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LIMITING CHARACTERIZATIONS OF LOW LEVEL SPACE COMPLEXITY CLASSES (*) (1)

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Abstract. – *The concept of limiting approximation, initially introduced for recursive functions, has been applied in [AP90] and in [APA91] in the framework of the computational complexity theory to determine meaningful characterizations of well known complexity classes. This investigation allowed to characterize PSPACE and other higher space complexity classes. In this paper we continue this study, considering space classes of lower complexity not studied yet from this point of view. In particular we characterize LOGSPACE and LINSPACE.*

Résumé. – *Le concept d'approximation à la limite, introduit initialement pour les fonctions récursives, a été appliqué dans [AP90] et [APA91] dans le cadre de la théorie de la complexité pour définir des caractérisations significatives de classes de complexité. Cette étude permet de caractériser PSPACE et d'autres classes de complexité de niveau supérieur. Dans cet article, nous poursuivons cette étude en considérant des classes de complexité de plus bas niveau qui n'ont pas encore été étudiées de ce point de vue. En particulier nous caractérisons LOGSPACE et LINSPACE.*

1. INTRODUCTION

The notions of limiting polynomially computable function and limiting polynomially decidable set were introduced in [AP90] (and then investigated in [APA91]) with the aim of extending the notion of feasibly computable functions and tractable sets; these new notions exploit the original idea due to Gold [G65] that enlarged the notion of recursiveness to functions and sets which are not recursive but may be approximated in the limit by a recursive

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function. In the case of limiting polynomially decidable sets it was shown how to approximate a set in PSPACE or in a higher space complexity class by a polynomially decidable set that satisfies some complexity conditions. In fact this approach is based on the idea of approximating sets in the limit using other sets which are easier to compute with respect to time and space.

In this paper we want to continue along this line of research trying to complete the characterization of the most important space complexity classes using limiting approximations. Since the characterizations until now were given for PSPACE and higher space complexity classes, we will study classes of lower complexity showing how to define in the limit LOGSPACE, LINSPACE. Of course in order to have meaningful limiting approximations of these classes we will have to consider approximating sets with complexity bounds that are different from those applied in [AP90] and in [APA91]. In particular, while in the case of PSPACE and higher complexity classes, the bounds on the approximating sets were essentially polynomial, now we will assume logarithmic or linear bounds.

We briefly sketch the content of the paper. In Section 2 we will recall the notions of limiting polynomially computable function and limiting polynomially decidable set and present a characterization of PSPACE in the limit. In Section 3 we will give a characterization of the sets belonging to the class LOGSPACE as limiting logarithmically decidable sets. Finally in Section 4 we will show how to extend this paradigm in order to deal with LINSPACE and P.

2. LIMITING DECIDABLE SETS AND COMPLEXITY BOUNDS

In this section we present some introductory definitions and results.

The notion of approximation of sets and functions we want to study started from the definition of limiting recursive function [G65]:

DEFINITION 2.1: A function f is said to be limiting recursive if there is a recursive function $G(x,n)$ such that, for every x, there exists $n_0(x)$ such that for every $n \geqq n_0(x)$

$$G(x, n)=f(x).$$

Moreover, the fact that f is limiting recursive is generally denoted in the following way:

$$\lim_{n \to \infty} G(x, n)=f(x).$$

The value $n_0(x)$ is called the *point of convergence* of the function f on input x. Therefore, given an input x, the value $n_0(x)$ determines the number of guesses that have to be performed in order to achieve the right value of $f(x)$.

