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UET FLOW SHOP SCHEDULING WITH DELAYS (*)

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Abstract. – *$F|UET$, delays $|C_{\max}$ is introduced and shown to be NP-complete.*

Résumé. – *Le problème du flow shop avec des temps de transport est introduit. Il est montré qu'il est NP-difficile même si les temps opératoires sont unitaires.*

1. INTRODUCTION

The usual flow shop problem can be described as follows.

Given are a set of n jobs and a set of m machines. Each machine can handle at most one job at a time and each job can be processed by at most one machine at a time. Each job consists of m tasks indexed by $1, \ldots, m$ and the i-th task of a job precedes its $(i+1)$-th task for $i = 1, \ldots, m-1$. Further, the i-th task of the j-th job has to be carried out on the i-th machine, during an uninterrupted period of a given length of time, l_{ij}. The purpose is to find a schedule of all the jobs which minimises the overall completion time.

Flow shop scheduling is shown to be NP-complete in the strong sense [1], even for the case $m = 3$. However, for the special case $m = 2$, there exists a polynomial time algorithm [2].

In this paper, we introduce the concept of an interprocessor time delay. This models the situation where there is a time delay when a job is transferred from one machine to another.

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Let d_{ij}, $1 \leq i \leq m-1$, $1 \leq j \leq n$, denote the time delay encountered in transferring job j from machine i to machine $i+1$ and c_{ij} and $s_{i+1\,j}$, respectively, denote the completion time of job j on machine i and the starting time of job j on machine $i+1$. Thus, the length of the i-th task of job j which must be scheduled on machine i is given by

$$l_{ij} = c_{ij} - s_{ij} + 1.$$

If in a given schedule,

$$s_{i+1\,j} \geq c_{ij} + d_{ij}, \quad \text{for } 1 \leq i \leq m-1, 1 \leq j \leq n,$$

then the schedule is called valid.

The delays are uniform if $d_{ij} = d$ for $1 \leq i \leq m-1$ and $1 \leq j \leq n$; otherwise the delays are non-uniform and, in general, this is the case we will be considering.

By setting $d_{ij} = 0$, for all $1 \leq i \leq m-1$, $1 \leq j \leq n$, it follows immediately from the NP-completeness of flow shop that flow shop with delays is NP-complete in the strong sense for any fixed $m \geq 3$. In [5] it is shown that the problem of flow shop with delays is NP-complete in the strong sense even for $m = 2$. However, if this problem is for a permutation flow shop, *i.e.* the order of the jobs is the same on all machines, then it can be solved in polynomial time [4].

When all the processing times are unit execution times (UET), the optimal flow shop schedule might not be achieved by a permutation flow shop nor by greedily ordering the jobs by nonincreasing delays even for $m = 2$. For example, consider four UET jobs on two machines with delays 5, 3, 3 and 1. The optimal schedule is of length 8 but the optimal permutation schedule is of length 10 and the greedy schedule is of length 9.

In section 2 of this paper, we prove that the problem of scheduling UET jobs with arbitrary delays in a (non-permutation) flow shop becomes NP-complete if we allow an arbitrary number of processors. The complexity of UET flow shop scheduling with delays for fixed $m \geq 2$ is open.

We summise that the two machine case is in *P* but we have been unable to prove this. A useful observation is that, for the two machine case only, there exists a valid schedule of optimal length such that the n jobs are processed continuously on machine 1 and continuously on machine 2. Moreover, for the two machine case, it is relatively straightforward to establish the bound

$$\omega_{\text{opt}} \geq \max\left\{\left\lceil\sum_{j=1}^{k} d_j/k\right\rceil + k + 1 : 1 \leq k \leq n\right\}.$$

However, this bound is not tight [3]; consider six jobs with delays 4, 4, 4, 0, 0, 0. The optimal schedule has length 10 which is greater than the bound of 9 given by the formula.

2. THE NP-COMPLETENESS RESULT

In this section, we prove the following decision problem is NP-complete.

UET FLOW SHOP WITH DELAYS (FUD)

Instance: number $p \in Z^+$ of processors, set J of jobs, for each job $j \in J$ and $1 \leq k < p$, a delay $d(j, k) \in Z^+$, and an overall deadline $D \in Z^+$.

Question: Is there a valid flow shop schedule of the jobs in J meeting the deadline where each job $j \in J$ has an associated UET task on each processor and such that for each $j \in J$ and each $1 \leq k < p$, if j is processed at time t on processor k then it is processed at time $\geq t + d(j, k) + 1$ on processor $k + 1$?

We will show that FUD is NP-complete. Our first step is to show that the following well-known NP-complete problem can be polynomially transformed into FUD.

