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PEBBLING DYNAMIC GRAPHS IN MINIMAL SPACE (*)

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Abstract. — Pebble game on dynamic graphs is studied as an abstract model of the incremental computations. Extreme explosion of the time complexity is related to small changes in the size of computational graphs. A class of dags \( \{G_n\} \) of the size \( n \) pebbleable in space \( O(n^{1/3}) \) is exhibited such that the standard pebble game requires superpolynomial time in \( n \) under the restriction that the minimal space is used. Moreover, by deleting just one edge in \( \{G_n\} \) a new class of graphs is obtained for which standard pebble game needs only polynomial time in \( n \) also in the case when the minimal space is used.

Résumé. — On étudie un jeu sur les graphes dynamiques qui constitue un modèle théorique de calcul incrémental. Une explosion considérable de la complexité temporelle a lieu pour de faibles modifications du graphe sur lequel on calcule. Ainsi on construit une classe de graphes acycliques \( \{G_n\} \) de taille \( n \) que l'on peut traiter avec une complexité en espace en \( O(n^{1/3}) \) et qui nécessite un temps superpolynomial en \( n \) si l'espace minimal est utilisé. De plus, en supprimant simplement un arc dans \( \{G_n\} \) une nouvelle classe est obtenue pour laquelle le jeu demande simplement un temps polynomial même si l'on utilise un espace minimal.

INTRODUCTION

Pebbling directed acyclic graphs (shortly dags) introduce an important paradigm in programming. Dags present an abstract computational structure of a given problem and pebbling performs the computation of this problem on the computational graph. Pebbling mediates us to model various types of computations and enables us to investigate the relationship between space and time complexity of these computations by means of a simple combinatorial game played with pebbles on dags. There are well known exploitations of pebbling in gaining time-space trade offs for concrete computational problems in various areas of computer science, such as compilers, database systems, parallel and distributed systems, programming methodology and others. In the theory of pebbling well known results have been obtained

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concerning space complexity and time-space trade offs for pebbling special classes of directed acyclic graphs. A survey of principal classical results and basic applications of pebbling is given by Pippenger (1980). Recent pebbling results are presented by Wilber (1988) and Venkateswaran, Tompa (1989).

Recently much effort has been devoted to the investigation of the incremental computations. We propose to study pebble game on dynamic graphs, acting as a model of incremental computations. From the theoretical point of view we are interested in the relationship between space or time complexity of pebbling and small incremental/decremental changes of graphs.

In this paper we concentrate on space optimal pebbling. We characterize extreme changes in the time complexity with respect to small changes in the size of computational graphs. We present a class of dags \( \{G_n\} \) of the size \( n \) which requires superpolynomial time in \( n \) for pebbling when the minimal space is used. But by deleting just one edge in \( \{G_n\} \) a new class of graphs is obtained which requires only polynomial time for pebbling in the minimal space.

1. BASIC NOTIONS

Throughout this paper we consider only directed acyclic graphs (dags) with in-degree bounded by 2. An input (an output) in a dag is a vertex with in-degree (out-degree) equal to 0. If a dag has exactly one output, then this vertex is called the root of the dag and the dag is called rooted. The size of a dag \( G \) (denoted as \( \text{size} (G) \)) is equal to the number of its edges.

Standard pebble game is performed with pebbles on dags using following rules:

**S-1:** A pebble can be removed from an arbitrary vertex of the graph.

**S-2:** If all direct predecessors of a vertex \( v \) are covered by pebbles, then a pebble can be laid also on the vertex \( v \).

**S-3:** If all direct predecessors of a vertex \( v \) are covered by pebbles, then a pebble can be moved from some predecessor of the vertex \( v \) onto the vertex \( v \).

A configuration is a subset of vertices, comprising just those vertices that have pebbles on them. A configuration is empty if there is no pebble on any vertex. A transition is an ordered pair of configurations \((C_1, C_2)\), where \( C_2 \) follows from \( C_1 \) according to one of the rules \( S-1, S-2, S-3 \). A computation is a sequence of configurations \( C_1, C_2, \ldots, C_m \), where \((C_k, C_{k+1})\) for \( 1 \leq k < m \) is a transition. A complete computation is one that begins and ends with the empty configuration and in which every vertex appears in
some configuration. The time $T$ is the number of transitions for complete computations and the space $S$ is the maximum number of vertices in any configuration for complete computations.

A dag $G$ is called open in a configuration $C$ if and only if there is a path in $G$ from an input to an output such that it does not contain a vertex in $C$. Otherwise $G$ is called closed in $C$.

