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W. Merzenich

L. Staiger

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FRACTALS, DIMENSION, AND FORMAL LANGUAGES (1)

by W. MERZENICH (2) and L. STAIGER (3)

Abstract – *We consider classes of sets of r adic expansions of reals specified by means of the theory of formal languages or automata theory It is shown how these specifications are used to calculate the Hausdorff dimension and Hausdorff measure of such sets*

Since the appearence of Mandelbrot's [Ma77] book "Fractals, Form, Chance and Dimension" Fractal Geometry as a means providing a theory describing many of the seemingly complex patterns in nature and the sciences has become popular not only in the sciences (cf. [PS88]), but also in Computer Science. Here Barnsley's [By88] "Computational Fractal Geometry" aims at a practical description of fractal patterns by so-called Iterated Function Systems (IFS). Besides IFS several other computational methods for the description (generation) of fractal images have been developed. Recently concepts involving methods of Automata or Formal Language Theory have become popular (see *e. g.* [BM89], [BN89], [CD90], [CD90/93], [CD93], [Fe93], [HKT93], [PLH88], or [Sm84]). Several of those concepts seem to originate not in Automata or Formal Language Theory, rather being developed earlier in connection with problems in Dynamical Systems Theory or Geometric Measure Theory (cf. [HPS92]). Nevertheless, providing a thorough consideration of finite computational devices, Automata or Formal Language Theory yields a new insight into problems dealing with the generation of fractal images as the above mentioned papers show.

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(2) Universitat Siegen, Fachbereich 12 (Elektrotechnik und Informatik), Postfach 101240, D-57068 Siegen, Germany Electronic mail merze@server informatik uni-siegen de

(3) Rhein -Westf Techn Hochschule Aachen, Lehrstuhl fur Informatik II, Ahornstr 55, D-52056 Aachen, Germany Electronic mail staiger@zeus informatik rwth-aachen de

What concerns the analysis of fractals it is known from geometric measure theory (cf. [Fa85]) that the estimation of the fractal dimension or measure of even rather simply definable sets is already a complicated task. It would be interesting to know whether Automata or Formal Language Theory can also contribute to the solution of problems arising there.

In this paper we show that this language or automata theoretical approach to image description does not only provide a method for their generation but also for the computation of their fractal or Hausdorff dimension and measure, that is, for their analysis, and even more generally, we can use our approach for the computation of the Hausdorff dimension and measure of certain constructively given subsets of the unit cube $[0, 1]^d$ in Euclidean space $\mathbb{R}^d$.

In order to specify subsets of the unit cube $[0, 1]^d$ constructively, we consider real numbers $b \in [0, 1]$ as infinite r-ary expansions $b = 0.\beta$ with $\beta \in Y^\omega$ where Y^ω is the set of all semi-infinite sequences of r-ary digits ($Y := \{0, 1, \ldots, r-1\}$, $r \in \mathbb{N}$, $r \geqq 2$). Then an expansion $\xi \in \underbrace{(Y \times \ldots \times Y)}_{d\,times}{}^\omega$ describes the point $\mathbf{p}(\xi) := (0.pr_1\,\xi, \ldots, 0.pr_d\,\xi) \in [0, 1]^d$.

In what follows let Y be a finite alphabet of cardinality $r := \operatorname{card} Y \geqq 2$, and let $X := \underbrace{Y \times \ldots \times Y}_{d\,times}$. We consider X^ω as a metric space with metric ρ defined by

$$\rho(\beta, \xi) = \inf\{r^{-|w|} : w \text{ is a common prefix of } \beta \text{ and } \xi\}.$$

This metric ρ satisfies the ultrametric inequality $\rho(\beta, \xi) \leqq \max\{\rho(\beta, \eta), \rho(\xi, \eta)\}$. Since X is finite, the space (X^ω, ρ) is compact.

The open (they are simultaneously closed) balls in (X^ω, ρ) are the sets of the form $w \cdot X^\omega$, where w is a finite string over X. Sets of finite strings (so-called *languages*) will play a major role in our investigations. Therefore, as usual we introduce X^* as the set of all finite strings (words) over X, including the empty word e. For $w \in X^*$ and $p \in X^* \cup X^\omega$ let $w \cdot p$ be their *concatenation.* This concatenation product extends in an obvious way to subsets $W \subseteqq X^*$ and $P \subseteqq X^* \cup X^\omega$. The *length* of a word w is denoted by $|w|$.

A word w is a *prefix* of some $p \in X^* \cup X^\omega$ (short $w \sqsubseteq p$) iff there is a $p' \in X^* \cup X^\omega$ such that $w \cdot p' = p$. Let $\mathbf{A}(P) := \{w : w \in X^* \wedge \exists p\,(p \in P \wedge w \sqsubseteq p)\}$ be the language of finite prefixes of P.

We call a language $V \subseteqq X^*$ *prefix-free* provided

$$\forall w, v (w \sqsubseteq v \wedge w, v \in V \rightarrow w = v).$$

The ball $w \cdot X^\omega$ has diameter $\operatorname{diam}(w \cdot X^\omega) = r^{-|w|}$. Thus balls $w \cdot X^\omega$ in X^ω correspond to r-ary subcubes $[i_1 \cdot r^{-|w|}, (i_1 + 1) \cdot r^{-|w|}] \times \cdots \times [i_d \cdot r^{-|w|}, (i_d + 1) \cdot r^{-|w|}]$ in $[0, 1]^d$. *Open* sets in X^ω are sets of the form $W \cdot X^\omega = \bigcup_{w \in W} w \cdot X^\omega$ where $W \subseteqq X^*$. A subset $E \subseteqq X^\omega$ is *closed* if its complement is open (or if its elements do not have any prefix in some $W' \subseteqq X^*$).

So far we have made clear which connection between subsets of the unit cube in $\mathbb{R}^d$ and languages of finite or infinite words we have in mind. The following section makes these connections more precise, and derives several fundamental relations between the so-called entropy of languages and the Hausdorff dimension in the space (X^ω, ρ).

Then, in Section 2 subsets of X^ω definable by finite linear systems of equations are introduced and a relation between the adjacency matrix of the underlying regular language and the Hausdorff dimension of the defined subset of X^ω is derived.

This approach is extended further in Section 3 and yields an effective procedure for computing the Hausdorff measure of a subset of X^ω definable by a finite linear system of equations. This procedure is based upon bounds on the Hausdorff measure derived also in this section.

The last section deals with a close relationship between the topological density of the subsets of X^ω investigated so far and their Hausdorff dimension and measure. This leads to the conclusion that for black-and-white fractals definable by finite systems of equations the blackness of a picture is directly related to its Hausdorff dimension and measure. This fact is illustrated in the appendix, where we present several high-resolution (in comparison with the number of defining equations) pictures of fractals and the dimension and measure as computed by our algorithm.

1. HAUSDORFF DIMENSION AND THE ENTROPY OF LANGUAGES

In this section we consider subsets of X^ω which can be obtained from languages via certain operations. We show that the close connection between languages and open subsets of X^ω can be utilized to estimate the Hausdorff dimension of these sets.

First we recall the definition of the Hausdorff dimension in the space (X^ω, ρ): An r^{-n}-*cover* of a set $F \subseteqq X^\omega$ is a family $(v \cdot X^\omega)_{v \in V}$ of balls such that $V \cdot X^\omega \supseteqq F$ and whose *diameters diam* $(v \cdot X^\omega)$ do not exceed r^{-n}, that is, $\inf \{|v| : v \in V\} \geqq n$.

Let $L_\alpha(F; V) := \sum_{v \in V} (\operatorname{diam} v \cdot X^\omega)^\alpha = \sum_{v \in V} r^{-\alpha \cdot |v|}$ for a cover $(v \cdot X^\omega)_{v \in V}$ of $F \subseteqq X^\omega$. Then

$$L_\alpha(F) := \lim_{n \to \infty} (\inf \{L_\alpha(F; V) : V \cdot X^\omega \supseteqq F \wedge \inf \{|v| : v \in V\} \geqq n\})$$

is the α-*dimensional outer measure* of F, and the following properties are satisfied [Fa85]:

PROPERTY 1: *1. If* $\inf \{\rho(\xi, \beta) : \xi \in F \text{ and } \beta \in E\} > 0$ *for (not necessarily measurable) subsets* E, $F \subseteqq X^\omega$ *then* $L_\alpha(F \cup E) = L_\alpha(F) + L_\alpha(E)$.

2. L_α *is a measure on the Borel subsets of* X^ω.

From Property 1 one easily infers the following useful identity.

