$, $(x^{-1}\rho)\sim_L \in X^*\phi$ for every $x \in X$ and so ϕ is surjective.

Let $u, v \in X^*$ and suppose that $u\phi = v\phi$. Then $(u\rho) \sim_L (v\rho)$ and so, for all $a, b \in X^*$, we have $aub \in P \Leftrightarrow aub \in L\theta^{-1} \Leftrightarrow (a\rho)(u\rho)(b\rho) \in L \Leftrightarrow (a\rho)(v\rho)(b\rho) \in L \Leftrightarrow avb \in L\theta^{-1} \Leftrightarrow avb \in P$. Thus $Ker\,\phi \subseteq \sim_P$ and so $X^* \sim_P$ is a homomorphic image of $FIM(X)/\sim_L$, hence inverse.

If the conditions of the theorem hold, it is obvious that $L \in RecFIM(X) \Rightarrow L\theta^{-1} \in Rec(X \cup X^{-1})^* \Rightarrow L\theta^{-1} \cap X^* \in Rec(X \cup X^{-1})^* \Rightarrow P \in Rec(X \cup X^{-1})^* \Rightarrow P \in RecX^*$. Conversely, if $P \in RecX^*$, then $P\phi$ is finite and so when we choose $L = P\phi\psi^{-1}$ we obtain $L \in RecFIM(X)$. Therefore we have $P \in RecX^* \Leftrightarrow L \in RecFIM(X)$ is L is defined as in the proof of the theorem.

We note hat being a zigzag (free) submonoid is not enough to secure an inverse syntactical monoid, as the next example shows.

EXAMPLE 7.3: Let $X = \{x, y\}$ and let $P = \{x, xy\}^*$. It is easy to check that P is a zigzag submonoid of X^*. Moreover, a simple verification shows that $X^*/\sim_P$ is a regular monoid. However, idempotents in $X^*/\sim_P$ do not commute (namely, $(xy) \sim_P$ and $x \sim_P$) and so $X^*/\sim_P$ is not an inverse monoid.

A language $P \subseteq X^*$ is said to be a *cross* language if

$$ab,\ cd,\ cb \in P \quad \Rightarrow \quad ad \in P$$

holds for all $a, b, c, d \in P$. If P is also a submonoid of X^*, it is said to be a cross submonoid.

Let P be a cross submonoid of X^*. If we consider the particular cases $c = d = 1$ and $a = b = 1$, we obtain the implications

$$ab,\ b \in P \quad \Rightarrow \quad ad \in P$$

$$cd,\ c \in P \quad \Rightarrow \quad d \in P.$$

If follows that P is a free submonoid and the basis of P is a bifix code [1].

THEOREM 7.4: *Let P be a submonoid of X^*. Then the following conditions are equivalent:*

(i) *P is a cross language;*

(ii) $P = \langle P\theta \rangle^\omega \theta^{-1} \cap X^*$;

(iii) *$P = L\theta^{-1} \cap X^*$ for some closed inverse submonoid L of $FIM(X)$.*

(iv) *$P = G \cap X^*$ for some subgroup G of R_X.*

Proof: (i)⇒(ii). Suppose that P is a cross language. Let $u \in \langle P\theta \rangle^\omega \theta^{-1} \cap X^*$. Then $u\theta \geq (p_0 q_1^{-1} p_1 \dots q_n^{-1} p_n)\theta$ for some $p_i, q_j \in P$, $n \geq 0$, and we can assume that such n is minimal. Let $v = p_0 q_1^{-1} p_1 \dots q_n^{-1} p_n$.

Suppose that $n \geq 1$. Since $(p_0 q_1^{-1} p_1 \dots q_n^{-1} p_n)\iota = u\iota = u \in X^*$, one of the following must necessarily happen:

(1) $(p_{i-1} q_i^{-1} p_i)\iota \in X^*$ for some $i \in \{1, \dots, n\}$;

(2) $(q_{j-1}^{-1} p_{j-1} q_j^{-1})\iota \in (X^{-1})^*$ for some $j \in \{2, \dots, n\}$.

Suppose that (1) is verified. Then there exist $a, b, c, d \in X^*$ such that $p_{i-1} = ab$, $q_i = cb$ and $p_i = cd$. Hence $ab, cd, cb \in P$ and since P is a cross language, it follows that $ad \in P$. Since $ad = (p_{i-1} q_i^{-1} p_i)\iota$, we have $(p_0 q_1^{-1} \dots q_{i-1}^{-1} (ad) q_{i+1}^{-1} \dots p_n)\iota = v\iota = u$ and it follows easily that $u\theta \geq (p_0 q_1^{-1} \dots q_{i-1}^{-1} (ad) q_{i+1}^{-1} \dots p_n)\theta$, contradicting the minimality of n. If (2) is verified, then we have $(q_j p_{j-1}^{-1} q_{j-1})\iota \in X^*$ and we proceed similarly to the previous case, contradicting again the choice of n. Therefore $n = 0$ and so $v \in P$. Now we have $u\theta \geq v\theta$ and $u, v \in X^*$, hence $u = v$ and $u \in P$. Thus $\langle P\theta \rangle^\omega \theta^{-1} \cap X^* \subseteq P$. Since the converse inclusion is trivial, it follows that $P = \langle P\theta \rangle^\omega \theta^{-1} \cap X^*$.