2. TIME RESULTS FOR THE MINIMAL SPACE PEBBLING ON DYNAMIC GRAPHS

The influence of small changes (performed e.g. by inserting or deleting an edge) in computational graphs onto the time complexity of standard pebbling is studied. As it has been previously noticed, we are interested only in the space optimal pebbling.

First of all we determine the space complexity of standard pebbling in the case when there is only a little change in the underlying dag.

**Proposition 2.1:** Inserting an edge to an arbitrary dag can increase the space complexity of standard pebbling by at most 1.

The proposition directly follows from the fact that if a new dag $\hat{G}$ is obtained from the original dag $G$ by adding an edge $(v, u)$ to $G$, then a complete computation on $\hat{G}$ can be constructed from a complete computation on $G$ by placing a permanent pebble on the vertex $v$.

Now, we present an upper bound on time complexity for pebbling a graph of size $n$, which can be pebbled with $S(n)$ pebbles.

**Proposition 2.2:** Let $G_n$ be a rooted dag with $n$ vertices and let $S(n)$ be the number of pebbles sufficient for pebbling $G_n$. Then for the time $T(n)$ needed to pebble the graph $G_n$ with $S(n)$ pebbles it holds

$$T(n) = 2^{O(S(n) \log n / S(n))}.$$ 

**Proof:** Each complete computation on rooted dags can be transformed onto the complete computation without repeated configurations.

Consider a complete computation with $S(n)$ pebbles in the form starting with $S(n)$ applications of the rule $S-2$, following with a sequence of applications either of the rule $S-3$ or of the rule $S-1$ followed by the rule...
S-2 and ending with $S(n)$ applications of the rule S-1. The worst-case time complexity of these three phases of computation is estimated as follows

$$T(n) \leq S(n) + 2 \cdot \frac{n}{S(n)} + S(n).$$

Applying Stirling formula $n! \geq \sqrt{2\pi n} (n/e)^n$ one can obtain the time estimation

$$T(n) \leq \frac{\sqrt{2}}{\sqrt{\pi} S(n)} \cdot 2^{S(n) \log e \cdot n / S(n)} + 2 \cdot S(n)$$

and hence the asymptotic time-space trade-off formula is in the form

$$T(n) = 2^{O(S(n) \log n / S(n))}.$$

In a special case when a graph can be pebbled with a constant number of pebbles the following upper bound on time complexity can be derived.

**Consequence 2.3:** Let $G_n$ be a rooted dag with $n$ vertices and let $k$ be a constant number of pebbles necessary and sufficient to pebble $G_n$. Then for the time $T(n)$ needed to pebble the graph $G_n$ using $k$ pebbles it holds

$$T(n) = O(n^k).$$

Now, we present a lower bound on the time complexity of the minimal space pebbling up to space $O(n^{1/3})$ which is asymptotically tight to the previously given upper bound. Basic to all graphs discussed later in this paper is the pyramidal graph $P_k = (V, E)$, where

$$V = \{u_{i,j} \mid 1 \leq i \leq j \leq k\}$$

$$E = \{(u_{i,j}, u_{s,t}) \mid s = i + 1, t \in \{j, j + 1\}, 1 \leq i \leq j, s \leq t \leq k\}$$

**Lemma 2.4** (Cook, 1974): Each complete computation of the standard pebble game on the pyramidal graph $P_k$ needs at least $k$ pebbles.

Consider 3 graph components $P_i, R_j, H_l$, where $P_i$ is a pyramidal graph with $i$ inputs, 1 output and $1/2 i (i + 1)$ vertices, $R_j$ is a chain graph with 1 input, 1 output and $j$ vertices and $H_l$ is a combination graph with 1 input, 2 outputs and $l (l + 2)$ vertices. A pyramid $P_i$ is a fragment of rectangular grid. A chain $R_j$ is a sequence of $j$ vertices, where there is an edge oriented from the $k$-th vertex to the $(k + 1)$-st one. A combination graph $H_l$ consists of two pyramids $P_l$ and a chain $R_{2l}$, where the $i$-th vertex of $R_{2l}$ for $1 \leq i \leq l$
is connected with oriented edge to the \(i\)-th input vertex of the first pyramid and the \(j\)-th vertex of \(R_{2l}\) for \(l + 1 \leq j \leq 2l\) is connected with oriented edge to the \((j - l)\)-th input vertex of the second pyramid.