PROPERTY 2: *Let* $V \subseteqq X^*$ *be a prefix-free language and* $(F_v)_{v \in V}$ *be a family of subsets of* X^ω. *Then*

$$L_\alpha\Big(\bigcup_{v \in V} v \cdot F_v\Big) = \sum_{v \in V} r^{-\alpha \cdot |v|} \cdot L_\alpha(F_v).$$

Proof: Since V is prefix-free, for every pair $v, w \in V$, $v \neq w$ it holds $\inf \{\rho(\xi, \beta) : \xi \in v \cdot X^\omega \wedge \beta \in w \cdot X^\omega\} \geqq \max\{r^{-|v|}, r^{-|w|}\} > 0$. Thus by Property 1 $L_\alpha\Big(\bigcup_{v \in V'} v \cdot F_v\Big) = \sum_{v \in V'} r^{-\alpha \cdot |v|} \cdot L_\alpha(F_v)$ for every finite subset V' of V. If V is infinite, taking limits on both sides yields $L_\alpha\Big(\bigcup_{v \in V} v \cdot F_v\Big) = \sum_{v \in V} L_\alpha(v \cdot F_v)$

The proof is finished by the easily verified identity $L_\alpha(v \cdot F) = r^{-\alpha \cdot |v|} \cdot L_\alpha(F)$.

Q.E.D.

Now, consider $L_\alpha(F)$ as a function of α. Then there is an $\alpha(F) \in [0, \infty]$ such that

$$L_\alpha(F) = \begin{cases} \infty, & \text{if } \alpha < \alpha(F), \\ 0, & \text{if } \alpha > \alpha(F). \end{cases}$$

This number $\alpha(F)$ is called the *Hausdorff dimension* $\dim F$ of F, that is, the Hausdorff dimension of F is given by

$$\dim F = \sup\{\alpha : L_\alpha(F) = \infty\} = \inf\{\alpha : L_\alpha(F) = 0\}.$$

Remark: It should be noted that, since our metric ρ does not coincide with the usual metric in $\mathbb{R}^d$, our measure L_α differs from the usual α-dimensional outer measure $\mathcal{H}^\alpha$ in $\mathbb{R}^d$. Since $\operatorname{diam} w \cdot X^\omega$ is the edge length of the r-ary subcube $[i_1 \cdot r^{-|w|}, (i_1+1) \cdot r^{-|w|}] \times \cdots \times [i_d \cdot r^{-|w|}, (i_d+1) \cdot r^{-|w|}]$, our measure L_α is up to certain ambiguities due to double expansions of numbers $i \cdot r^{-j}$ a scaled by $\left(1/\sqrt{d}\right)^\alpha$ version of the net measure $\mathcal{M}^\alpha$ in $[0, 1]^d$ (cf. Section 5 in [Fa85]). Thus, in particular, $\dim F = \dim\{\mathbf{p}(\xi) : \xi \in F\}$.

Note that dim is monotone and countably stable, that is,

$$F \subseteqq F' \subseteqq X^\omega \to 0 \leqq \dim F \leqq \dim F' \leqq d,$$

and

$$\dim \bigcup_{i \in \mathbb{N}} F_i = \sup_{i \in \mathbb{N}} \dim F_i$$

for any countable family of sets $F_i \subseteqq X^\omega$.

Next we shall introduce the entropy of languages (cf. [Ku70]) which has close relations to the Hausdorff dimension of subsets $F \subseteqq X^\omega$.

Let $\mathbf{s}_V(n) := \operatorname{card}\{v : \ v \in V \wedge |v| = n\}$ for $n \in \mathbb{N}$ and $V \subseteqq X^*$. The *entropy* of a language $V \subseteqq X^*$ is defined as follows.

$$H_V := \begin{cases} 0, & \text{if } V \text{ is finite,} \\ \limsup\limits_{n\to\infty} \dfrac{1}{n} \log_r \mathbf{s}_V(n), & \text{otherwise.} \end{cases}$$

For $F \subseteqq X^\omega$ let $\mathbf{s}_F := \mathbf{s}_{\mathbf{A}(F)}$ and $H_F := H_{\mathbf{A}(F)}$.

In order to relate the entropy of languages to the Hausdorff dimension of subsets $F \subseteqq X^\omega$ (so-called ω-languages) we need operations transforming languages to ω-languages (for some examples of those operations and their general properties, see *e. g.* [LS77] or [St87]) and, thereby, yielding estimates of the Hausdorff dimension of the transformed ω-language via the entropy of the original language.

The following two operations prove to be useful in this respect (cf. also [St89/93]). The first one introduced in [SW74] and [LS77], usually called *adherence* of languages (cf. [BN80]), is defined as follows

$$\text{(1)} \qquad \mathbf{ls}\, W := \{\xi : \xi \in X^\omega \wedge \mathbf{A}(\xi) \subseteqq \mathbf{A}(W)\}.$$

The second one, Davis' [Da64] δ-limit of $V \subseteq X^*$ is

$$V^\delta := \{\xi : \xi \in X^\omega \wedge \xi \text{ has infinitely many prefixes in } V\}.$$

It is well-known that a subset F of X^ω is closed iff $F = \mathbf{ls}\, W$ for some language $W \subseteq X^*$, and that $E \subseteq X^\omega$ is a countable intersection of open sets [a so-called G_δ-set in (X^ω, ρ)] if and only if there is a $V \subseteq X^*$ such that $E = V^\delta$.

In connection with these topological properties we mention still that $\mathbf{ls}\, \mathbf{A}(F) = (\mathbf{A}(F))^\delta = \{\xi : \mathbf{A}(\{\xi\}) \subseteq \mathbf{A}(F)\}$ is the *closure* of the set $F \subseteq X^\omega$, $\mathcal{C}(F)$, in the space (X^ω, ρ), that is $\mathcal{C}(F) = \mathbf{ls}\, \mathbf{A}(F) = (\mathbf{A}(F))^\delta$ is the smallest closed subset of (X^ω, ρ) containing F.

We obtain our first result.

LEMMA 3: ([St89/93]) $\dim F = \inf\{H_W : W^\delta \supseteq F\}$, *in particular* $\dim V^\delta \leq H_V$, *and* $\dim \mathbf{ls}\, V \leq H_{\mathbf{A}(V)}$.

Then $\mathbf{ls}\, \mathbf{A}(F) = (\mathbf{A}(F))^\delta = \mathcal{C}(F) \supseteq F$ implies the following.

COROLLARY 4: *If* $F \subseteq X^\omega$ *then* $\dim F \leq H_F$.

In order to formulate the next results, we define $W^* := \{w_1 \cdots w_n : n \in \mathbb{N} \wedge w_i \in W\}$ as the set of all finite products of words from W, and $W^\omega := \{w_1 \cdots w_n \cdots : w_i \in W \setminus \{e\}\}$ as the set of infinite strings formed by concatenating words in W. The operations $\mathbf{A}$, $\mathbf{ls}$ and $^\delta$ share some common properties:

Let op be one of the operations $\mathbf{A}$, $\mathbf{ls}$ or $^\delta$. Then

$$\operatorname{op}(W \cup V) = \operatorname{op}(W) \cup \operatorname{op}(V), \tag{2}$$

$$\operatorname{op}(W \cdot V) = \operatorname{op}(W) \cup W \cdot \operatorname{op}(V) \quad \text{if} \quad e \in V, \tag{3}$$

$$\operatorname{op}(W^*) = W^\omega \cup W^* \cdot \operatorname{op} W \quad \text{if } \operatorname{op} = \mathbf{ls} \quad \text{or} \quad \operatorname{op} = {}^\delta \tag{4}$$

and

$$\mathbf{A}(W^*) = \{e\} \cup W^* \cdot \mathbf{A}(W). \tag{5}$$

THEOREM 5: ([St89/93]) *For every* $W \subseteq X^*$ *it holds* $\dim W^\omega = \dim (W^*)^\delta = H_{W^*}$, *and if* $\alpha = \dim W^\omega$ *then* $L_\alpha(W^\omega) \leq L_\alpha((W^*)^\delta) \leq 1$. [1]

([1]) In fact, the last assertion of was proved in Proposition 6.6 of [St89/93] only for $L_\alpha(W^\omega)$, but a simple modification of the proof given there works also in the case of $L_\alpha((W^*)^\delta)$.

THEOREM 6: *Let* $V \subseteqq X^*$ *be a regular language. Then* $L_\alpha((V^*)^\delta) = L_\alpha(C(V^\omega))$ *for all* $\alpha \in [0, d]$. *If, moreover, V is prefix-free then* $L_\alpha(V^\omega) = L_\alpha(C(V^\omega))$ *for all* $\alpha \in [0, d]$.

Proof: Since $C(V^\omega) = \mathbf{ls}\, V^* = (\mathbf{A}(V)^*)^\delta = C((V^*)^\delta)$, it suffices to show that for regular languages V the inequality $\dim(\mathbf{ls}\, V^* \backslash (V^*)^\delta) < \dim (V^*)^\delta = H_{V^*}$ holds.