VERTEX COVER (VC)

Instance: Graph $G = (V, E)$ and a positive integer, $k < |V|$.

Question: Is there a vertex cover of size $\leq k$ for G, *i.e.* a subset $V' \subset V$ with $|V'| \leq k$ where each edge in E is adjacent to some element in V'?

LEMMA 1: $VC \propto FUD$.

Proof: Let an instance of VC comprise a graph, $G = (V, E)$ with $|V| = n$, $|E| = m < n^2$ and a positive integer $k < n$. Assume $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$.

We construct an instance of FUD as follows.

$$\begin{aligned}
p &= 2m+3,\\
J &= \{x, b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_{k+1}\} \cup V,\\
d(x, 1) &= k+1,\\
d(c_i, 1) &= 3k+1, \ 1 \le i \le k+1,\\
d(b_i, 1) &= 0, \ 1 \le i \le k,\\
d(v_i, 1) &= 0, \ 1 \le i \le n,
\end{aligned}$$

and then, for $1 \le r \le m$,

$$\begin{aligned}
d(x, 2r) &= 2k+1,\\
d(c_i, 2r) &= k, \ 1 \le i \le k+1,\\
d(b_i, 2r) &= k-1, \ 1 \le i \le k,\\
d(x, 2r+1) &= 0,\\
d(c_i, 2r+1) &= k+1, \ 1 \le i \le k+1,\\
d(b_i, 2\ +1) &= k+2, \ 1 \le i \le k,
\end{aligned}$$

and, for $1 \le i \le n$, we define

$$d(v_i, 2r) = \begin{cases} 0, & \text{if } v_i \text{ is adjacent to } e_r,\\ k, & \text{otherwise} \end{cases}$$

and

$$d(v_i, 2r+1) = \begin{cases} 2k, & \text{if } v_i \text{ is adjacent to } e_r,\\ k, & \text{otherwise.} \end{cases}$$

Finally,

$$\begin{aligned}
d(x, 2m+2) &= 4k+n+3,\\
d(c_i, 2m+2) &= 2k+n+3-2i, \ 1 \le i \le k+1,\\
d(b_i, 2m+2) &= 3k+n+2-2i, \ 1 \le i \le k,\\
d(v_i, 2m+2) &= 0, \ 1 \le i \le n.
\end{aligned}$$

The deadline, D, is set by

$$D = 5k + 2mk + 3m + n + 7.$$

Since k is $O(n)$, this is clearly a polynomial transformation.

Our first observation concerns x. The total delays for x are

$$(k+1)+m(2k+1)+4k+n+3.$$

The total computation time for job x, as for all jobs, is $p = 2m + 3$. The sum of these two values is exactly D. Thus, the deadline is met iff x is the first job processed on machine 1 and the last job processed on machine p. Moreover, the processing time on every intervening machine is also determined uniquely by the delays.

Now, we consider the job c_i. This has a total delay of

$$\begin{aligned}3k+1+m(2k+1)+2k+n+3-2i\\ =5k+2mk+m+n+4-2i\end{aligned}$$

and a processing time of $p = 2m + 3$. The sum of these is

$$5k+2mk+3m+n+7-2i = D-2i.$$

A simple induction argument can then be used to determine that the deadline is met iff c_i is processed at time $1 + i$ on machine 1 and at time $D - i$ on machine p. Moreover, the processing times of each c_i on every intervening machine are also determined uniquely by the delays.

Next, consider the job, b_i. The earliest it can be processed on machine 1 is $k + 3$ and the latest it can be processed on machine p is

$$D-k-2 = 4k+2mk+3m+n+5.$$

The total of the delays for b_i is

$$m(2k+1)+3k+n+2-2i.$$

Adding the processing time of $2m + 3$ gives a total of

$$3k+2mk+3m+n+5-2i.$$

Again, a simple induction argument shows that b_i must be processed at time $k + 2 + i$ on machine 1 and at time $D - k - i - 1$ on machine p with all the intervening times uniquely determined by the delays.

The necessary scheduling of the job x and those of types b and c is described in figure 1. This shows that a valid schedule of all these jobs is achievable within the deadline.