(ii)⇒(iii). Immediate.

(iii)⇒(iv). Suppose that $P = L\theta^{-1} \cap X^*$, where L denotes a closed inverse submonoid of $FIM(X)$. Let $G = L\iota \subseteq R_X$. Since L is an inverse submonoid of $FIM(X)$, we have $1 \in L$ and so $1 \in L\iota$. Further, let u, $v \in G$. Then there exist u', $v' \in L$ such that $u = u'\iota$ and $v = v'\iota$. Hence $(uv)\iota = (u'v')\iota \in L\iota = G$ and $u^{-1} = (u'\iota)^{-1} = u'^{-1}\iota \in L\iota = G$. Thus G is a subgroup of R_X.

Let $u \in P$. Then $u \in X^*$ and $u\theta \in L$. If follows that $u = u\iota = (u\theta)\iota \in L\iota = G$ and so $P \subseteq G \cap X^*$.

Conversely, suppose that $u \in G \cap X^*$. Then $u = u'\iota$ for some $u' \in L$, and so $u\theta \geq u' \in L$. Since L is closed, it follows that $u\theta \in L$ and so $u \in L\theta^{-1} \cap X^* = P$. Thus $G \cap X^* \subseteq P$ and so $P = G \cap X^*$.

(iv)⇒(i). Suppose that $P = G \cap X^*$, where G denotes a subgroup of R_X. Let a, b, c, $d \in X^*$ be such that ab, cd, $cb \in P$. Since $P \subseteq G$, we have $b^{-1} c^{-1} \in G$ and it follows that $((ab)(b^{-1} c^{-1})(cd))\iota \in G$, that is, $ad \in G$. Thus $ad \in G \cap X^* = P$ and so P is a cross language.

Corollary 7.5: *Let P be a cross submonoid of X^*. Then $P \in RecX^*$ if and only if $\langle P\theta \rangle^\omega \in iRecFIM(X)$.*

Proof: If $P \in RecX^*$, then $P \in Rec(X \cup X^{-1})^*$ and so $\langle P\theta \rangle^\omega \in iRecFIM(X)$ by Theorem 6.5.

Conversely, suppose that $\langle P\theta \rangle^\omega \in iRecFIM(X)$. Then $\langle P\theta \rangle^\omega \theta^{-1} \in Rec(X \cup X^{-1})^*$. Since $X^* \in Rec(X \cup X^{-1})^*$, it follows that $\langle P\theta \rangle^\omega \theta^{-1} \cap X^* \in Rec(X \cup X^{-1})^*$ and so $P \in Rec(X \cup X^{-1})^*$ by Theorem 7.4. Since $P \subseteq X^*$, it is immediate that $P \in RecX^*$.

We denote by **ECom** the pseudovariety of all finite monoids with commuting idempotents.

Theorem 7.6: *Let $P \in RecX^*$ be a cross submonoid. Then $X^*/\sim_P \in$* **ECom**.

Proof: Let $L = \langle P\theta \rangle^\omega$ and let $M = (X \cup X^{-1})^*/\sim_{L\theta^{-1}}$. By Theorem 3.2, M is an inverse monoid. Since $P \in RecX^*$, it follows from Corollary 7.5 that $L \in iRecFIM(X)$. Therefore $L\theta^{-1} \in Rec(X \cup X^{-1})^*$ and so M is a finite inverse monoid. Let $N = \{u \sim_{L\theta^{-1}};\ u \in X^*\}$. Obviously, N is a submonoid of M. We define a mapping $\Phi : N \to X^*/\sim_P$ by $u\Phi = u \sim_P$, for $u \in X^*$.

To show that Φ is well-defined, let u, $v \in X^*$ be such that $u_{\sim_{L\theta^{-1}}} = v_{\sim_{L\theta^{-1}}}$. Let a, $b \in X^*$. Suppose that $aub \in P$. Then $aub \in L\theta^{-1}$ and so $avb \in L\theta^{-1}$. Since $avb \in X^*$, it follows from Theorem 7.2 that $avb \in P$. Similarly, we show that $avb \in P$ implies $aub \in P$, hence $u \sim_P = v \sim_P$ and Φ is well-defined.

Now it is immediate that Φ is a surjective homomorphism and so $X^*/ \sim_P$ is the homomorphic image of a submonoid of the finite inverse monoid M. Since $M \in \mathbf{ECom}$, if follows that $X^*/ \sim_P \in \mathbf{ECom}$.

The syntactic monoid of a recognizable cross submonoid of X^* does not have to be inverse, as the next example shows.

EXAMPLE 7.7: Let $X = \{x, y, z\}$ and let $P = \{x^2, y, xyz\}^*$. It is easy to check that P is a cross submonoid of X^*. However, a simple verification shows that $X^*/ \sim_P$ is not regular (namely, $(xy) \sim_P$ is not a regular element). Therefore $X^* \sim_P$ is not an inverse monoid.

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