To this end we recall for V^* the well-known representation of regular languages by prefix-free regular languages:

$$V^* = \bigcup_{i=1}^{u} W_i \cdot V_i^*,$$

for suitable $n \in \mathbb{N}$, regular and prefix-free languages V_i, W_i where $W_i \neq \emptyset$. Thus using the above Equations (2), (3), (4) and (5), and taking into account that $U^\delta = \emptyset$ if U is prefix-free we obtain

$$(V^*)^\delta = \bigcup_{i=1}^{n} W_i \cdot V_i^\omega,$$

and

$$\mathbf{ls}\, V^* = \bigcup_{i=1}^{n} W_i \cdot V_i^\omega \cup \bigcup_{i=1}^{n} W_i \cdot V_i^* \cdot \mathbf{ls}\, V_i \cup \bigcup_{i=1}^{n} \mathbf{ls}\, W_i.$$

Since dim in countably stable, it suffices to show that $\dim \mathbf{ls}\, W_i$, $\dim \mathbf{ls}\, V_i < \dim (V^*)^\delta = H_{V^*}$. In virtue of Lemma 3 we have $\dim \mathbf{ls}\, U \leqq H_{\mathbf{A}(U)}$ for $U \subseteqq X^*$. Utilizing the fact that $H_{\mathbf{A}(U)} = H_U < H_{U^*}$, if U is prefix-free and regular (see *e. g.* Property 2.7 of [St89/93]) we obtain our assertion from the obvious inclusions $W_i \subseteqq W_i^* \subseteqq V^*$, $W_i \cdot V_i \subseteqq (W_i \cdot V_i)^* \subseteqq V^*$ and the inequality $H_{W_i \cdot V_i} \geqq H_{V_i}$.

The second assertion follows from the first one and (4), because $V^\delta = \emptyset$.

Q.E.D.

2. FINITE STATE ω-LANGUAGES AND ADJACENCY MATRICES

As we have seen in the previous section there are several relations between the Hausdorff dimension of subsets of X^ω (so-called ω-languages) and the entropy of languages. In order to compute $\dim F$ or $L_\alpha(F)$ for an ω-language $F \subseteqq X^\omega$ exactly we have to specify the ω-language F in a constructive way. Here we use the specification via systems of equations which resembles in some way the subdivision of sets in $\mathbb{R}^d$ into r-ary subcubes. The following example illustrates this fact.

Example A: (to be continued) Consider the fractal in Figure 1.

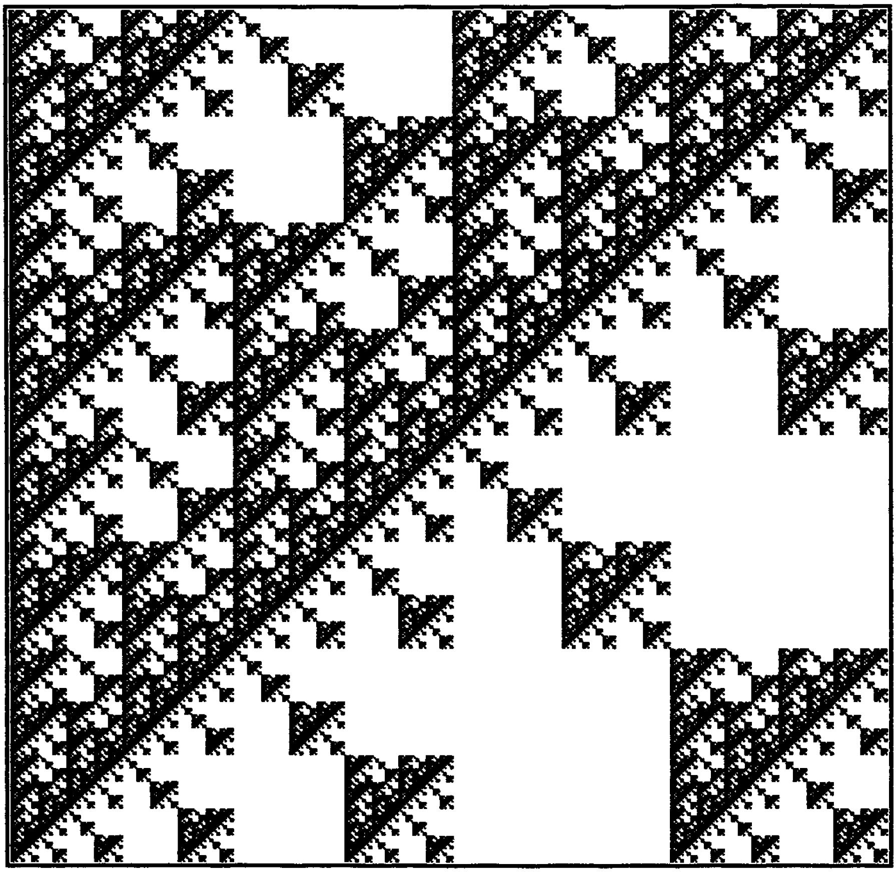

Figure 1.

The northeast (NE) and the southwest (SW) of its binary subsquares are both similar to the fractal itself (denoted by S_1), whereas the northwest (NW) and southeast (SE) subsquares and the original fractal are pairwise different (denoted by S_2 and S_3, respectively). This yields the first equation.

$$S_1 = SE \cdot S_3 \cup SW \cdot S_1 \cup NE \cdot S_1 \cup NW \cdot S_2$$

If we consider the northwest and southeast subsquares in the same way we obtain that the first one has subsquares similar to itself (SE), to the original fractal (SW and NE) and to the southeast subsquare of the original fractal (NW) and the second one has subsquares similar to itself (NW) and to the original fractal (SE), whereas its remaining subsquares are empty (completely white). Thus we may write the system of equations for the fractal S_1 in the following way:

$$S_1 = SE \cdot S_3 \cup SW \cdot S_1 \cup NE \cdot S_1 \cup NW \cdot S_2$$
$$S_2 = SE \cdot S_2 \cup SW \cdot S_1 \cup NE \cdot S_3 \cup NW \cdot S_3$$
$$S_3 = SE \cdot S_1 \cup NW \cdot S_3 \quad \square$$

In general those systems of equations correspond to certain normal form grammars (or deterministic automata) specifying ω-languages. Here we are dealing mainly with finite systems of equations.

Therefore, for a word w and a set $P \subseteqq X^* \cup X^\omega$ we call $P/w := \{p : w \cdot p \in P\}$ the *state* of P derived by the word w, and we call a set P *finite-state* provided $\{P/w : w \in X^*\}$ is a finite set. Finite-state languages are also known as regular languages (languages accepted by finite automata), whereas finite-state ω-languages form a larger class than ω-languages accepted by finite automata. The interrelations between both classes are investigated in more detail in [St83].

Since $F/w = (F \cap w \cdot X^\omega)/w$, the state of F derived by the word w is the $r^{|w|}$-fold magnification of the r-adic subcube of F specified by the word w, $F \cap w \cdot X^\omega$. Thus the equality of states $F/w = F/v$ is equivalent to the similarity of the corresponding subcubes $F \cap w \cdot X^\omega$ and $F \cap v \cdot X^\omega$.

As in the example above finite-state subsets of X^ω can be characterized as solutions of systems of linear equations of the form:

$$S_i = \bigcup_{x \in X_i} x \cdot S_{j\,(i,\,x)} \qquad \emptyset \neq X_i \subseteqq X, \quad 1 \leqq i \leqq k \tag{6}$$

where j maps $\{1, \ldots, k\} \times X$ to $\{1, \ldots, k\}$ (cf. [LS77]). Observe that $S_{j\,(i,\,x)} = S_i/x$. In order to avoid unnecessary equations in (6), we require that for every $i \in \{1, \ldots, k\}$ there is a $w \in X^*$ such that $S_i = S_1/w$.

A system of the form (6) is closely related to a finite automaton, but it may have many solutions. [2] If we, however, confine to solutions which are closed subsets of (X^ω, ρ) we have the following.

LEMMA 7: ([LS77]) *If we require that every $S_i \neq \emptyset$ then for $k \geqq 1$ a system (6) has a unique nonempty closed solution S_1. Moreover, S_1 is the maximum solution of (6).*

(2) In fact, already the simple system $S = X \cdot S$ has $2^{2^{\aleph_0}}$ solutions (cf. [St83]).

The next lemma shows another connection between Hausdorff dimension and the entropy of languages (cf. Corollary 4 above).

LEMMA 8: ([St85/89]) *If* $F \subseteqq X^\omega$ *is finite-state and closed then* $\dim F = H_F$.

Now the result of Lemma 8 reduces the computation of $\dim S$ to the calculation of the entropy of the corresponding languages of finite prefixes, $H_{\mathbf{A}(S)}$. This latter problem can be reduced to the computation of the maximum eigenvalue $\lambda_{\max}$ of the adjacency matrix of $\mathbf{A}(S)$, $\mathcal{A}_S = (a_{ij})_{1 \leqq i, j \leqq k}$ where $a_{ij} := \operatorname{card}\{x : j(i, x) = j\}$ (cf. [Ku70], [LS77] or [St85]). [3] It turns out that $\dim S = \log_r \lambda_{\max}$.

