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## FINITE DEGREES OF AMBIGUITY IN PATTERN LANGUAGES (\*)

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Abstract – *The paper investigates nondeterminism and degrees of nondeterminism in representing words according to a pattern given a priori. The issues involved belong to the basic combinatorics of words. Our main results concern decidability and construction of finite degrees of nondeterminism.*

### 1. INTRODUCTION

There has been much interest recently in *patterns* and *pattern languages*. (See, for instance, [3], [5], [6], [8] and their references). Indeed, a natural way of describing a given sample of words is to exhibit a common pattern for the words. Such an approach is especially appropriate if the sample is growing, for instance, through some *learning* process. Finding patterns for a sample sets is, thus, a typical problem of *inductive inference*.

Pattern languages in the sense understood in this paper were introduced in [1]. The essential difference between the two cases, where the empty word  $\lambda$  can or cannot be substituted for the variables, was studied in [5]. It was also observed that many problems in combinatorics of words, ranging from the classical ones discussed in [11] to the more recent ones discussed in [2] and [7], can be expressed in terms of the inclusion problem for pattern languages. The same holds true for certain problems in term rewriting, [8]. From this point of view it is not surprising that the inclusion problem turned out to be undecidable, [6].

Given a terminal word  $w$  and a pattern  $\alpha$ , it may happen that  $w$  “follows” the pattern  $\alpha$  in several ways. In other words, there are several assignments for

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the variables in  $\alpha$ , each of which gives rise to  $w$ . This kind of *nondeterminism* or *ambiguity* in patterns will be investigated in this paper. Indeed, the classical language-theoretic notions of *unambiguity*, *inherent ambiguity* and *degrees of ambiguity* (see [10]) find their natural counterparts in the context of patterns. The proofs make use of various aspects in combinatorics of words. The case of a finite degree of ambiguity greater than one turns out to be rather involved. In decidability issues, modifications of the result by Makanin, [9], can be used.

## 2. DEFINITIONS AND PRELIMINARY RESULTS

Let  $\Sigma$  be an alphabet (of *terminals*) and  $V$  an alphabet (of *variables*) such that  $\Sigma \cap V = \emptyset$ . Let  $H(\Sigma, V)$  (resp.  $H_+(\Sigma, V)$ ) be the set of all morphisms (resp. nonerasing morphisms)

$$h : (\Sigma \cup V)^* \rightarrow \Sigma^*$$

such that  $h(a) = a$  for all  $a \in \Sigma$ .

Nonempty words  $\alpha$  over  $\Sigma \cup V$  are referred to as *patterns*. A pattern  $\alpha \in (\Sigma \cup V)^+$  defines the *languages*:

$$\begin{aligned} L_E(\alpha) &= \{w \mid h(\alpha) = w, \text{ for some } h \in H(\Sigma, V)\} \\ L_{NE}(\alpha) &= \{w \mid h(\alpha) = w, \text{ for some } h \in H_+(\Sigma, V)\}. \end{aligned}$$

The languages  $L_E(\alpha)$  and  $L_{NE}(\alpha)$  are referred to as *pattern languages*. Sometimes we speak of *E-patterns* and *NE-patterns* (“erasing” and “nonerasing”) to indicate which of the languages we are interested in. Also the alphabet  $\Sigma$  may be indicated in the notation:  $L_E(\alpha, \Sigma)$  or  $L_{NE}(\alpha, \Sigma)$ . This is the case especially if  $\Sigma$  is not visible from  $\alpha$ , that is, all letters of  $\Sigma$  do not occur in  $\alpha$ .

We now come to the central notions of this paper. It may happen that a word  $w$  in  $L_E(\alpha)$  or  $L_{NE}(\alpha)$  has several “representations”, that is, there are several morphisms  $h$  satisfying  $w = h(\alpha)$ . For instance, the terminal word  $w = a^7 ba^7$  possesses 8 representations in terms of the pattern  $\alpha = xyx$ . (The number is 7 if  $\alpha$  is viewed as an *NE*-pattern.) We express this by saying that the *degree of ambiguity* of  $w$  with respect to  $\alpha$  equals 8. Whenever important, we indicate whether we are dealing with the *E*- or *NE*-case.

The *degree of ambiguity of a pattern*  $\alpha$  equals the maximal degree of ambiguity of words  $w$  in the language of  $\alpha$ , or infinity ( $\infty$ ) if no such maximal degree exists. More formally, we associate to a pattern  $\alpha$  over  $\Sigma \cup V$  and a word  $w \in \Sigma^+$  the subset  $S(\alpha, w, \Sigma)$  of  $H(\Sigma, V)$ , consisting

of morphisms  $h$  such that  $h(\alpha) = w$ . The *cardinality* of this subset is denoted by  $\text{card}(\alpha, w, \Sigma)$ . (We make here the *convention* that morphisms differing only on variables not present in  $\alpha$  are not counted as different). The *degree of ambiguity* of  $\alpha$  equals  $k \geq 1$  iff

$$\text{card}(\alpha, w, \Sigma) \leq k, \quad \text{for all } w \in L_E(\alpha),$$

and

$$\text{card}(\alpha, w', \Sigma) = k, \quad \text{for some } w' \in L_E(\alpha).$$

If there is no such  $k$ , then the *degree of ambiguity* of  $\alpha$  equals  $\infty$ . For  $k = 1$ ,  $\alpha$  is also termed *unambiguous* and, for  $k > 1$ ,  $\alpha$  is termed *ambiguous*.

*Remark:* The terminals actually appearing in  $\alpha$  constitute a subset  $\Sigma'$ , maybe empty, of  $\Sigma$ . Indeed, any pattern over  $\Sigma' \cup V$  is a pattern also over  $\Sigma \cup V$ , where  $\Sigma' \subseteq \Sigma$ . In the definition of the degree of ambiguity we actually specified the pair  $(\alpha, \Sigma)$ . However, it is pleasing to observe that, in fact, it suffices to specify only  $\alpha$  because the degree is independent of the choice of  $\Sigma$ . The following argument justifies this observation.

If  $\alpha$  contains no terminals (that is,  $\Sigma'$  is empty), then the degree of ambiguity of  $\alpha$  is 1 or  $\infty$ , depending on whether  $\alpha$  contains occurrences of one or more than one variable. If  $\Sigma'$  contains at least one terminal  $a$ , we denote by  $g: \Sigma^* \rightarrow \Sigma'^*$  the morphism keeping the letters of  $\Sigma'$  fixed and mapping the letters of  $\Sigma - \Sigma'$  into  $a$ . Clearly, the degree of ambiguity does not decrease if the terminal alphabet  $\Sigma'$  is replaced by  $\Sigma$ . But it does not increase either because, whenever  $w$  has  $m$  representations according to  $\alpha$ , then  $g(w)$  has at least  $m$  representations according to  $\alpha$ .  $\square$

By the above remark, we speak of the degree of ambiguity of a pattern  $\alpha$  (without specifying the alphabet). The above definitions were carried out in the  $E$ -case. The  $NE$ -case is analogous.