A class of dags \(\{G_{k,m} \mid k \geq 1, m \geq 2\}\) is defined as follows. \(G_{k,m}\) is constructed from \(k\) levels \(U_{1}^{m}, \ldots, U_{k}^{m}\), where the \(i\)-th level \(U_{i}^{m}\) for \(1 \leq i \leq k\) consists of the serial connection of three components \(P_{i}, R_{m}, H_{i}\) (i.e., the output of \(P_{i}\) is connected with oriented edge to the input of \(R_{m}\) and the output of \(R_{m}\) is connected to the input of \(H_{i}\)). The graph \(G_{k,m}\) is completed using the following connections of levels \(U_{i}^{m}\) and \(U_{i+1}^{m}\) for \(1 \leq i < k\). The first (second) output of \(U_{i}^{m}\) is connected with oriented edges to all odd (even) vertices in the components \(R_{m}\) of the level \(U_{i+1}^{m}\). See the Figure 1.

\[
\begin{align*}
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(ii) the minimal number of pebbles needed for complete computation on the graph $G_{k,m}$ is equal to $k$.

Proof: Case (i). The size of the graph $G_{k,m}$ fulfills the following recurrent relation
\[
\text{size}(G_{k,m}) = a_1 \cdot m + \text{size}(G_{k-1,m})
\]
for $k$ in the range $\omega(1)$ to $O(\sqrt{m})$ and some constants $a_1, a_2 > 1$ and
\[
\text{size}(G_{k,m}) = a_3 \cdot k^2 + \text{size}(G_{k-1,m})
\]
for $k$ in the range $\Omega(\sqrt{m})$ to $m^{O(1)}$ and a constant $a_3 > 1$.

The solution is in the form
\[
\text{size}(G_{k,m}) = \Theta(k^3 + k \cdot m)
\]

Case (ii): The proposition follows directly from the Lemma 2.4. □

Consider now the $k$-th level of the graph $G_{k,2m}$. Denote by $u$ the output of the first component $P_k$ of the $k$-th level $U^2_{k,m}$ and by $x_0, x_1, \ldots, x_{2m-1}$ the sequence of vertices in the second component $R_{2m}$ of the $k$-th level $U^2_{k,m}$ and by $\alpha_0, \alpha_1$ the output vertices of the $(k-1)$-st level $U^2_{k-1,m}$ in the graph $G_{k,2m}$. The following proposition holds:

**Lemma 2.6:** In any computation on the graph $G_{k,2m}$ using $k$-pebbles for pebbling the vertex $x_i$ (for $0 \leq i \leq 2m-1$) there have to be an open path from an input vertex to $x_{i+1}$ leading through $\alpha_{(i+1) \mod 2}$ at some configuration in which $x_i$ is pebbled.

Proof: Case $i = 0$: Consider an arbitrary computation $S_0$ on $G_{k,2m}$ with $k$ pebbles starting with the empty configuration and ending with a configuration in which a pebble is placed on $x_0$ for the first time. From Lemma 2.4 it follows that $k$ pebbles are needed to close all paths from inputs to the vertex $u$ in the pyramid $P_k$. Furthermore, if there is at least one path opened in the pyramid $P_k$, then $k$ pebbles are needed to close $P_k$. Two important facts follow.

Firstly, a computation on the subgraph $G_{k-1,2m}$ of the graph $G_{k,2m}$ can be started after closing the pyramid $P_k$ with the root $u$ because at the moment of closing the pyramid the subgraph $G_{k-1,2m}$ is empty.
Secondly, the pyramid $P_k$ with the root $u$ have to be maintained as closed during the rest of the computation $\mathcal{S}_0$ because otherwise the computation $\mathcal{S}_0$ would be repeated from some configuration before closing $P_k$. At least 1 pebble has to be permanently placed on $P_k$ in order to maintain the pyramid $P_k$ closed. Thus all path to $x_0$ through $u$ are closed. Hence, there have to be a configuration (say $C$) in the computation $\mathcal{S}_0$ such that 1 pebble is placed on the pyramid $P_k$ with the root $u$ and $k - 1$ pebbles are placed on the pyramid $P_{k-1}$ with the root $\alpha_0$. That means that in the configuration $C$ there is no pebble on a path leading to the vertex $\alpha_1$ in $G_{k-1,2m}$ and thus there is an open path to $x_{i+1}$ through $\alpha_1$.

Case $0 < i < 2m - 1$: It is sufficient to repeat arguments used in case $i = 0$ with the addition that permanent pebble keeping closeness of $P_k$ lay on $x_{i-1}$ and thus it maintains all path to $x_i$ through $x_{i-1}$ closed. So Lemma 2.6 is proved.