Gold introduced this definition with the aim of enlarging the set of computable sets and investigated some properties of this notion and its relationship with Kleene's arithmetical hierarchy. In [AP90] it was studied whether it was possible to extend this definition to the field of the computational complexity theory putting some resource bounds on $G(x, n)$ and on the point of convergence $n_0(x)$ in such a way to characterize some complexity classes beyond the polynomial level. While Gold extended the notion of recursiveness, now the aim was to extend the notion of tractability. Starting from the general assumption that the tractable problems are those solved in polynomial time, it was tried to approximate untractable sets by means of sets computable in polynomial time. However in [AP90] it was shown that, in order to characterize in the limit meaningful untractable sets, the approximating sets have to satisfy not only a polynomial time bound but also some other conditions. The following definitions were given to reach this aim:

DEFINITION 2.2: A function f is said to be *$(h_1(n), h_2(|x|))$-limiting polynomially computable* if there exists a recursive function $G(x, n)$ with running time τ_G and work space σ_G such that

1. $f(x)=\lim G(x, n)$
2. $\tau_G(x, n) \leqq p_n(|x|)$ for a suitable polynomial p_n and for every n
3. $\sigma_G(x, n) \leqq p(|x|, h_1(n))$ for a suitable polynomial p and for every n
4. the point of convergence $n_0(x)$ satisfies $n_0(x) \leqq c\, h_2(|x|)$ for a suitable constant c.

According to this definition a function f is approximated by a function which is computable in polynomial time in $|x|$; instead the space necessary to compute the function $G(x, n)$ and the point of convergence depends on the choice of the functions $h_1(n)$ and $h_2(|x|)$. In this way, considering different pairs of $h_1(n)$ and $h_2(|x|)$ functions, important complexity classes were characterized. In order to recall some of this characterizations we need other definitions.

DEFINITION 2.3: A set S is said to be *$(h_1(n), h_2(|x|))$-limiting polynomially decidable* if its characteristic function satisfies Definition 2.2.

DEFINITION 2.4: Let C_1 and C_2 be two families of functions. (C_1, C_2)-*LPD* will denote the class of all $(h_1(n), h_2(|x|))$ limiting polynomially decidable sets for $h_1 \in C_1$, $h_2 \in C_2$.

We will use the following notations for denoting some families of functions: lin (linear functions), poly (polynomials), log (logarithms), exp (exponential functions).

THEOREM 2.1 ([AP90]): *PSPACE* $\equiv$ *(lin, poly)-LPD*;

The theorem shows that a set is in PSPACE if and only if it can be approximated in the limit by a recursive function $G(x, n)$ having polynomial time complexity in $|x|$ and requiring space bounded by a polynomial in $|x|$ and n. Moreover the number of guesses that have to be performed in order to achieve the right value of $f(x)$ is polynomially bounded.

Results similar (and stronger) to Theorem 2.1 were given in [APA91] characterizing the most important space complexity classes which are higher than PSPACE, starting from EXPSPACE to the Grzegorczyk hierarchy.

3. A LIMITING CHARACTERIZATION OF LOGSPACE

The study pursued in [AP90] and [APA91] was performed with the aim of extending in the limit the notion of tractable set using a computable function $G(x, n)$ that has to satisfy some polynomial bounds. Now we want to go on along this line of research trying to approximate sets in the limit that we already know to be tractable but that we would like to approximate with sets which are much easier to compute.

The first class that we study is LOGSPACE (set of languages recognized by a deterministic Turing machine in logarithmic space) and now we are going to present a characterization in the limit that allows to capture LOGSPACE. To obtain this characterization we need to introduce a definition of function similar to that of $(h_1(n), h_2(|x|))$-limiting polynomially computable function but suitable to the level of complexity we are now dealing with.

DEFINITION 3.1: A function f is said to be $(h_1(n), h_2(n))$-*limiting logarithmically computable* if there exists a recursive function $G(x, n)$ with running time τ_G and space σ_G such that

1. $f(x) = \lim G(x, n)$
2. $\tau_G(x, n) \leq h_1(n) \log|x|$ for every n
3. $\sigma_G(x, n) \leq h_2(n) + \log|x|$ for every n
4. the point of convergence $n_0(x)$ satisfies $n_0(x) \leq \log|x|$.

DEFINITION 3.2: A set S is $(h_1(n), h_2(n))$*-limiting logarithmically decidable* if its characteristic function f_S is limiting logarithmically computable.

DEFINITION 3.3: Let C_1 and C_2 be two families of functions. (C_1, C_2)-*LLogD* will denote the class of all $(h_1(n), h_2(n))$ limiting logarithmically decidable sets for $h_1 \in C_1$, $h_2 \in C_2$.