The notions are now naturally extended to concern languages. We do this in the  $E$ -case. A *pattern language*  $L$  is *ambiguous of degree*  $k \geq 1$  if  $L = L_E(\alpha)$ , for some pattern  $\alpha$  ambiguous of degree  $k$ , but there is no pattern  $\beta$  of degree less than  $k$  such that  $L = L_E(\beta)$ . Here  $k$  is a natural number or  $\infty$ . Again, if  $k = 1$  we say that  $L$  is *unambiguous*. Otherwise,  $L$  (*inherently*) *ambiguous*.

It was shown in [1] that two  $NE$ -patterns are *equivalent* (in the sense that they generate the same language) exactly in the case they are identical up to a possible renaming of variables. This yields immediately the following result:

**THEOREM 1:** *The degree of ambiguity of an  $NE$ -pattern  $\alpha$  equals the degree of ambiguity of the pattern language  $L_{NE}(\alpha)$ .*

Theorem 1 does not hold for  $E$ -patterns:  $L_E(X) = L_E(XY)$  but the degrees of ambiguity of the patterns  $X$  and  $XY$  are 1 and  $\infty$ , respectively.

**THEOREM 2:** *For every  $E$ -pattern (containing at least one variable), there is an equivalent  $E$ -pattern whose degree of ambiguity is  $\infty$ . Conversely, there are  $E$ -patterns (for instance,  $XY Y X$ ) such that the degree of ambiguity of every equivalent  $E$ -pattern, and hence also the degree of ambiguity of the generated language, equals  $\infty$ .*

*Proof:* To prove the first sentence, it suffices to replace all occurrences of a variable  $X$  in the pattern with  $X_1 X_2$ . The second sentence follows by a simple case analysis concerning patterns equivalent to  $XY Y X$ .  $\square$

In what follows we do not make any distinction between  $E$ - and  $NE$ -patterns because the results hold in both cases.

**THEOREM 3:** *Every pattern containing occurrences of only one variable  $X$  is unambiguous. Every pattern containing occurrences of at least two variables  $X$  and  $Y$  but of at most one terminal  $a$  is ambiguous of degree  $\infty$ .*

*Proof:* The first sentence follows by a length argument: for every  $w$ , the value  $h(X)$  is uniquely determined. To prove the second sentence, we first replace in the given pattern the other variables (if any) with  $a$ . The resulting pattern contains  $m \geq 1$   $X$ 's,  $n \geq 1$   $Y$ 's and  $p \geq 0$   $a$ 's. Given any  $k$ , we can find a  $z_k$  such that

$$mx + ny + p = z_k$$

has more than  $k$  positive solutions  $(x, y)$ . This means that  $a^{z_k}$  has more than  $k$  representations according to the given pattern.  $\square$

### 3. DETERMINISM AND NONDETERMINISM

We now continue the study begun in the preceding section and characterize some basic cases of ambiguity and nonambiguity.

**THEOREM 4:** *Every pattern  $\alpha$  satisfying the following two conditions is of  $\infty$  degree of ambiguity. (i)  $\alpha$  contains occurrences of at least two variables. (ii) Some variable occurs in  $\alpha$  only once.*

*Proof:* Let  $Z$  be the variable that has only one occurrence in  $\alpha$ . We'll consider the following two possibilities:

*Case 1:* The pattern  $\alpha$  starts or ends with  $Z$ . Assume that  $\alpha$  ends with  $Z$  (the situation  $\alpha$  starts with  $Z$  is symmetric). Hence,  $\alpha = \beta Z$ , where  $\beta$  is a pattern that contains at least one variable, but  $Z$  does not occur in  $\beta$ . Let  $X$  be the leftmost variable in  $\beta$ , i. e.  $\beta = X \gamma$ , where  $\gamma$  is a pattern. (We assume without loss of generality that  $\beta$  starts with a variable). Therefore,  $\alpha = X \gamma Z$  and  $Z$  does not occur in  $X \gamma$ .

Now, assume to the contrary that  $\alpha$  has the degree of ambiguity  $k$ , where  $k < \infty$ . Let  $w$  be a terminal word that has  $k$  different decompositions with respect to  $\alpha$ . Let  $u$  be a fixed terminal word. Consider the morphism  $f$ , defined as:  $f(X) = w$  and  $f(Y) = u$ , for any variable  $Y$ ,  $Y \neq X$ . We obtain  $f(\alpha) = t = wv$ .

The terminal word  $t$  has one decomposition with respect to  $\alpha$  corresponding to the morphism  $f$  and, moreover,  $t$  has  $k$  other decompositions with respect to  $\alpha$ , corresponding to the  $k$  possible decompositions of  $w$ , each such decomposition being modified as follows: if  $Z$  was substituted by  $r$  in the originally considered decomposition of  $w$ , then  $Z$  is substituted by  $rv$ , in order to obtain  $t$ , and any other variable  $Y$  continues to be substituted as in the originally considered decomposition of  $w$ . Thus,  $t$  has  $k + 1$  different decompositions with respect to  $\alpha$ , contrary to the assumption that  $\alpha$  has the degree of ambiguity  $k$ .

*Case 2:* The variable  $Z$  has only one occurrence in  $\alpha$  and this occurrence is neither the leftmost nor the rightmost occurrence of a variable in  $\alpha$ . Hence,  $\alpha = X \beta Z \gamma Y$ , where  $\beta, \gamma$  are patterns,  $X, Y, Z$  variables (possibly  $X = Y$ ) and  $Z$  does not occur in  $X \beta \gamma Y$ . Again, assume to the contrary that  $\alpha$  has the degree of ambiguity  $k$ , where  $k < \infty$ , and let  $w$  be a terminal word that can be decomposed in  $k$  different ways with respect to  $\alpha$ . Hence, there are  $p_i, q_i, r_i, i = 1, \dots, k$ , such that  $w = p_i r_i q_i$  and  $Z$  was substituted by  $r_i$ . Consider the morphism  $f$ , such that:  $f(X) = w$ ,  $f(Y) = w$ , and  $f(Q) = u$ , for any  $Q, Q \neq X, Q \neq Y$ , and  $u$  is a fixed, arbitrary terminal word. (Note that, if  $X = Y$ , then  $f$  continues to be well-defined.) Let  $t$  be the terminal word  $f(\alpha)$ . Hence,  $t = wsw$ , where  $s = f(\beta Z \gamma)$ . Note that  $t$  has one decomposition, with respect to  $\alpha$ , corresponding to the morphism  $f$ . Moreover,  $t = p_i r_i q_i s p_i r_i q_i$ , for  $i = 1, \dots, k$ . Each such decomposition of  $t$  is corresponding to the substitution of  $Z$  in  $\alpha$  with the terminal word  $r_i q_i s p_i r_i$ , and the remaining variables are substituted as in the original decomposition of  $w$ . Therefore, altogether,  $t$  has  $k + 1$  different

decompositions with respect to  $\alpha$ , contrary to the assumption that the degree of ambiguity of  $\alpha$  is  $k$ .  $\square$