Related results concerning so-called graph directed constructions in $\mathbb{R}^d$ involving more complicated cases of similarities (not only those of r-ary subcubes) were obtained by Bandt ([Ba88] and [Ba89]) and Mauldin and Williams ([MW88]).

As a final result in this section we recall a relation between the structure function $\mathbf{s}_F(n)$ of a finite-state ω-language F and the maximum eigenvalue of $\mathcal{A}_F$, $\lambda_{\max}$, which is derived in Lemma 7 through Corollary 9 of [St85].

LEMMA 9: *Let* $F \subseteqq X^\omega$ *be an* ω*-language having* k *states. Then* $F' := F \setminus \bigcup_{H_{F/w} < H_F} w \cdot X^\omega$ *has at most* k *states and the structure functions* $\mathbf{s}_F$ *and* $\mathbf{s}_{F'}$ *satisfy*

$$\mathbf{s}_{F'}(n) \leqq \operatorname{card}\{w : |w| = n \wedge H_{F/w} = H_F\} \tag{7}$$

$$\mathbf{s}_F(n) \geqq \mathbf{s}_{F'}(n) \geqq \frac{1}{(\operatorname{card} X)^k} \lambda_{\max}^n \qquad \text{for all } n \in \mathbb{N}. \tag{8}$$

3. THE COMPUTATION OF THE MEASURE L_α

In this section we are going to show that for finite-state closed ω-languages S their Hausdorff measure $L_\alpha(S)$ $(\alpha = \dim S)$ is also computable.

First we apply the mapping L_α to the system (6) and we obtain the following system:

$$L_\alpha(S_i) = \sum_{x \in X_i} r^{-\alpha} \cdot L_\alpha(S_{j(i, x)}) \qquad 1 \leqq i \leqq n \tag{9}$$

(3) Since $\mathcal{A}_S$ is a nonnegative matrix, it has a nonnegative eigenvalue of maximum modulus $\lambda_{\max}$.

Thus the vector $\vec{L}_\alpha(S) := (L_\alpha(S_1), \ldots, L_\alpha(S_n))^\top$ solves the equation

$$\vec{L}_\alpha(S) = r^{-\alpha} \cdot \mathcal{A}_S \cdot \vec{L}_\alpha(S).$$

In other words, $\vec{L}_\alpha(S)$ is an eigenvector of the nonnegative matrix $\mathcal{A}_S$ corresponding to the eigenvalue $r^\alpha = \lambda_{\max}$. From the Perron-Frobenius-Theory of nonnegative matrices (see *e. g.* [Ga58], [La69]) we know that $\mathcal{A}_S$ has at least one nonnegative eigenvector corresponding to $\lambda_{\max}$. The difficulties in solving our problem, however, consist in the following items:

1. The Perron-Frobenius-Theory is not fully applicable here, because $\vec{L}_\alpha(S)$ may contain infinite entries.

2. The eigenspace of $\mathcal{A}_S$ corresponding to $\lambda_{\max}$ may have dimension > 1, and

3. even if the eigenspace is one-dimensional, we do not know which one of the nonnegative eigenvectors is our solution.

Our approach to get the actual solution $\vec{L}_\alpha(S)$ consists in deriving some bounds on the entries of the vector $\vec{L}_\alpha(S)$, that is, on the values $L_\alpha(S/w)$, where $w \in \mathbf{A}(S)$.

The first result excludes the all-zero-vector as solution for $\vec{L}_\alpha(S)$.

LEMMA 10: ([Ba89], [MW88], [St89]) *If $F \subseteqq X^\omega$ is nonempty, finite-state and closed then $L_\alpha(F) > 0$.*

The next result gives a lower bound to the maximum entry of $\vec{L}_\alpha(S)$.

THEOREM 11: *Let $L_\alpha(F) > 0$. Then for every $\varepsilon > 0$ there is a $w \in \mathbf{A}(F)$ such that $L_\alpha(F/w) > 1 - \varepsilon$.*

We use the following auxiliary lemma.

LEMMA 12: *Let $E \subseteqq X^\omega$ satisfy $L_\alpha(E) > 0$ and $L_\alpha(E) \geqq L_\alpha(E/w) - \varepsilon$ for some $\varepsilon > 0$ and all $w \in X^*$. Then $L_\alpha(E) \geqq 1 - \varepsilon$.*

Proof: Let $\varepsilon' > 0$ and let $W \subseteqq X^*$ be a prefix-free language such that $E \subseteqq W \cdot X^\omega$ and $L_\alpha(E; W) = \sum_{w \in W} r^{-\alpha|w|} \leqq L_\alpha(E) + \varepsilon'$. According to Property 1 we have

$$L_\alpha(E) = \sum_{w \in W} r^{-\alpha|w|} \cdot L_\alpha(E/w) \leqq \sum_{w \in W} r^{-\alpha|w|} \cdot (L_\alpha(E) + \varepsilon).$$

Consequently, $L_\alpha(E) \leqq (L_\alpha(E) + \varepsilon') \cdot (L_\alpha(E) + \varepsilon)$. Since ε' can be made arbitrarily small, the assertion follows.

Q.E.D.

Proof of Theorem 11: If $c := \sup \{L_\alpha(F/w) \mid w \in X^*\} \geqq 1$ we are done. Assume now $c < 1$. Then for every $\varepsilon > 0$ there is a $u \in X^*$ such that $L_\alpha(F/u) \geqq c - \varepsilon$. Thus F/u satisfies the hypothesis of Lemma 12 and hence $L_\alpha(F/u) \geqq 1 - \varepsilon$.

Q.E.D.

For finite-state ω-languages we obtain immediately.

COROLLARY 13: *If $F \subseteqq X^\omega$ is finite-state and $L_\alpha(F) > 0$ then there is a $w \in \mathbf{A}(F)$ such that $L_\alpha(F/w) \geqq 1$.*

Consequently, our vector $\vec{L}_\alpha(S)$ contains an entry $\geqq 1$.

Utilizing Corollary 13 we obtain a general lower bound on the nonzero measures $L_\alpha(F/w)$.

COROLLARY 14: *If $F \subseteqq X^\omega$ has at most k states and $L_\alpha(F) > 0$ then for all $w \in \mathbf{A}(F)$ such that $L_\alpha(F/w) > 0$ we have $L_\alpha(F/w) \geqq r^{-\alpha(k-1)}$.*

Proof: If $L_\alpha(F/w) > 0$ then according to Corollary 13 there is a $v \in X^*$ such that $L_\alpha(F/w \cdot v) \geqq 1$. Using well-known techniques from automata theory we get such a $v \in X^*$ having $|v| \leqq k - 1$. Then the assertion follows from the obvious inequality $L_\alpha(F/w) \geqq r^{-\alpha \cdot |v|} \cdot L_\alpha(F/w \cdot v)$.

Q.E.D.

To obtain a general bound from above on the entries of $\vec{L}_\alpha(S)$ other than $L_\alpha(S_i) \leqq \infty$ is not possible. Therefore, we confine to a particular case of ω-languages from which we can combine the general solution.

We call an ω-language $F \subseteqq X^\omega$ *strongly connected* provided for every $w \in \mathbf{A}(F)$ there is a $v \in X^*$ such that $F/w \cdot v = F$. Observe also that every state F/w of a strongly connected ω-language $F \subseteqq X^\omega$ is again strongly connected. In order to characterize strongly connected ω-languages F we introduce an auxiliary language U_F as in [St83]. [4]

$$
\begin{aligned}
U_F := \{u :\ & u \in \mathbf{A}(F) \setminus \{e\} \wedge F/u = F \\
& \wedge \forall w \, (e \sqsubset w \sqsubset u \rightarrow F/w \neq F\}
\end{aligned}
\tag{10}
$$

Clearly, U_F is prefix-free, and U_F is regular provided F is finite-state.

Strongly connected finite-state closed ω-languages can be represented by prefix-free regular languages in the following way.

([4]) Remark that there is a slight change in the definition. In contrast to [St83] we have here $U_F = \varnothing$ if $F = \varnothing$.

LEMMA 15: *Let an ω-language $\emptyset \neq F \subseteqq X^\omega$ be finite-state and closed. Then the following conditions are equivalent:*

1. F is strongly connected.

2. $F = \mathcal{C}(U_F^\omega)$

3. The matrix $\mathcal{A}_F$ is irreducible.

Now from the identity $L_\alpha(V^\omega) = L_\alpha(\mathcal{C}(V^\omega))$ (cf. Theorem 6) we obtain via Theorem 5 the following.