We are now ready to present how to approximate LOGSPACE in the limit. In the following the logarithms are to base 2.

THEOREM 3.1: *LOGSPACE ≡ (exp, lin)-LLogD*

Proof: (i) (exp, lin)-LLogD $\subseteq$ LOGSPACE.

Let f be the characteristic function of a set S in (exp, lin)-LLogD. We may compute f in the following way: Given x, determine $f(x) = G(x, \log|x|)$.

Then the space needed to compute f is so bounded:

$$\sigma_f(x) = \sigma_G(x, \log|x|) \leqq \log|x| + \log|x| \leqq 2\log|x|.$$

Therefore the set S belongs to LOGSPACE.

(ii) LOGSPACE $\subseteq$ (exp, lin)-LLogD

Let f be the characteristic function of a set S in LOGSPACE. Namely, let $\sigma_f(x) \leqq \log|x|$.

Now we have to exhibit a function $G(x, n)$ such that the conditions of Definition 3.1 are satisfied. Let us define $G(x, n)$ in the following way:

$$G(x, n) = \begin{cases} 1 & \text{if } \sigma_f(x) \leqq n \quad \text{and} \quad f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f(x) = \lim G(x, n)$ and

– $\tau_G(x, n) \leqq 2^n \log|x|$;

– $\sigma_G(x, n) \leqq n + \log n$ where the term $\log n$ takes into account the space needed to check the bound n on the running time;

– $n_0(x) \leqq \log|x|$.

Hence $f \in$ (exp, lin)-LLogD.

QED

4. LIMITING LINEARLY DECIDABLE SETS

In this section we continue our investigation on the limiting properties of low level complexity classes. More precisely we will study what complexity properties the approximating function $G(x, n)$ has to satisfy in order to characterize the classes P and LINSPACE.

P and LINSPACE can be characterized in the limit inside a unified framework. To reach this aim we give some definitions similar to those presented in Definitions 3.1, 3.2, 3.3 but at different levels of complexity.

DEFINITION 4.1: A function f is said to be $(h_1(n), h_2(n))$*-limiting linearly computable* if there exists a recursive function $G(x, n)$ with running time τ_G and space σ_G such that

1. $f(x) = \lim G(x, n)$
2. $\tau_G(x, n) \leqq h_1(n)|x|$ for every n
3. $\sigma_G(x, n) \leqq h_2(n) + c_1|x|$ for every n and for a suitable constant c_1
4. the point of convergence $n_0(x)$ satisfies $n_0(x) \leqq c_2|x|$ for a suitable constant c_2.

DEFINITION 4.2: A set S is said to be $(h_1(n), h_2(n))$*-limiting linearly decidable* if its characteristic function satisfies Definition 4.1.

DEFINITION 4.3: Let C_1 and C_2 be two families of functions. (C_1, C_2)*-LLinD* will denote the class of all $(h_1(n), h_2(n))$ limiting linearly decidable sets for $h_1 \in C_1$, $h_2 \in C_2$.

We are going to prove that the complexity classes P (set of languages recognized by a deterministic Turing Machine in polynomial time) and LINSPACE (set of languages recognized by a Turing Machine in linear space) are both characterizable as (C_1, C_2)-LLinD classes.

THEOREM 4.1: $P \equiv$ *(poly, poly)-LLinD*

Proof: (i) (poly, poly)-LLinD $\subseteq$ P

Let f be the characteristic function of a set S in (poly, poly)-LLinD. According to Definitions 4.1, 4.2, 4.3, $\tau_f(x) \leqq p(n)|x|$ for a suitable polynomial p.

We may compute f in the following way: Given x, determine $f(x) = G(x, c_2|x|)$. Then the time needed to compute f is so bounded: $\tau_f(x) = \tau_G(x, c_2|x|) \leqq p(c_2|x|)|x| = r(|x|)$ for a suitable polynomial r.

Therefore the set S belongs to P.

(ii) $P \subseteq$ (poly, poly)-LLinD

Let f be the characteristic function of a set S in P. Namely, let $\tau_f(x) \leqq q(|x|)$ for a suitable polynomial q.