Theorems 3 and 4 determine the degree of ambiguity of all patterns except patterns  $\alpha$  satisfying each of the following three conditions: (i)  $\alpha$  contains occurrences of at least two variables. (ii)  $\alpha$  contains occurrences of at least two terminals. (iii) Every variable occurs in  $\alpha$  at least twice. Indeed, all tricky cases fall among such patterns  $\alpha$ . Let us consider patterns with two occurrences of two variables, separated by terminal words. Such patterns belong to one of the three types

$$X w_1 Y w_2 X w_3 Y, \quad X w_1 Y w_2 Y w_3 X, \quad X w_1 X w_2 Y w_3 Y.$$

We mention without proof that the first two types are always of degree of ambiguity 1 or  $\infty$ , whereas a finite degree  $\neq 1$  is possible in the third type. We will return to this matter in Section 5.

The following theorem serves as a basis in many constructions.

**THEOREM 5:** *The pattern  $X a Y X b Y$  is unambiguous.*

*Proof:* Assume that for some terminal words  $u_1, v_1, u_2, v_2$  there is the equality:

$$(1) \quad u_1 a v_1 u_1 b v_1 = u_2 a v_2 u_2 b v_2$$

Note that  $|u_i a v_i| = |u_i b v_i|$ ,  $i = 1, 2$ . Thus, the equality (1) leads to the next two equalities:

$$(2) \quad u_1 a v_1 = u_2 a v_2 \quad \text{and} \quad u_1 b v_1 = u_2 b v_2$$

Without loss of generality, we can assume that  $|u_1| \leq |u_2|$  and, consequently,  $|v_1| \geq |v_2|$ . Thus, there are terminal words  $u_3$  and  $v_3$  such that  $u_2 = u_1 u_3$  and  $v_1 = v_3 v_2$ . Hence, from (2) we deduce that:

$$a v_3 = u_3 a \quad \text{and} \quad b v_3 = u_3 b.$$

The above system of equations has the unique solution  $u_3 = v_3 = \lambda$ . Therefore, we obtain  $u_1 = u_2$  and  $v_1 = v_2$  and thus, the pattern is unambiguous.  $\square$

*Composition* can be applied to patterns in the natural fashion: variables are uniformly substituted by patterns. If in the pattern of Theorem 5 the variable

$X$  is replaced by  $X_1 a Y_1 X_1 b Y_1$  (that is, the original pattern with renamed variables) and the variable  $Y$  is left unchanged, we obtain the pattern

$$X_1 a Y_1 X_1 b Y_1 a Y X_1 a Y_1 X_1 b Y_1 b Y.$$

Clearly, also this pattern is unambiguous, by Theorem 5. In fact, the next theorem is a corollary of Theorem 5.

**THEOREM 6:** *Compositions of unambiguous patterns are unambiguous. Unambiguous patterns of arbitrarily many variables can be effectively constructed.*

#### 4. DECIDABILITY

Using the general theorem of Makanin, [9], the following results can be obtained quite independently of our other results.

**THEOREM 7:** *The following problems are decidable, given a pattern  $\alpha$  and a natural number  $k$ . Is the degree of ambiguity of  $\alpha$  equal to  $k$ , greater than  $k$  or less than  $k$ ? Consequently, it is decidable whether or not  $\alpha$  is unambiguous.*

*Proof:* It was shown in [4] how Makanin's decidability can be extended to concern systems of equations and *inequalities*. Inequalities  $x \neq x'$  are essential in expressing that a given equation possesses two solutions. The details of the argument are left to the reader.  $\square$

Theorem 7 does not yield a method of deciding whether or not the degree of ambiguity of  $\alpha$  is  $\infty$ . Indeed, this is an open decision problem. As regards decision methods for pattern languages, the results of Theorems 1 and 7 can be combined for  $NE$ -patterns. The situation is trickier for  $E$ -patterns. In fact, even the decidability of the equivalence problem is open for  $E$ -patterns.

We mention, finally, that Theorem 6 gives a simple way of going from a system of equations to a single equation. Consider a system of equations

$$(*) \quad \alpha_i = \beta_i, \quad i = 1, \dots, n,$$

where  $\alpha_i$  and  $\beta_i$  may contain variables and constants (that is, terminals). Choose an unambiguous pattern  $P(X_1, \dots, X_n)$  of  $n$  variables  $X_i$ . Then (\*) has a solution exactly in the case the equation

$$P(\alpha_1, \dots, \alpha_n) = P(\beta_1, \dots, \beta_n)$$

has a solution.

## 5. FINITE DEGREE OF AMBIGUITY

It is rather difficult to exhibit patterns with the degree of ambiguity  $k > 1$ , where  $k$  is finite. Indeed, it was our conjecture for a long time that 1 and  $\infty$  are the only possible degrees.

*Notations:* If  $t \in \Sigma^*$  then  $first(t)$  ( $last(t)$ ) denotes the leftmost (rightmost) letter of  $t$ . Moreover,  $pref(t)$  ( $suff(t)$ ) is the set of all proper prefixes (suffixes) of  $t$ .

**DEFINITION 8:** Let  $\alpha = \alpha(X_1, \dots, X_n)$  be a pattern. A *nontrivial solution* of the equation:

$$\alpha(X_1, \dots, X_n) = \alpha(X'_1, \dots, X'_n)$$

is a  $2n$ -tuple of terminal words,  $(x_1, \dots, x_n, x'_1, \dots, x'_n)$ , such that:

$$\alpha(x_1, \dots, x_n) = \alpha(x'_1, \dots, x'_n)$$

and, moreover,  $(x_1, \dots, x_n) \neq (x'_1, \dots, x'_n)$ .  $\square$

**LEMMA 9:** *The pattern,*

$$\alpha = XuvXvwuYuvwY$$

*has the degree of ambiguity 2, if  $u, v, w$  are nonempty words over the alphabet  $\Sigma = \{a, b, c\}$  such that:*

- (i)  $|u| = |v| = |w|$ ,
- (ii)  $first(t) = last(t)$ , for any  $t \in \{u, v, w\}$ ,
- (iii)  $pref(uvw) \cap suf(uvw) = \emptyset$  and  $pref(v) \cap suf(w) = \emptyset$ ,
- (iv) if  $t_1 \in pref(uv)$  and  $t_2 \in suf(uv)$ , then  $t_1 t \neq t t_2$  for any  $t \in \Sigma^*$ .