THEOREM 16: *If $F \subseteqq X^\omega$ is strongly connected, finite-state and closed then for all $w \in X^*$ we have $L_\alpha(F/w) \leqq 1$.*

Our theorem bounds the entries in $\vec{L}_\alpha(S)$ from above by 1 provided S is strongly connected. This yields the following procedure for calculating $\vec{L}_\alpha(S)$ when S is strongly connected.

PROCEDURE 1: *If the adjacency matrix $\mathcal{A}_S$ of the closed ω-language S given by the system (6) is irreducible then $\vec{L}_\alpha(S)$ is the positive eigenvector with maximum entry* 1 *correspnding to the maximum eigenvalue* [5] $\lambda_{\max}$.

Example A: (continued) Consider again the fractal given in the previous section. Its adjacency matrix is obtained as:

$$\mathcal{A}_{S_1} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Its maximum eigenvalue is $\lambda_{\max} = 2 + \sqrt{2}$, hence $\dim S_1 = \log_2\left(2 + \sqrt{2}\right)$, and the measure vector results in

$$\vec{L}_\alpha(S_1) = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} - 1 \end{pmatrix} \quad \square$$

In order to treat the general case we introduce the decomposition of an ω-language F with respect to its *connected part* $\mathrm{cn}(F) := F \cap \mathcal{C}(U_F^\omega)$.

$$F = \mathrm{cn}(F) \cup \bigcup_{w \in U_F^0} w \cdot F/w \tag{11}$$

([5]) According to the Perron-Frobenius-Theory, for an irreducible matrix $\mathcal{A}_S$, this eigenvector exists and is unique.

where $U_F^0 := \mathbf{A}(\mathrm{cn}(F)) \cdot X \setminus \mathbf{A}(\mathrm{cn}(F)) = \{w : \forall v\,(F/w \cdot v \neq F) \wedge \forall u\,(u \sqsubset w \rightarrow \exists v\,(F/u \cdot v = F))\}$, into pairwise disjoints parts $\mathrm{cn}(F)$ and $w \cdot (F/w)$ $(w \in U_F^0)$.

Observe, that $\mathrm{cn}(F)$ is closed if F is closed, and $\mathrm{cn}(F)$ is finite-state if F is finite-state. From Property 2 and the obvious fact that $L_\alpha(F/w) = 0$ implies $L_\alpha(F/w \cdot v) = 0$ we obtain immediately.

PROPERTY 17: *Let $F \subseteqq X^\omega$. Then $L_\alpha(F) = 0$ iff $L_\alpha(\mathrm{cn}(F)) = 0$ and $L_\alpha(F/w) = 0$ for all $w \in U_F^0$.*

Moreover, for the sake of convenience, we assume the states S_i numbered according to the following rules:

1. $S_1 = S$.

2. The states in strongly connected components (SCCs) $C_j := \{S_i : \exists w\, \exists v\,(S_i/w = S_{i_j} \wedge S_{i_j}/v = S_i)\}$ have consecutive numbers. (Observe that, since $S_{i_j}/e = S_{i_j}$, we assume every state S_i to be contained in some SCC.)

3. Otherwise, $S_i/w = S_j$ implies $j \geqq i$.

If we denote by $C \vdash C'$ the accessibility of SCCs, that is, $C \vdash C'$ iff $C \neq C'$ and $\exists w\, \exists S\, \exists S'$ $(S \in C \wedge S' \in C' \wedge S/w = S')$ then $C \vdash C'$ means that states in C' have higher numbers than states in C. Let k_j be the number of the first state S_{k_j} in the SCC C_j. Using these rules, $\mathcal{A}_S$ will be in upper block diagonal form, that is,

$$\mathcal{A}_S = \begin{pmatrix} \mathcal{A}_1 & \dots & \mathcal{A}_{1,\kappa} \\ O & \ddots & \vdots \\ O & O & \mathcal{A}_\kappa \end{pmatrix}$$

where κ is the number of SCCs, and the square matrices on the main diagonal, $\mathcal{A}_j$, correspond to the strongly connected components C_j. Note that each matrix $\mathcal{A}_j$ is of size (card $C_j \times$ card C_j), and is zero or irreducible. In particular, $\mathcal{A}_\kappa \neq O$ and if $\mathcal{A}_j = O$ then it is a (1×1)-matrix.

Moreover, one easily observes, that $\mathcal{A}_j$ is the adjacency matrix of $\mathrm{cn}(S_{k_j})$, and that the nonzero entries of the matrices $\mathcal{A}_{j,j'}$ correspond to words $w \in U_{S_{k_j}}^0$. Let further denote λ_j the maximum eigenvalue of the matrix $\mathcal{A}_j$, and observe that $\lambda_j \leqq \lambda_{\max}$.

Now, decompose our solution $\vec{L}_\alpha(S)$ of Equation 9 as

$$\vec{L}_\alpha(S) = (\vec{\mathbf{x}}_1^\top, \dots, \vec{\mathbf{x}}_\kappa^\top)^\top$$

where $\vec{x}_j$ is a column vector of size card C_j. Then Equation 9 splits into κ equations

$$(12) \qquad \mathcal{A}_l \cdot \vec{x}_l + \sum_{j=l+1}^{\kappa} \mathcal{A}_{l,j} \cdot \vec{x}_j = \lambda_{\max} \cdot \vec{x}_l, \quad l \in \{1, \ldots, \kappa\}$$

We show that this system can be uniquely solved according to the constraints given by the structure of S and the properties of the measure L_α proceeding in the following way in a bottom up direction.

We start with $l = \kappa$ and observe that, since $\mathcal{A}_\kappa \neq O$ is irreducible, in case $\lambda_\kappa = \lambda_{\max}$ our Procedure 1 is applicable to the set S_{k_κ}, and we obtain the partial solution $\vec{x}_\kappa$. In case $\lambda_\kappa < \lambda_{\max}$ we have dim S_{k_κ} < dim S, hence $\vec{x}_\kappa = \vec{o}$. [6]

We may now assume that the system (12) be already solved for $j < l$, that is, $\sum_{j=l+1}^{\kappa} \mathcal{A}_{l,j}. \vec{x}_j$ is a vector $\vec{b}$ with entries in $[0, \infty]$, and we may further assume that $\vec{x}_j = \vec{o}$ or $\vec{x}_j > \vec{o}$. [7]

If $\mathcal{A}_l = O$ then $\vec{x}_l = \lambda_{\max}^{-1} \cdot \vec{b}$ and, since $\mathcal{A}_l$ is a 1×1-matrix, $\vec{x}_l = 0$ or $\vec{x}_l > 0$.

If $\mathcal{A}_l \neq O$ then $\mathcal{A}_l$ is irreducible, whence $\vec{x}_l > \vec{o}$ or $\vec{x}_l = \vec{\infty}$ whenever $\vec{x}_l$ contains an entry $x_m > 0$ or $x_m = \infty$, respectively. If $\lambda_l < \lambda_{\max}$, that is, $\dim \mathrm{cn}\,(S_{k_l}) < \dim S$ the matrix $\lambda_{\max} \cdot \mathcal{I} - \mathcal{A}_l$ is invertible [8], and the unique solution $\vec{x}_l$ is readily obtained from Equation 12. It should be mentioned that $\vec{x}_l \geqq \sum_{j=l+1}^{\kappa} \mathcal{A}_{l,j} \cdot \vec{x}_j \geqq \vec{o}$.

Now, it remains to deal with the case $\lambda_l = \lambda_{\max}$.

If $\vec{b} := \sum_{j=l+1}^{\kappa} \mathcal{A}_{l,j} \cdot \vec{x}_j \neq \vec{o}$ then according to the Perron-Frobenius-Theory of nonnegative matrices (*see* [Ga58] or [La69]) the equation $\mathcal{A}_l \cdot \vec{x} + \vec{b} = \lambda_{\max} \cdot \vec{x}$ has no solution $\vec{o} \leqq \vec{x} < \vec{\infty}$. Hence, following the above remark concerning the irreducibility of $\mathcal{A}_l$, we have $\vec{x} = \vec{\infty}$.

(6) We denote by $\vec{o}$ and $\vec{\infty}$ the vectors having all entries 0 or ∞, respectively, of appropriate size.

(7) As usual, by $>$ or $\geqq$ we denote the fact that $>$ or $\geqq$ holds for all entries of the vectors or matrices.

(8) Here $\mathcal{I}$ is the unity matrix of appropriate size.

Consider the case $\vec{\mathbf{b}} = \vec{\mathbf{o}}$. This means that $\mathcal{A}_{l,j} \cdot \vec{\mathbf{x}}_j = \vec{\mathbf{o}}$, that is, $S_{k_l}/w \notin C_j$ for $w \in U^0_{S_k}$, if $\mathcal{A}_{l,j} = O$, or if $\mathcal{A}_{l,j} \neq O$ then, since $\vec{\mathbf{x}}_j \neq \vec{\mathbf{o}}$ implies $\vec{\mathbf{x}}_j > \vec{\mathbf{o}}$, we have $\vec{\mathbf{x}}_j = \vec{\mathbf{o}}$, that is $L_\alpha(S_{k_j}/w) = 0$ for all $w \in U^0_{S_{k_l}}$. Consequently Property 17 proves $L_\alpha(S_{k_l}) = L_\alpha(\mathrm{cn}(S_{k_l}))$ and hence $\vec{\mathbf{x}}_l = \vec{L}_\alpha(\mathrm{cn}(S_{k_l}))$. So we may again apply Procedure 1 to obtain $\vec{\mathbf{x}}_l > \vec{\mathbf{o}}$.