**Theorem 2.7:** Let $S(n)$ be a space in the range $\omega(1)$ to $O(n^{1/3})$. Then there is a class of dags $\{G_n\}$, with $n$ vertices and with $S(n)$ pebbles necessary and sufficient to pebble $G_n$, such that for the time $T(n)$ needed to pebble $G_n$ with $S(n)$ pebbles it holds

$$ T(n) = 2^{\Omega(S(n) \cdot \log n / S(n))}. $$

**Proof:** Consider the graph $G_{k,2m}$ with $k = S(n)$ and $m = c_1 \cdot n / (S(n))$ for appropriately chosen constant $c_1 > 0$. We show that the following recurrent relation (2.1) holds

$$ T(G_{k,2m}) \geq b_1 \cdot m \cdot T(G_{k-1,2m}) $$

$$ T(G_{1,2m}) \geq b_2 \cdot m $$

It follows directly that (2.1) holds for $i = 1$. Suppose that (2.1) holds also for $i = 2, 3, \ldots, k - 1$. Following Lemma 2.6 the time needed for pebbling $G_{k,2m}$ with $k = S(n)$ pebbles is lower estimated by

$$ T(G_{k,2m}) \geq d_2 \cdot k^2 + \sum_{i=0}^{2m-1} \left[ \frac{d_1}{2} T(G_{k-1,2m}) \right] $$

because for pebbling the pyramid $P_k$ it is needed $1/2 k (k + 1)$ computational steps and for pebbling $2m$ vertices $x_0, x_1, \ldots, x_{2m-1}$ it is needed $1/2 d_1 \cdot T(G_{k-1,2m})$ computational steps. Choose $k = S(n) = c_2 \cdot n^{1/3}$.
for appropriate constant \( c_2 > 0 \). So we have \( m = c_3 \cdot n^{2/3} \) and thus for appropriately chosen constant \( b_1 > 0 \) it is obtained

\[
T(G_{k,2m}) \geq b_1 \cdot m \cdot T(G_{k-1,2m}).
\]

Solving (2.1) one can obtain \( T(G_{k,2m}) \geq d \cdot m^k \) for some constant \( d > 0 \) and thus for \( G_n \equiv G_{S(n),n/S(n)} \) it holds

\[
T(n) = 2^{\Omega(S(n) \log n/S(n))}.
\]

If the minimum space necessary to pebble a \( n \)-vertex dag is a constant, then minimum space and polynomial time are achieved simultaneously and the polynomial time is asymptotically tightly expressed in the following form.

**Consequence 2.8:** Let \( T(G_{k,m}) \) be the time needed to pebble \( G_{k,m} \) with the minimum number of pebbles \( S(G_{k,m}) = k \) and let \( k \) be a constant. Then

\[
T(G_{k,m}) = \Theta(m^k).
\]

Finally we show that the class of dags requiring exponential time for the minimal space standard pebbling is not “resistant” according to the “edge deleting”.

**Proposition 2.9:** Let \( S(n) \) be a space in the range \( \omega(1) \) to \( O(n^{1/3}) \). Then there is a class of dags \( \{\tilde{G}_n\} \) with \( n \) vertices constructed from \( \{G_n\} \) by removing one edge and pebbled using the minimal number of pebbles \( S(n) \) in time

\[
T(n) = O(n^2).
\]

In the graph \( G_n \equiv G_{k,m} \) it is sufficient to remove the edge outgoing from the output vertex of the component \( R_m \) at the level \( U_{k-2} \) in order to obtain the required proposition.

3. CONCLUSIONS

The class of dags \( \{G_n\} \) of size \( n \) pebbleable in space \( O(n^{1/3}) \) is considered. An asymptotically optimal time complexity for the minimal space standard pebbling is presented. Moreover, it is proved that dags requiring exponential time for the minimal space standard pebbling are not tolerant to the “edge deleting”.

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An interesting problem might to be find a class of dags of size $n$ which
a. can be pebbled in space $\omega(n^{1/3})$;
b. requires exponential time for the minimal space standard pebbling;
c. is not tolerant to the "edge deleting".

A possible attempt to solve this problem is to construct a class of rooted
dags $\{D_k\}$ requiring polynomial time in the minimal space $k$ and having
size $o(k^2)$.

Another fruitful problem would be to characterize extreme changes in
the time complexity with respect to the "edge deleting" in dags for other
variants of the pebble game.

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