*Proof:* First, observe that the pattern  $\alpha$  is ambiguous. For example:

$$\begin{aligned} \alpha(wu, uvwvw) &= \overbrace{wu} uv \overbrace{wu} vwu \overbrace{uvwvw} uvw \overbrace{uvwvw} \\ &= \overbrace{wuuvw} uv \overbrace{wuuvw} vwu \overbrace{vw} uvw \overbrace{vw} \\ &= \alpha(wuuvw, vw). \end{aligned}$$

It remains to prove that there is no terminal word  $t$  that has 3 or more decompositions with respect to  $\alpha$ .

**CLAIM  $A_1$ :** *The general form of nontrivial solutions of the equation:*

$$\alpha(X_1, Y_1) = \alpha(X_2, Y_2)$$

is:

$$\begin{aligned}x_1 &= w (\nu uvw)^i \zeta, & y_1 &= \mu uvw (\nu vwu)^j \eta, \\x_2 &= w (\nu uvw)^i \zeta uvw \nu, & y_2 &= (\nu vwu)^j \eta,\end{aligned}$$

where  $\zeta$  is a proper prefix of  $\nu uvw$  or  $\lambda$ ,  $\eta$  is a proper prefix of  $\nu vwu$  or  $\lambda$ ,  $i, j \geq 0$  and, moreover,

$$\zeta vwu \mu = \nu uvw \zeta \quad \text{and} \quad \eta uvw \mu = \nu vwu \eta.$$

*Proof of Claim A<sub>1</sub>:* Assume that  $x_1, y_1, x_2, y_2$  are nonempty words over  $\Sigma$  such that  $|x_1| < |x_2|$  and

$$(1) \quad x_1 uvx_1 vwu y_1 uvw y_1 = x_2 uvx_2 vwu y_2 uvw y_2.$$

Hence, there are  $x'_2, y'_1 \in \Sigma^+$  such that  $x_2 = x_1 x'_2$ ,  $y_1 = y'_1 y_2$  and  $|x'_2| = |y'_1| > 0$ . It follows from (1) that

$$(2) \quad uvx_1 vwu y'_1 y_2 uvw y'_1 = x'_2 uvx_1 x'_2 vwu y_2 uvw.$$

Reading in the above equality a prefix of length  $|x'_2 uv|$  and a suffix of length  $|uvw y'_1|$ , we obtain

$$(3) \quad uvx_3 = x'_2 uv \quad \text{and} \quad uvw y'_1 = y_3 uvw$$

for some  $x_3, y_3 \in \Sigma^+$ .

Note that  $x'_2$  cannot be a proper prefix of  $uv$ . (Otherwise  $x_3$  is a proper suffix of  $uv$  and for  $t = uv$  this contradicts condition (iv).)

Assume that  $x'_2 = uv$ . From (i) and (3) (second equality) we deduce that  $y'_1 = vw$  and hence  $uvw = y_3 u$ . Using (i) it follows that  $w = u$ , a contradiction.

Now, assume that  $x'_2 = uv p_1$ ,  $y'_1 = q_2 vw$  for some words  $p_1, q_2$ , with  $0 < |p_1| = |q_2| < |u|$ . From the second equality of (3) we obtain

$$(4) \quad uvw q_2 = y_3 u.$$

Hence,  $q_2$  is a proper suffix of  $u$ , i. e.,  $u = q_1 q_2$  for some  $q_1 \in \Sigma^+$ . The equality (4) becomes  $uvw = y_3 q_1$ . Thus  $q_1$  is a proper suffix of  $w$  and a proper prefix of  $u$ . But this contradicts the first part of condition (iii).

Concluding, we deduce that  $|x'_2| = |y'_1| \geq |uvw|$ . Hence,  $x'_2 = uvg$ ,  $y'_1 = huvw$ , for some  $g, h \in \Sigma^*$  such that  $|g| = |h| + |w|$ . Therefore, the relation (2) becomes:

$$x_1 vwuhuvwy_2 uvwh = guvx_1 uvgvwy_2.$$

Considering prefixes and suffixes of the same length in the above equality, we obtain:

$$(5) \quad x_1 vwuh = guvx_1 \quad \text{and} \quad wy_2 uvwh = gvwy_2.$$

From the second equality in (5), it follows that  $g$  cannot be a proper prefix of  $w$ . (Otherwise, after simplification, a proper prefix of  $v$  is equal to a proper suffix of  $w$ , contrary to the second part of condition (iii).)

Hence,  $g = wg'$ , for some  $g' \in \Sigma^*$ , with  $|g'| = |h|$ . Thus, the relations (5) become:

$$(6) \quad x_1 vwuh = wg'uvx_1 \quad \text{and} \quad y_2 uvwh = g'vwuy_2.$$

By a similar argument,  $x_1$  cannot be a proper prefix of  $w$ . Thus,  $x_1 = wx''$ , for some  $x'' \in \Sigma^*$ . Denote  $y_2 = y''$ . From (6), it follows that

$$(7) \quad x''vwuh = g'uvwx'' \quad \text{and} \quad y''uvwh = g'vwuy''.$$

Denoting  $g' = \nu$  and  $h = \mu$ , the relations (7) become:

$$x''(vwu\mu) = (\nu uvw)x'' \quad \text{and} \quad y''(uvw\mu) = (\nu vwu)y''.$$

where  $|\nu| = |\mu|$ .

Using a well-known result often called Lyndon's Theorem, it follows that

$$(8) \quad x'' = (\nu uvw)^i \zeta \quad \text{and} \quad y'' = (\nu vwu)^j \eta$$

where  $i, j \geq 0$  and:

$$(9) \quad \nu uvw \zeta = \zeta vwu \mu, \quad \eta uvw \mu = \nu vwu \eta.$$

Using (8) and (9) we obtain the general form of  $x_1, x_2, y_1, y_2$  as in the Claim  $A_1$ .

End of the proof of Claim  $A_1$ .