This procedure can be presented in a more concise manner. To this end we recall that nonzero entries in the vector $\vec{L}_\alpha(S)$ appear only either as a result of the application of Procedure 1 or if the vector $\vec{\mathbf{b}}$ is nonzero. Thus it holds the following.

FACT 18: *Let $S_i \in C$. We have $L_\alpha(S_i) = 0$ iff the maximal eigenvalue λ of the diagonal matrix $\mathcal{A}$ corresponding to the SCC C is smaller than $\lambda_{\max}$ and for every C' such that $C \vdash C'$ and every $S_j \in C'$ the identity $L_\alpha(S_i) = 0$ holds.*

In the same way one observes that infinite entries in $\vec{L}_\alpha(S)$ appear only if $\vec{\mathbf{b}} \neq \vec{\mathbf{o}}$ and the corresponding matrix $\mathcal{A}_l$ has eigenvalue $\lambda_l = \lambda_{\max}$.

FACT 19: *Let $S_i \in C$. We have $L_\alpha(S_i) = \infty$ iff there are SCCs C', C'' such that $C = C'$ or $C \vdash C'$, and $C' \neq C'' \wedge C' \vdash C''$ and the square matrices $\mathcal{A}'$ and $\mathcal{A}''$ corresponding to C' and C'', respectively, have maximum eigenvalues $\lambda' = \lambda'' = \lambda_{\max}$.*

In particular, if $L_\alpha(S_i/w) = \infty$ then $L_\alpha(S_i) = \infty$.

This yields the following procedure.

PROCEDURE 2: *Compute for every SCC C_j the maximal eigenvalue λ_j of the corresponding matrix $\mathcal{A}_j$.*

1. We start with terminal SCCs, that is, SCCs C_j for which there is no C' satisfying $C_j \vdash C'$: For all terminal SCCs C_j do:

(a) If $\lambda_j < \lambda_{\max}$ then set $L_\alpha(S_i) := 0$ for every $S_i \in C_j$.

(b) Otherwise solve the equation $\mathcal{A}_j \cdot \vec{\mathbf{x}} = \lambda_{\max} \cdot \vec{\mathbf{x}}$ in such a way that $\vec{\mathbf{x}}$ is positive and has maximum entry 1. Insert the entries of $\vec{\mathbf{x}}$ into the corresponding places of $\vec{L}_\alpha(S)$.

2. Complete as much of the entries of $\vec{L}_\alpha(S)$ according to Facts 18 and 19 as possible. (The resulting incomplete solution to the vector $\vec{L}_\alpha(S)$ will be called $\vec{\mathbf{x}}$.)

3. Solve the matrix equation $\mathcal{A}_S \cdot \vec{\mathbf{x}} = \lambda_{\max} \cdot \vec{\mathbf{x}}$ for the remaining entries of $\vec{L}_\alpha(S)$.

From our above consideration it follows that throughout the first items we had gained enough information in order to solve this last equation.

4. BLACKNESS AND DENSITY

In this last section we derive a connection between topological density and Hausdorff dimension. It turns out that density and dimension are closely related for finite-state and closed ω-languages F having finite and locally positive α-measure $L_\alpha(F)$ $(\alpha = \dim F)$.

The result obtained indicates that for fractals represented by finite-state and closed ω-languages F the blackness of the picture (in a sufficiently high resolution) increases as Hausdorff dimension $\alpha = \dim F$ and Hausdorff measure $L_\alpha(F)$ increase. The fractals presented in the appendix give evidence of this fact. For these fractals dimension and measure vector were computed using the procedures of the previous section.

We recall that a set $F \subseteq X^\omega$ has *finite α-measure* iff $L_\alpha(F) < \infty$, and F has *locally positive α-measure* iff $0 < L_\alpha(F \cap w \cdot X^\omega)$ whenever $F \cap w \cdot X^\omega \neq \emptyset$. Clearly a set F having finite and locally positive α-measure has Hausdorff dimension $\dim F = \alpha$. Moreover those sets allow for a simple derivation of an upper bound to their structure function $\mathbf{s}_F$.

PROPERTY 20: *Let $F \subseteq X^\omega$ be closed and have locally positive α-measure. If F has no more than k states then $\mathbf{s}_F(n) \leqq r^{\alpha(n+k-1)} \cdot L_\alpha(F)$.*

Proof: According to Corollary 14, since F has locally positive α-measure, we have $r^{\alpha(1-k)} \leqq L_\alpha(F/w)$ for every $w \in \mathbf{A}(F)$. Then $L_\alpha(F) \geqq \mathbf{s}_F(n) \cdot r^{-\alpha n} \cdot r^{\alpha(1-k)}$ for all $n \in \mathbb{N}$.

Q.E.D.

Taking into account Eq. (8) in Lemma 9 we obtain the following property of the structure function of finite-state closed sets F having both finite and locally positive α-measure.

$$(13) \quad \exists c_1, c_2 (0 < c_1 \leqq c_2 \wedge \forall n (n \in \mathbb{N} \rightarrow c_1 \cdot r^{\alpha n} \leqq \mathbf{s}_F(n) \leqq c_2 \cdot r^{\alpha n}))$$

After deriving these simple properties we proceed to the main result relating topological density and Hausdorff dimension.

Topological density is based on the following notion. A set E is *nowhere dense* in $F \subseteq X^\omega$ provided $\mathcal{C}(F \setminus \mathcal{C}(E)) = \mathcal{C}(F)$, that is, if $\mathcal{C}(E)$ does not contain a nonempty subset of the form $F \cap w \cdot X^\omega$.

THEOREM 21: *Let $F \subseteqq X^\omega$ be a nonempty, finite-state and closed ω-language having both finite and locally positive α-measure. Then for every finite-state and closed subset $E \subseteqq F$ the following conditions are equivalent:*

1. E *is nowhere dense in* F,

2. $L_\alpha(E) = 0$,

3. $\dim E < \alpha = \dim F$, and

4. $\limsup_{n\to\infty} (\mathrm{s}_E(n)/\mathrm{s}_F(n)) = 0$

Proof: In virtue of Lemma 10 conditions 2. and 3. are equivalent for arbitrary finite-state and closed ω-languages.

If $\dim E < \dim F$ then according to Lemma 8 we have $\mathrm{s}_E(n) < r^{\gamma n} < \mathrm{s}_F(n)$ for $\gamma = (\dim F - \dim E)/2$ and large $n \in \mathbb{N}$. Thus 3. implies 4. also in case of arbitrary finite-state and closed ω-languages.

If 4. holds and F has finite and locally positive α-measure then Property 20 implies that $\mathrm{s}_E(n) \geqq \varepsilon \cdot r^{\alpha n}$ cannot hold for $\varepsilon > 0$ and infinitely many $n \in \mathbb{N}$. Now Eq. (8) and Lemma 8 prove that $\dim E < \alpha$.

It remains to show that 1. is equivalent to one of the conditions 2., 3. or 4.

If E is not nowhere dense in F then $E \supseteqq F \cap w \cdot X^\omega \neq \emptyset$ for some $w \in X^*$. Consequently, $L_\alpha(E) \geqq L_\alpha(F \cap w \cdot X^\omega) > 0$.

Now assume E to be nowhere dense in F, and let $m := \max\{\limsup_{n\to\infty} (\mathrm{s}_{E/v}(n)/\mathrm{s}_{F/v}(n)) : v \in \mathbf{A}(F)\}$. (The maximum exists, because E and F are finite-state.) In the sequel let $w \in \mathbf{A}(F)$ be chosen such that

$$\limsup_{n\to\infty} \frac{\mathrm{s}_{E/w}(n)}{\mathrm{s}_{F/w}(n)} = \max\left\{\limsup_{n\to\infty} \frac{\mathrm{s}_{E/v}(n)}{\mathrm{s}_{F/v}(n)} : v \in \mathbf{A}(F)\right\}.$$

Since E is nowhere dense in F, there is a u such that $E/wu = \emptyset$ and $F/wu \neq \emptyset$. Set $U := \{v : |v| = |u| \wedge v \in \mathbf{A}(F)\}$ and $U' := \{v : |v| = |u| \wedge v \in \mathbf{A}(E)\}$. Then $U' \subseteqq U$ and $u \in U \setminus U'$

In virtue of the general identity $\mathrm{s}_M(n) = \sum_{|v|=k} \mathrm{s}_{M/v}(n-k)$ we obtain the following:

$$m = \limsup_{n\to\infty} \frac{\sum_{v\in U} \mathrm{s}_{E/wv}(n)}{\sum_{v\in U} \mathrm{s}_{F/wv}(n)} = \limsup_{n\to\infty} \frac{\sum_{v\in U'} \mathrm{s}_{E/wv}(n)}{\sum_{v\in U'} (\mathrm{s}_{F/wv}(n) + s(n))}$$

where $s(n) = 1/\operatorname{card} U' \cdot \sum_{v \in U \backslash U'} s_{F/wv}(n)$.