We note that on machine $2r$, $(1 \leq r \leq m+1)$, x is processed at time

$$\begin{aligned} t_{2r} &= 2r + \sum_{j=1}^{2r-1} d(x, j) \\ &= 2r + k + 1 + (r-1)(2k+1) \\ &= 3r + 2rk - k, \end{aligned}$$

b_i is processed at time

$$\begin{aligned} & k + 1 + i + 2r + \sum_{j=1}^{2r-1} d(b_i, j) \\ &= k + 1 + i + 2r + (r-1)(2k+1) \\ &= t_{2r} + i \end{aligned}$$

and c_i is processed at time

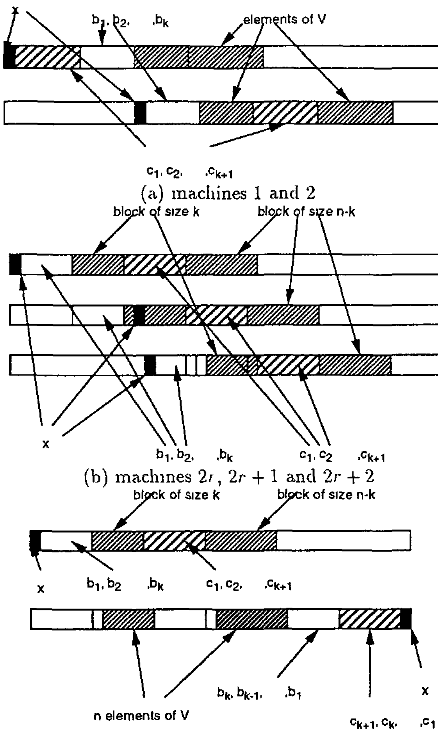
$$\begin{aligned} & i + 2r + \sum_{j=1}^{2r-1} d(c_i, j) \\ &= i + 2r + 3k + 1 + (r-1)(2k+1) \\ &= t_{2r} + 2k + i. \end{aligned}$$

On machine $2r+1$,

$$\begin{aligned} & b_i \text{ is proceesed at } t_{2r} + k + i, \\ & x \text{ is processed at } t_{2r} + 2k + 2, \quad \text{and} \\ & c_i \text{ is processed at } t_{2r} + 3k + i + 1. \end{aligned}$$

On machine $2r+1$, spare processing time is thus available in a one unit slot at time $t_{2r} + 2k + 1$ and in a $(k-1)$ continuous block, $t_{2r} + 2k + 3, \ldots, t_{2r} + 3k + 1$. On machine $2m+3$, all the available times $\geq D - 2k - 1$ are allocated jobs. The lastest a job in V could be processed on machine $2m+3$ is

$$D - 2k - 2 = 3k + 2mk + 3m + n + 5$$



(a) machines 1 and 2

(b) machines $2r$, $2r+1$ and $2r+2$

(c) machines $2m+2$ and $2m+3$

Figure 1. – The schedule structure

and hence on machine $2m+2$ is

$$\begin{aligned} &3k + 2mk + 3m + n + 4 \\ &\quad = 3(m+1) + 2(m+1)k - k + 2k + 2 + n - 1 \\ &\quad = t_{2m+2} + |J| - 1. \end{aligned}$$

Thus on machine $2m+2$, the block of size k between the b-jobs and the c-jobs must be allocated to k jobs in V. The remaining $n-k$ jobs in V must be processed immediately after c_{k+1}. This partitions the jobs in V into two sets V_1 and V_2. The k jobs in V_1 are processed on machine $2m+2$ in the early block of size k.

If an element $v \in V$ is processed at time $< t_{2r}$ on machine $2r$ for any $1 < r \leq m$ then, since the b-block on machine $2r-1$ is fixed, this v must be processed before that block on machine $2r-1$ and hence at time $< t_{2r-2}$ on machine $2r-2$. Hence, we can deduce that such a v would be processed at time $< t_2$ on machine 2 which is impossible. We thus know every $v \in V$ is processed at time $> t_{2r}$ on machine $2r$ for all $1 \leq r \leq m+1$.

Having established that the block of size k on machine $2m+2$ must be allocated to V_1, an induction argument can then be used to show that V_1 always occupies the k locations between the b-block and the c-block on machines $2m$, $2m-2, \ldots, 2$.

Now, consider machine $2r+1$ $(1 \leq r \leq m)$. The single, isolated location between the location allocated to b_k and that allocated to x can be used iff V_1 contains a vertex adjacent to e_r. A schedule is valid iff this location is used. Hence, we have a valid schedule within the deadline iff V_1 is a vertex cover. Thus, the lemma is established.

It is now easy to establish

THEOREM 1: *FUD is NP-complete.*

Proof: Having established that an NP-complete problem polynomially transforms to FUD, all we need to establish is that FUD $\in$ NP.

Given the $|J|$ jobs and p machines, we simply guess an integer, $1 \leq t_{j,r} \leq D + r - p$, for each $j \in J$, $1 \leq t \leq p$. Then, in polynomial time, we check that

1. $t_{j,r} = t_{j',r} \Rightarrow j = j'$, for all $1 \leq r \leq p$, and
2. $t_{j,r+1} \geq t_{j,r} + d(j, r)$ for each $j \in J$ and $1 \leq r < p$.

An instance of FUD is a yes-instance iff there is some guess which passes all these checks.

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