Assume now the existence of an intermediate solution,  $(x', y')$ , of the equation:

$$\alpha(X_1, Y_1) = \alpha(X_2, Y_2).$$

i. e.,  $|x_1| < |x'| < |x_2|$ ,  $|y_2| < |y'| < |y_1|$  and

$$\alpha(x_1, y_1) = \alpha(x', y') = \alpha(x_2, y_2).$$

CLAIM  $A_2$ : If  $x_1, x', x_2, y_1, y', y_2$  are terminal words with the above properties, then:

$$\begin{aligned} x_1 &= w \zeta, & y_1 &= \rho uvw \delta uvw \eta, \\ x' &= w \zeta uvw \theta, & y' &= \delta uvw \eta, \\ x_2 &= w \zeta uvw \theta uvw \pi, & y_2 &= \eta, \end{aligned}$$

where  $\theta, \delta, \eta, \rho, \zeta, \pi$  are terminal words that satisfy the system of equations  $E = \{e_1, e_2, e_3, e_4, e_5\}$  with: —

$$\begin{aligned} (e_1) \quad & \theta vwu \delta = \delta uvw \theta \\ (e_2) \quad & \eta uvw \rho = \theta uvw \eta \\ (e_3) \quad & \zeta vwu \rho = \theta uvw \zeta \\ (e_4) \quad & \zeta uvw \delta = \pi uvw \zeta \\ (e_5) \quad & \eta uvw \delta = \pi vwu \eta \end{aligned}$$

and, moreover,  $|\pi| = |\delta|$  and  $|\theta| = |\rho|$ .

*Proof of Claim  $A_2$* : From Claim  $A_1$  we obtain that

$$(10) \quad x' = w (\nu uvw)^i \zeta \varphi \quad \text{and} \quad y' = \psi (\nu vwu)^j \eta,$$

for some nonempty terminal words  $\varphi$  and  $\psi$ , such that  $\varphi$  is a proper prefix of  $uvw\nu$ ,  $\psi$  is a proper suffix of  $\mu uvw$  and  $|\varphi| + |\psi| = |\nu| + |uvw| = |\mu| + |uvw|$ .

Moreover, we obtain that:

$$(11) \quad \alpha(x', y') = w (\nu uvw)^i \zeta \varphi uvw (\nu uvw)^i \zeta \varphi vwu \psi (\nu vwu)^j \eta uvw \psi (\nu vwu)^j \eta.$$

From the equality  $\alpha(x_1, y_1) = \alpha(x', y')$  and (11) it follows that:

$$(12) \quad \begin{aligned} uvw (\nu uvw)^i \zeta vwu \mu uvw (\nu vwu)^j \eta uvw \mu uvw \\ = \varphi uvw (\nu uvw)^i \zeta \varphi vwu \psi (\nu vwu)^j \eta uvw \psi. \end{aligned}$$

Note that  $\varphi$  cannot be a proper prefix of  $uvw$ . (Otherwise, after simplification of  $\varphi$ , it follows that a proper suffix of  $uvw$  must be a proper prefix of  $uvw$ , contrary to the condition (iii).)

Hence,  $\varphi = uvw\theta$  and  $\psi = \delta uvw$ , for some words  $\theta, \delta \in \Sigma^*$  such that  $\theta$  is a prefix of  $\nu$  with  $|\theta| < |\nu|$ , and  $\delta$  is a suffix of  $\mu$  with  $|\delta| < |\mu|$ .

The equality (12) becomes:

$$(13) \quad (\nu uvw)^i \zeta vwu \mu uvw (\nu vwu)^j \eta uvw \mu \\ = \theta uvw (\nu uvw)^i \zeta uvw \theta vwu \delta uvw (\nu vwu)^j \eta uvw \delta$$

where:

$$(14) \quad |\theta| + |\delta| + 3|u| = |\mu| = |\nu| \quad \text{and} \quad |\theta| < |\mu|$$

Note that  $|\mu| \geq |\delta| + 3|u| = |uvw\delta|$ .

Reading the suffixes of the equality (13), we deduce that:

$$(15) \quad \mu = \rho uvw \delta$$

for some word  $\rho$ , with  $|\rho| = |\theta|$ . It follows from (13) (reading prefixes/suffixes of the same length) that

$$(16) \quad (\nu uvw)^i \zeta vwu \rho = \theta uvw (\nu uvw)^i \zeta$$

and

$$\delta uvw (\nu vwu)^j \eta uvw \rho = \theta vwu \delta uvw (\nu vwu)^j \eta.$$

Now, assuming that  $i \geq 1$  and using (16), we have:

$$(\nu uvw)^{i-1} \nu uvw \zeta vwu \rho = \theta uvw (\nu uvw)^{i-1} \nu uvw \zeta.$$

Using the equality  $\zeta vwu \mu = \nu uvw \zeta$  (see Claim  $A_1$ ) and using (15), the above equality becomes:

$$(\nu uvw)^{i-1} \zeta vwu \rho uvw \delta vwu \rho \\ = \theta uvw (\nu uvw)^{i-1} \zeta vwu \rho uvw \delta.$$

Note that  $|\delta vwu \rho| = |\rho uvw \delta|$ . From the suffixes of the above equality, we obtain:

$$uvw \delta vwu \rho = vwu \rho uvw \delta.$$

But, this is contradicts  $u \neq v$ .

Therefore, the only possibility is  $i = 0$ . By a similar proof, we obtain that also  $j = 0$ .

Note that, from (14),  $|\nu| \geq |\theta| + 3|u| = |\theta uvw|$  and from prefixes of the equality  $\alpha(x', y') = \alpha(x_2, y_2)$  we obtain that  $\nu = \theta uvw\pi$ , for some word  $\pi$ , with  $|\pi| = |\delta|$ .

Thus, from Claim  $A_1$ , (10) and (13), we obtain the first part of Claim  $A_2$ , *i. e.*:

$$\begin{aligned} x_1 &= w\zeta, & y_1 &= \rho uvw \delta uvw \eta \\ x' &= w\zeta uvw \theta & y' &= \delta uvw \eta, \\ x_2 &= w\zeta uvw \theta uvw \pi, & y_2 &= \eta. \end{aligned}$$

It remains to verify the conditions satisfied by the terminal words  $\theta$ ,  $\delta$ ,  $\eta$ ,  $\rho$ ,  $\zeta$ ,  $\pi$ , *i. e.*, the second part of Claim  $A_2$ .