Since $u \in U \backslash U'$ and $L_\alpha(F/wu) > 0$ we have $s(n) \geqq \varepsilon \cdot r^{\alpha \cdot n}$ for some $\varepsilon > 0$.

Thus

$$m \leqq \max \left\{ \limsup_{n \to \infty} \frac{s_{E/wv}(n)}{s_{F/wv}(n) + s(n)} : v \in U' \right\} \leqq \frac{m}{1+\varepsilon} < m.$$

This contradicts the assumption $m > 0$.

Q.E.D.

Finally, we will provide examples showing that the requirements concerning the finiteness and the local nonnull behaviour of the α-measure of F are really essential. Throughout the examples let $X := \{a, b, c, d\}$ where a, b, c, d are shorthands for $(0, 0), \ldots, (1, 1)$ (or $SW, \ldots, NE$), respectively.

Example B: Let $F := \{a, b, c\}^* \cdot d \cdot \{a, b, c\}^\omega$ and $E := d \cdot \{a, b, c\}^\omega$. Then condition 4. is fulfilled, but since $E = F \cap d \cdot X^\omega$ none of the other conditions is true. Observe that $L_\alpha(F) = \infty$ for $\alpha = \dim F$. □

Example C: Consider $F := \{a, b, c\}^\omega \cup \{a, b, c\}^* \cdot d \cdot \{a, b\}^\omega$, $E := \{a, b, c\}^\omega$ and $E' := d \cdot \{a, b\}^\omega$. Then E is nowhere dense in F, because $\forall w \, (w \in \mathbf{A}(E) \rightarrow wd \cdot X^\omega \cap E = \varnothing)$, but none of the other conditions is true.

In contrast to this, E' is not nowhere dense in F, but $\dim E' < \dim F$.

Here we have $L_\alpha(F) < \infty$ but $L_\alpha(F \cap d \cdot X^\omega) = L_\alpha(d\{a, b\}^\omega) = 0$. □

Example D: Let $F := \{a, b, c\}^\omega \cup \bigcup_{w \in \{a, b, c\}^*} w \cdot d^{|w|} \cdot \{a, b, c\}^\omega$ and $E := \{a, b, c\}^\omega$. One easily checks that $3^n = s_E(n) \leqq s_F(n) \leqq 3^{n+1}$ and accordingly $1 = L_\alpha(E) \leqq L_\alpha(F) \leqq 3$ for $\alpha = \dim E = \dim F = \log_2 3$, but E is nowhere dense in F. Here F is not finite-state though closed. □

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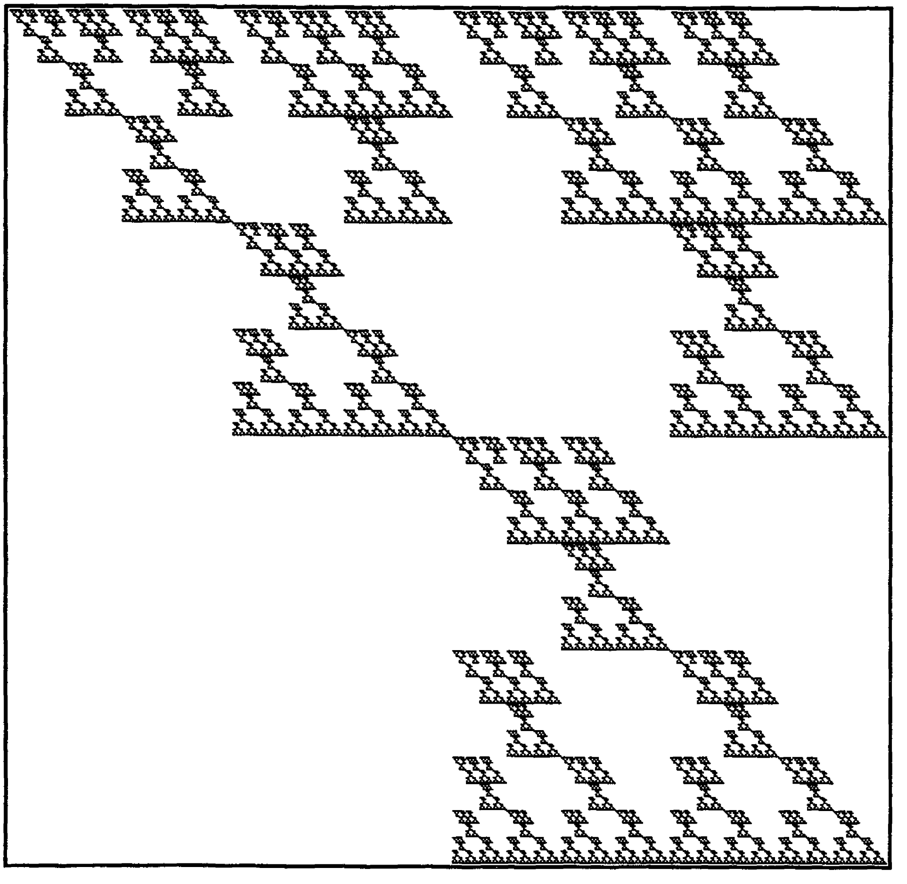
APPENDIX



System Table

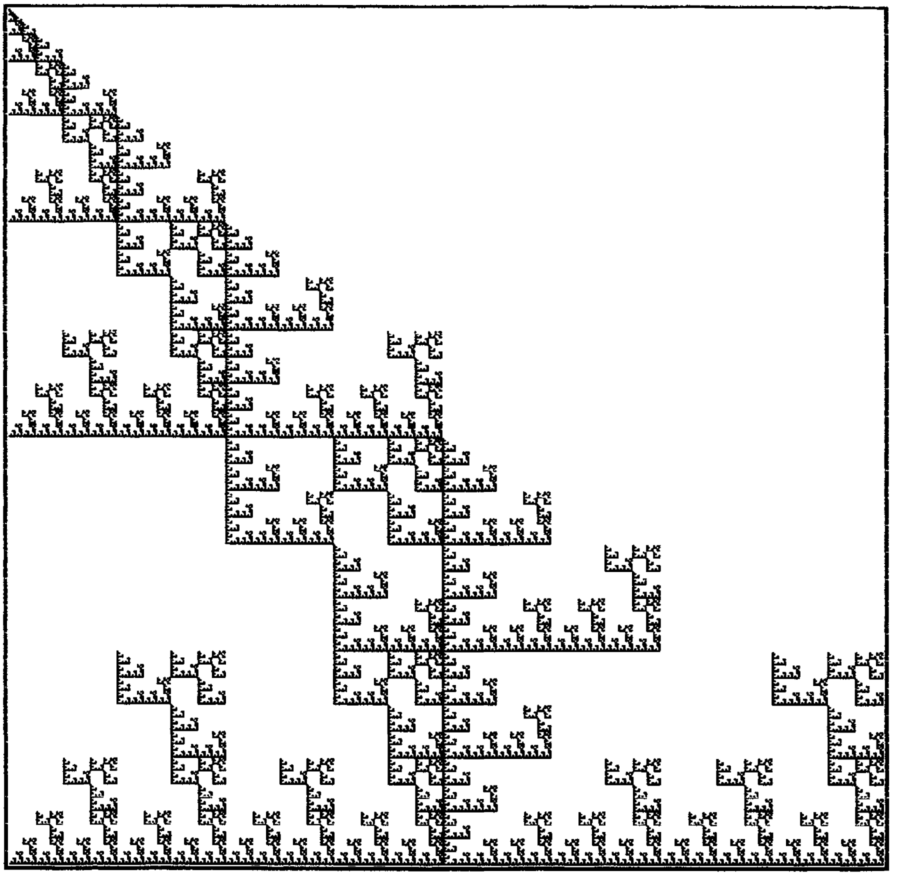
	SE	SW	NE	NW
1	2	0	3	1
2	2	2	0	3
3	2	0	2	1

Adjacency Matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

λ_{max}: 3.0000
Hausdorff dimension: 1.5850
Measure vector: [1.000, 1.000, 1.000]

Figure 2.