Now we present a recursive function $G(x, n)$ such that the conditions of Definition 4.1 are satisfied. Let us define, for $n \in N$

$$G(x, n)=\begin{cases} 1 & \text{if } \quad |x| \leqq n \quad \text{and} \quad f(x)=1 \\ 0 & \text{otherwise} \end{cases}$$

We may show that $G(x, n)$ satisfies conditions 1 to 4 in Definition 4.1.

1. First of all, by construction $f(x)=\lim G(x, n)$ and therefore condition 1 is satisfied.

2. In order to compute $G(x, n)$ we have to decide if $|x| \leqq n$; this takes at most $|x|$ steps; if the answer is positive, we also have to compute f with a cost bounded by $q(n)$ because $\tau_f(x) \leqq q(|x|)$ and $|x| \leqq n$. Globally we obtain that $\tau_G(x, n)+|x|$ and therefore condition 2 is satisfied.

3. Since $\sigma_G(x, n) \leqq \tau_G(x, n) \leqq q(n)+|x|$, condition 3 is satisfied.

4. Finally, for $n \geqq |x|$, $G(x, n)=f(x)$ and so condition 4 is satisfied.

Q.E D.

THEOREM 4.2: *(exp, lin)-LLinD ≡ LINSPACE*

Proof: (i) (exp, lin)-LLinD ⊆ LINSPACE

Let f be the characteristic function of a set S in (exp,lin)-LLinD. According to Definitions 4.1, 4.2, 4.3, $\sigma_G(x, n) \leqq n+c_1|x|$ for a suitable constant c_1.

We may compute f in the following way: Given x, determine $f(x)=G(x, c_2|x|)$.Then the space needed to compute f is so bounded: $\sigma_f(x)=\sigma_G(x, c_2|x|) \leqq c_1|x|+c_2|x|$.

Therefore the set S belongs to LINSPACE.

(ii) LINSPACE ⊆ (exp, lin)-LLinD

Let f be the characteristic function of a set S in LINSPACE. Namely, let $\sigma_f(x) \leqq c|x|$ for a suitable constant c.

Now we have to exhibit a recursive function $G(x, n)$ such that the conditions of Definition 4.1 are satisfied. Let us define, for $n \in N$

$$G(x, n)=\begin{cases} 1 & \text{if } \quad \tau_f(x) \leqq 2^n|x| \quad \text{and} \quad f(x)=1 \\ 0 & \text{otherwise} \end{cases}$$

Now we verify that the four conditions of Definition 4.1 are satisfied.

1. First of all, by construction $f(x)=\lim G(x, n)$ and therefore condition 1 is satisfied.

2. $\tau_G(x, n) \leqq 2^n|x|$ and therefore condition 2 is satisfied.

3. $\sigma_G(x, n) \leqq \max\{c|x|, \log(2^n|x|)\}=\max\{c|x|, n+\log(|x|)\} \leqq c|x|+n$.

4. Since $\sigma_f(x) \leqq c|x|$, we have that $\tau_f(x) \leqq 2^{c|x|}$; hence, for every $n \gtreqqless c|x|$, we obtain $f(x) = G(x, n)$.

Q.E.D.

Finally we note that comparing Theorems 3.1 and 4.1 we are able to give a new separation condition with respect to LOGSPACE and P. In fact we have:

LOGSPACE ≡ (exp, lin)-LLogD ⊆ (lin, lin)-LLinD ⊆ (poly, poly)-LLinD ≡ P.

5. CONCLUSIONS

In this paper we have continued the research started in [AP90] showing how to characterize in the limit the classes LOGSPACE, LINSPACE and P. Taking into account the preceding results, the investigation of the complexity classes between P and PSPACE still remains open in the framework of the limiting approach. Therefore a natural area of research concerns the study of the relationships between limiting approximable sets and Meyer-Stockmeyer's polynomial hierarchy. More generally, a deepened investigation of suitable limiting approximable sets with respect to deterministic or probabilistic hierarchies at polynomial level seems worth to be pursued.

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