Reading prefixes and suffixes of the same length in the equalities

$$\alpha(x_1, y_1) = \alpha(x', y') = \alpha(x_2, y_2)$$

we obtain the following 5 equalities:

$$(17) \quad \zeta vwu \rho = \theta uvw \zeta$$

(from prefixes in  $\alpha(x_1, y_1) = \alpha(x', y')$ ),

$$(18) \quad \zeta uvw \theta vwu \delta = \pi uvw \zeta uvw \theta$$

(from prefixes in  $\alpha(x', y') = \alpha(x_2, y_2)$ ),

$$(19) \quad \zeta vwu \rho uvw \delta = \theta uvw \pi uvw \zeta$$

(from prefixes in  $\alpha(x_1, y_2) = \alpha(x_2, y_2)$ ),

$$(20) \quad \eta uvw \rho uvw \delta = \theta uvw \pi vwu \eta$$

(from suffixes in  $\alpha(x_1, y_1) = \alpha(x_2, y_2)$ ),

$$(21) \quad \eta uvw \delta = \pi vwu \eta$$

(from suffixes in  $\alpha(x', y') = \alpha(x_2, y_2)$ ).

From (17) and (19) we obtain:

$$(22) \quad \zeta uvw \delta = \pi vwu \zeta.$$

From (20) and (21) it follows that

$$(23) \quad \eta uvw \rho = \theta uvw \eta.$$

From (22) and (18) we obtain:

$$(24) \quad \theta vwu \delta = \delta uvw \theta.$$

Therefore, from (24), (23), (17), (22) and (21) we can conclude that the terminal words  $\theta, \delta, \eta, \zeta, \pi, \rho$  should satisfy the following set  $E$  of equations:

$$(e_1) \quad \theta vwu \delta = \delta uvw \theta$$

$$(e_2) \quad \eta uvw \rho = \theta uvw \eta$$

$$(e_3) \quad \zeta vwu \rho = \theta uvw \zeta$$

$$(e_4) \quad \zeta uvw \delta = \pi uvw \zeta$$

$$(e_5) \quad \eta uvw \delta = \pi vwu \eta$$

with the supplementary conditions  $|\theta| = |\rho|$  and  $|\pi| = |\delta|$ .

*End of the proof of CLaim A<sub>2</sub>.*

We will complete the proof of Lemma 9, proving the following:

**CLAIM A<sub>3</sub>:** *The system E from Claim A<sub>2</sub>, with the supplementary conditions  $|\theta| = |\rho|$  and  $|\pi| = |\delta|$ , does not have solutions.*

*Proof of Claim A<sub>3</sub>:* First, it is easy to observe that from conditions (ii) and (iii) of Lemma A, it follows that for any two different words  $t, t' \in \{u, v, w\}$ ,  $first(t) \neq first(t')$  and  $last(t) \neq last(t')$ . Hence, without loss of generality, we can assume that:  $first(u) = last(u) = a$ ,  $first(v) = last(v) = b$ ,  $first(w) = last(w) = c$ .

Consider now the following two sets:

$$A = \{\theta, \delta, \eta, \zeta, \pi\}$$

and

$$B = \{\theta, \delta, \eta, \zeta, \rho\}.$$

From the system  $E$ , we can deduce that the words from the set  $A$  have the following important property: If  $t \in A$ , then for any  $t' \in A$ ,  $first(t) = first(t')$ . Similarly, if  $t \in B$ , then for any  $t' \in B$ ,  $last(t) = last(t')$ . The above property can be extended for prefixes of words from  $A$  (for suffixes of words from  $B$ ).

Thus, we'll start a discussion concerning the possible shortest word among the words:  $\theta, \delta, \eta, \zeta, \pi, \rho$ .

Case  $I_1$ : The shortest word is  $\theta$  and, moreover,

$$|\theta| < |\delta| = |\pi|, |\theta| < |\eta|, |\theta| < |\zeta|.$$

Note that  $\theta, \rho \in B$ ,  $|\theta| = |\rho|$  and therefore  $\theta = \rho$ . Let's denote:  $\theta = \rho = \xi_1$ . Hence,  $\delta = \delta'_1 \xi_1$ ,  $\zeta = \zeta'_1 \xi_1$ , for some nonempty words,  $\delta'_1, \zeta'_1$ . The equalities  $(e_1)$ ,  $(e_3)$  and  $(e_4)$  from  $E$  become:

$$(e_{11}) \quad \xi_1 vwu \delta'_1 = \delta'_1 \xi_1 uvw$$

$$(e_{31}) \quad \zeta'_1 \xi_1 vwu = \xi_1 uvw \zeta'_1$$

$$(e_{41}) \quad \zeta'_1 \xi_1 uvw \delta'_1 = \pi uvw \zeta'_1$$

From  $(e_{11})$  it follows that  $last(\delta'_1) = last(w) = c$ . From  $(e_{31})$  we obtain that  $last(\zeta'_1) = last(u) = a$  and from  $(e_{41})$  it follows that  $last(\delta'_1) = last(\zeta'_1)$ , i. e.  $a = c$ , a contradiction.

Case  $I_2$ : The shortest word is  $\delta$  and, moreover,

$$|\delta| < |\theta| = |\rho|, |\delta| < |\eta|, |\delta| < |\zeta|.$$

Note that  $\delta, \pi \in A$ ,  $|\delta| = |\pi|$  and therefore  $\delta = \pi$ . Let's denote:  $\delta = \pi = \xi_2$ . Hence,  $\theta = \xi_2 \theta'_2$ ,  $\eta = \xi_2 \eta'_2$ , for some nonempty words,  $\theta'_2, \eta'_2$ . The equalities  $(e_1)$ ,  $(e_2)$  and  $(e_5)$  from  $E$  become:

$$(e_{12}) \quad \theta'_2 vwu \xi_2 = uvw \xi_2 \theta'_2$$

$$(e_{22}) \quad \eta'_2 uvw \rho = \theta'_2 uvw \eta'_2$$

$$(e_{52}) \quad \eta'_2 uvw \xi_2 = vwu \xi_2 \eta'_2$$

From  $(e_{12})$  it follows that  $first(\theta'_2) = first(u) = a$ . From  $(e_{52})$  we obtain that  $first(\eta'_2) = first(v) = b$  and from  $(e_{22})$  it follows that  $first(\eta'_2) = first(\theta'_2)$ , i. e.  $a = b$ , a contradiction.

Case  $I_3$ : The shortest word is  $\eta$ .