System Table

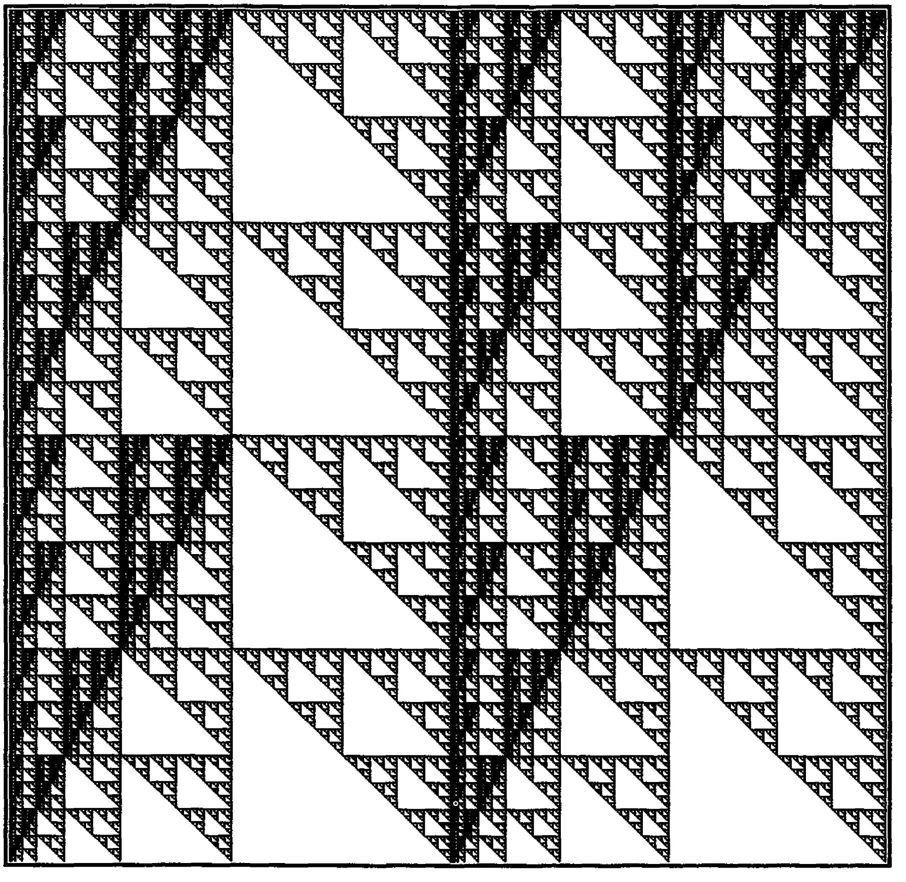
	SE	SW	NE	NW
1	3	2	0	1
2	2	2	4	0
3	2	3	0	3
4	3	0	4	3

λ_{max}: 3.0000
Hausdorff dimension: 1.5850
Measure vector: [1.000, 1.000, 1.000, 1.000]

Adjacency Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Figure 3.



System Table

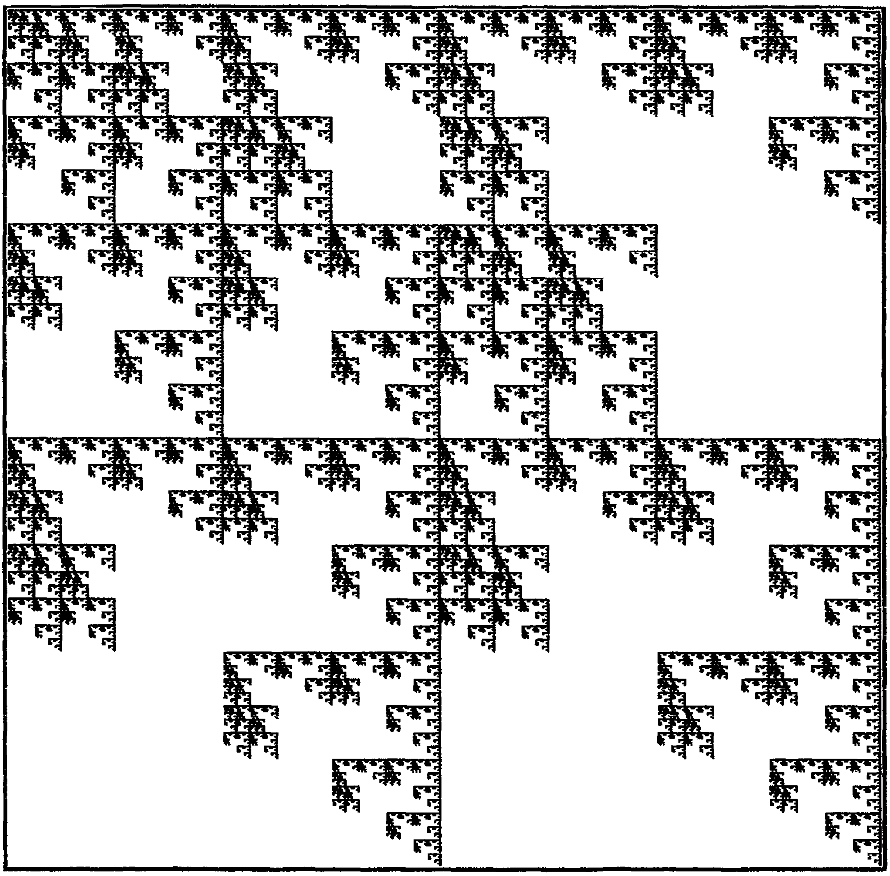
	SE	SW	NE	NW
1	2	2	1	2
2	3	2	3	1
3	3	0	3	3

λ_{max}: 3.0000
Hausdorff dimension: 1.5850
Measure vector: [6.000, 4.000, 1.000]

Adjacency Matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Figure 4.



System Table

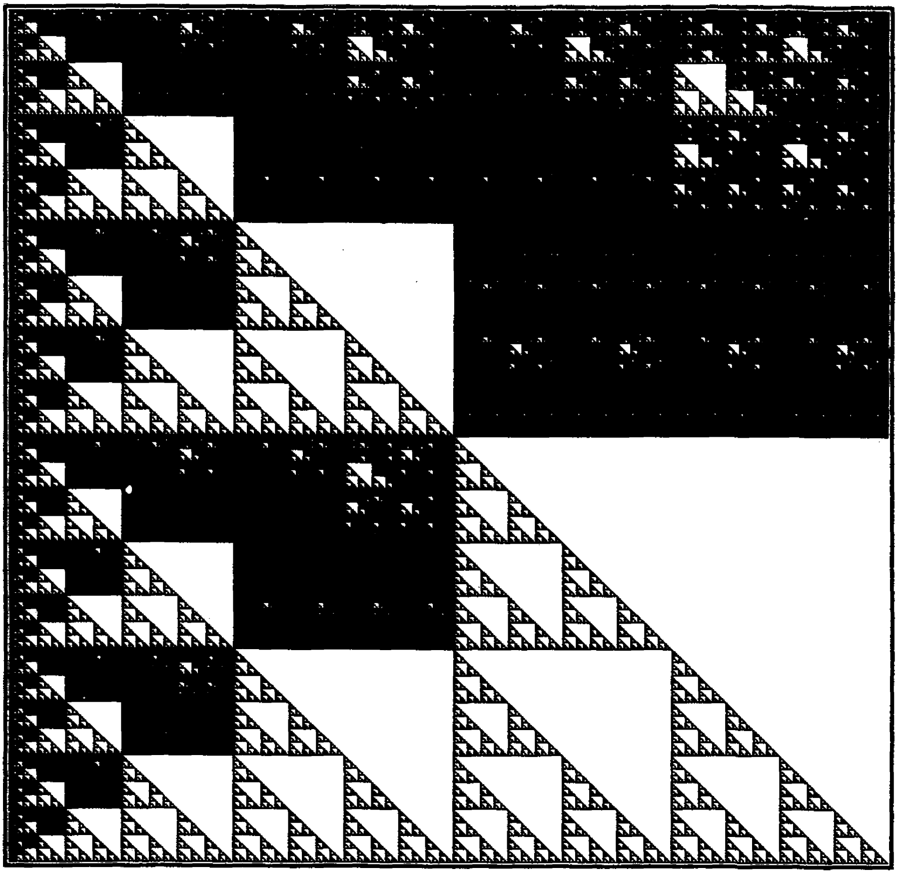
	SE	SW	NE	NW
1	3	3	2	1
2	0	1	3	2
3	3	0	3	2

λ_{max}: 3.1479
Hausdorff dimension: 1.6544
Measure vector: [1.000, 0.783, 0.682]

Adjacency Matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Figure 5.



System Table

	SE	SW	NE	NW
1	7	1	2	1
2	3	3	4	2
3	2	2	3	3
4	4	4	4	5
5	5	7	6	6
6	5	6	5	6
7	7	7	0	7

Adjacency Matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

λ_{max}: 3.5616
Hausdorff dimension: 1.8325
Measure vector: $[\infty, \infty, \infty, 1.390, 0.781, 1.000, 0.000]$

Figure 6.