Denote  $\eta = \xi_3$  and observe that  $\delta = \delta'_3 \xi_3$ ,  $\rho = \rho'_3 \xi_3$ ,  $\zeta = \zeta'_3 \xi_3$ , for some nonempty words,  $\delta'_3, \rho'_3, \zeta'_3$ . The equalities  $(e_2)$ ,  $(e_3)$ ,  $(e_4)$  and  $(e_5)$  from  $E$  become:

$$(e_{23}) \quad \xi_3 uvw \rho'_3 = \theta \xi_3 uvw$$

$$(e_{33}) \quad \zeta'_3 \xi_3 vwu \rho'_3 = \theta \xi_3 uvw \zeta'_3$$

$$(e_{43}) \quad \zeta'_3 \xi_3 uvw \delta'_3 = \pi uvw \zeta'_3$$

$$(e_{53}) \quad \xi_3 uvw \delta'_3 = \pi vwu$$

From  $(e_{33})$  and  $(e_{43})$  it follows that  $last(\rho'_3) = last(\zeta'_3) = last(\delta'_3)$ . From  $(e_{23})$  we obtain that  $last(\rho'_3) = c$  and from  $(e_{53})$  it follows that  $last(\delta'_3) = a$ . Therefore, we obtain  $a = c$ , a contradiction.

*Case I<sub>4</sub>*: The shortest word is  $\zeta$ .

Denote  $\zeta = \xi_4$  and observe that  $\theta = \xi_4 \theta'_4$ ,  $\eta = \xi_4 \eta'_4$ ,  $\pi = \xi_4 \pi'_4$ , for some nonempty words,  $\theta'_4$ ,  $\eta'_4$ ,  $\pi'_4$ . The equalities  $(e_2)$ ,  $(e_3)$ ,  $(e_4)$  and  $(e_5)$  from  $E$  become:

$$(e_{24}) \quad \eta'_4 uvw \rho = \theta'_4 uvw \xi_4 \eta'_4$$

$$(e_{34}) \quad vwu \rho = \theta'_4 uvw \xi_4$$

$$(e_{44}) \quad uvw \xi_4 \delta = \pi'_4 uvw \xi_4$$

$$(e_{54}) \quad \eta'_4 uvw \xi_4 \delta = \pi'_4 vwu \xi'_4$$

From  $(e_{24})$  and  $(e_{54})$  it follows that  $first(\eta'_4) = first(\theta'_4) = first(\pi'_4)$ . From  $(e_{34})$  we obtain that  $first(\eta'_4) = b$  and from  $(e_{44})$  it follows that  $first(\pi'_4) = a$ . Therefore, we obtain  $a = b$ , a contradiction.

Now we consider the situation when the shortest length of a word in the set  $C = \{\zeta, \eta, \theta, \delta\}$  is reached by exactly 2 words from this set. There are 6 possible combinations as follows:

*Case J<sub>1</sub>*: The shortest words are  $\zeta$  and  $\eta$ , i. e.  $|\zeta| = |\eta|$  and all others words from the set  $C$  are longer.

Because  $\zeta, \eta \in A$ , it follows that  $\zeta = \eta = \xi_5$  and  $\theta = \xi_5 \theta'_5$  for nonempty word  $\theta'_5$ .

The equalities  $(e_2)$  and  $(e_3)$  from  $E$  become:

$$(e_{25}) \quad uvw \rho = \theta'_5 uvw \xi_5$$

$$(e_{35}) \quad vwu \rho = \theta'_5 uvw \xi_5$$

Hence,  $first(\theta'_5) = a$  and  $first(\theta'_5) = b$ , a contradiction.

*Case J<sub>2</sub>*: The shortest words from  $C$  are  $\zeta$  and  $\theta$ .

The equality  $(e_3)$  from  $E$  becomes:

$$(e_{36}) \quad vwu \rho = uvw \zeta$$

Thus,  $first(v) = first(u)$ , a contradiction.

*Case J<sub>3</sub>*: The shortest words from  $C$  are  $\zeta$  and  $\delta$ .

Because  $\zeta, \delta \in A$ , and, moreover,  $|\delta| = |\pi|$  it follows that  $\zeta = \delta = \pi = \xi_7$  and  $\theta = \xi_7 \theta'_7$  for some nonempty word  $\theta'_7$ .

The equalities  $(e_1)$  and  $(e_3)$  from  $E$  become:

$$(e_{17}) \quad \theta'_7 vwu \xi_7 = uvw \xi_7 \theta'_7$$

$$(e_{37}) \quad vwu \rho = \theta'_7 uvw \xi_7$$

Hence,  $first(\theta'_7) = a$  and  $first(\theta'_7) = b$ , a contradiction.

*Case J<sub>4</sub>*: The shortest words from  $C$  are  $\eta$  and  $\theta$ .

Because  $\eta, \theta \in B$ , and, moreover,  $|\theta| = |\rho|$  it follows that  $\eta = \theta = \rho = \xi_8$  and  $\delta = \delta'_8 \xi_8$  for some nonempty word  $\delta'_8$ .

The equalities  $(e_1)$  and  $(e_5)$  from  $E$  become:

$$(e_{18}) \quad \xi_8 vwu \delta'_8 = \delta'_8 \xi_8 uvw$$

$$(e_{58}) \quad \xi_8 uvw \delta'_8 = \pi vwu$$

Hence,  $last(\delta'_8) = c$  and  $last(\delta'_8) = a$ , a contradiction.

*Case J<sub>5</sub>*: The shortest words from  $C$  are  $\eta$  and  $\delta$ .

The equality  $(e_5)$  from  $E$  becomes:

$$(e_{59}) \quad \eta uvw = \pi vwu$$

Thus,  $last(w) = last(u)$ , a contradiction.

*Case J<sub>6</sub>*: The shortest words from  $C$  are  $\theta$  and  $\delta$ .

The equality  $(e_1)$  from  $E$  becomes:

$$(e_{110}) \quad vwu = uvw$$

Thus, we obtain again a contradiction.

All the remaining cases lead to contradictions, because one can argue:  $\eta \neq \pi$  (see  $(e_5)$ ),  $\delta \neq \theta$  (see  $(e_1)$ ),  $\zeta \neq \rho$  (see  $(e_3)$ ) and also,  $\zeta \neq \theta$  (see  $(e_3)$ ).

This completes the proof of Claim  $A_3$  and the proof of Lemma 9, too.  $\square$

**COROLLARY 10:** *The pattern,*

$$\beta = X ab X bca Y abc Y$$

*has the degree of ambiguity 2.*

*Proof:* Take in Lemma 9,  $u = a$ ,  $v = b$  and  $w = c$  and note that conditions (i)-(iv) are satisfied.  $\square$

*Notations:* Let  $\gamma$  be the unambiguous pattern from Theorem 5 and let  $\alpha_1$  be the pattern  $\alpha$  from Lemma 9, for:

$$u = a^4 ba^8 ba^4, \quad v = ba^7 bcacaabcab, \quad w = caabcabcabcabcabc.$$

**LEMMA 11:** *The pattern,*

$$\sigma = \gamma(\alpha_1(X, Y), X\beta(Q, R))$$

*with the variables  $X, Y, Q, R$  has the degree of ambiguity 3.*

*Proof:* Let  $\alpha_0$  be the pattern from Lemma 9 for some fixed values  $u_0, v_0, w_0$  of the terminal words,  $u, v, w$ . Consider the pattern:

$$\sigma' = \gamma(\alpha_0(X, Y), X\beta(Q, R))$$

where  $\gamma$  is the unambiguous pattern from Theorem 5 and  $\beta$  is the pattern from Corollary 10.

CLAIM  $C_1$ : *The pattern  $\sigma'$  has the degree of ambiguity at most 3.*

*Proof of Claim  $C_1$ :* Observe that the equality

$$\sigma'(X, Y, Q, R) = t,$$

for some terminal word  $t \in \Sigma^*$ , leads to the equalities:

$$\alpha_0(X, Y) = t_1, \quad X\beta(Q, R) = t_2,$$

for some unique words  $t_1, t_2 \in \Sigma^*$  ( $\gamma$  is an unambiguous pattern).

Consider now the following two possible situations:

*Case  $K_1$ :* The equation  $\alpha_0(X, Y) = t_1$  has at most one solution,  $(x_0, y_0)$ .

It follows that the equation  $x_0\beta(Q, R) = t_2$  leads to at most one value of  $\beta(Q, R)$ . Consequently, because  $\beta$  has the degree of ambiguity 2, there are at most two pairs of terminal words,  $(q', r'), (q'', r'')$  that satisfy the equality  $x_0\beta(Q, R) = t_2$ . Hence, in this case,  $t$  has at most 2 possible decompositions with respect to  $\sigma'$ .

*Case  $K_2$ :* The equation  $\alpha_0(X, Y) = t_1$  has 2 solutions,  $(x_1, y_1), (x_2, y_2)$ .

It follows from Claim  $A_1$  (see the proof of Lemma 9) that

$$x_1 = w_0(\nu u_0 v_0 w_0)^i \zeta, \quad x_2 = w_0(\nu u_0 v_0 w_0)^i \zeta u_0 v_0 w_0 \nu,$$

for some terminal words  $\nu, \zeta$ , and  $i \geq 0$ .

From the equality  $x_2\beta(Q, R) = t_2$ , we obtain

$$(I) \quad \beta(Q, R) = t'_2$$

and from the equality  $x_1\beta(Q, R) = t_2$  it follows that

$$(II) \quad \beta(Q, R) = u_0 v_0 w_0 \nu t'_2.$$

The equation (I) has at most 2 solutions, say  $(q_1, r_1)$  and  $(q_2, r_2)$ . The equation (II) has at most 1 solution  $(q_3, r_3)$ , because the value of  $\beta(Q, R)$  has a prefix that starts with  $u_0 v_0 w_0$ . (See the proof of Lemma 9, Claim  $A_1$ .)

Therefore, altogether,  $t$  has at most 3 decompositions with respect to  $\sigma'$ , corresponding to:

$$(x_1, y_1, q_3, r_3), \quad (x_2, y_2, q_1, r_1), \quad (x_2, y_2, q_2, r_2).$$

*End of the proof of Claim C<sub>1</sub>.*

CLAIM C<sub>2</sub>: For the values:

$$u_0 = a^4 ba^8 ba^4, \quad v_0 = ba^7 bcacaabcab, \quad w_0 = caabcbcabcbcbcbcb,$$

the pattern  $\sigma'$  ( $= \sigma$ ) has the degree of ambiguity 3.

*Proof of Claim C<sub>2</sub>*: From Lemma 9, we have that:

$$\beta(ca, abcbc) = \beta(caabc, bc) = \Psi,$$

where  $\Psi = caabcbcaabcbcbcbcbcb$ .

In a similar way, we obtain that:

$$\alpha_0(wu, uvvww) = \alpha_0(wuuvw, v\bar{w}) = \Psi',$$

where  $\Psi' = h(\Psi)$  and the morphism  $h$  is defined by:  $h(a) = u$ ,  $h(b) = v$ ,  $h(c) = w$ .

Notations:

$$\begin{aligned} x_1 &= wu, & y_1 &= uvvww, \\ x_2 &= wuuvw, & y_2 &= vw, \\ q_1 &= ca, & r_1 &= abcbc, \\ q_2 &= caabc, & r_2 &= bc. \end{aligned}$$

Note that in all the above solutions we have  $\nu = \lambda$ .

Now we'll define the values of  $u$ ,  $v$ ,  $w$  in such a way that the equation

$$(*) \quad \beta(Q, R) = uvw \beta(q_1, r_1) = uvw \Psi$$

will have a solution  $(q_3, r_3)$ . (Note that this equation has at most 1 solution, because  $u$  is starting with  $a$ .) Moreover, the values of  $u$ ,  $v$ ,  $w$  will satisfy the conditions (i)-(iv) from Lemma 9.

The equation (\*) becomes:

$$(**) \quad Q ab Q bca R abc R = uvw \Psi.$$

For simplicity, we can consider that  $R = \Psi$ . Hence, we obtain from (\*\*):

$$(***) \quad Q ab Q bca \Psi abc = uvw.$$



(i)  $N(P, Z) \leq 1$ , for all  $Z$ .

(ii) There is a  $Z'$  such that  $N(P, Z') > 1$  but the numbers  $N(P, Z)$  possess an upper bound, that is, for some  $k$ ,  $N(P, Z) \leq k$  for all  $Z$ .

(iii) The numbers  $N(P, Z)$  possess no upper bound, that is, for every  $k$ ,  $N(P, Z') > k$  holds for some  $Z'$ .

We have been able to exhibit extensive classes of patterns for which (i) or (iii) holds. For instance, (i) holds if the number of variables  $n = 1$ , and (iii) holds if  $n > 1$  and  $P$  is "linear" with respect to some variable (see Theorems 3 and 4). According to our main results (Theorems 12 and 13), also (ii) is possible. However, it is an open problem, and in our estimation a very fundamental one in the theory of word equations, whether *all finite degrees* of ambiguity can actually be constructed. By theorem 6, it suffices to carry out the construction for prime degrees. We conjecture that such a construction is possible. Since arbitrarily large degrees can be obtained (Theorem 13), it would seem rather strange if some degrees were "missing".

The most interesting open decision problem is the decidability status of (iii). "Almost all" patterns seem to satisfy (iii), and yet Makanin's Theorem is not directly applicable to this case.

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