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PRESCRIBED ULTRAMETRICS (*)

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Abstract. – *Let* $G=(S, E)$ *be a subgraph of* $K_n=(S, F)$, *the complete graph on n vertices. Let v be a function from E to* R^+. *We prove two theorems on the extensibility of v. Every function v extends to a metric on F iff G is a forest. The function v extends to an ultrametric on F if and only if for all non-trivial cycles p in G,* mult $(p)>1$, *where* mult (p) *depends on the values of v on paths.*

Résumé. – *Soit* $G=(S, E)$ *un sous-graphe de* $K_n=(S, F)$, *le graphe complet sur n sommets. Soit v une fonction de E dans* R^+. *Nous prouvons deux théorèmes sur le prolongement de v. Toute fonction v se prolonge en une métrique sur F si et seulement si G est une forêt. La fonction v se prolonge en une ultramétrique sur F si et seulement si pour tout cycle non trivial p dans G, on a* mult $(p)>1$, *où* mult (p) *dépend des valeurs de v sur les chemins.*

INTRODUCTION

Let S be a set of points and u a non-negative real-valued function on $S\times S$. The function u is called a *metric* if

1. $u\,(x,\,y)\geqq 0$;
2. $u\,(x,\,y)=0$;
3. $u\,(x,\,y)=u\,(y,\,x)$;
4. $u\,(x,\,y)\leqq u\,(x,\,z)+u\,(z,\,y)$.

If for all z in S, u also satisfies

5. $u\,(x,\,y))\leqq \max\,\{u\,(x,\,y),\,u\,(z,\,y)\}$,

then u is called an *ultrametric*.

Ultrametrics satisfy more than the triangle inequality; inequality (5) prevents scalene triangles; that is, for any three points x, y, z of S, it is

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impossible that $u(x, y) < u(y, z) < u(x, z)$. To see why, note that (5) implies $u(x, z) \leqq \max\{u(x, y), u(y, z)\} = u(y, z)$, a contradiction. Thus, any three points in an ultrametric space determine either an *isosceles* triangle or an *equilateral* triangle.

Ultrametrics arise in the context of p–adic evaluations on infinite fields [5]. there is interest in creating arbitrary ultrametrics on finite sets, in particular, on K_n, the complete graph on n points [1 to 4]. Since many ultrametric extensions are known to be NP-complete [3], it is most interesting that one extension can be done in a polynomial number of steps.

THEOREM 1: *Let $G=(S, E)$ be a subgraph of the complete graph $K_n=(S, F)$ and let v be an arbitrary function from E to R^+. If G is a forest, then v extends to an ultrametric on F in at most $O(n^2)$ steps.*

Proof: Extend G to a spanning tree Q for K_n. Extend v to the edges of Q–G by assigning arbitrary positive number to each such edge. We use induction on n to extend v to an ultrametric u on all edges of K_n in at most $(n+1)(n-2)/2$ additional steps.

Basis: There is nothing to prove for $n=1$ or $n=2$. The case of $n=3$ is the so called isosceles restriction of an ultrametric. Namely, we define the ultrametric u on the missing edge to be the maximum of v on the other two sides. This extension takes one additional step.

Assume the result for n and consider the case $n+1$. There exists an end x of the tree Q. Let $U=S-\{x\}$. Let T be the restriction of Q to U. By induction, in at most $(n+1)(n-2)/2$ additional steps, we can find an ultrametric extension u to U of the restriction of v to T. As x is an end, there exists a unique y in U with (x, y) in Q. Let $w=v(x, y)$. For each z in $U-\{y\}$, set $u(x, z)=\max\{w, u(y, z)\}$. The number of steps to create this extension is at most $n+((n+1)(n+2)/2)=(n+2)(n+1)/2$ as claimed.

To check that our extension u is an ultrametric, we need only verify $u(a, b) \leqq \max\{u(a, c), u(b, c)\}$ for all choices of distinct a, b, c in S. There are two cases: (1) x is not in $\{a, b, c\}$. (2) x is in $\{a, b, c\}$. In case (1), the inequality holds as u is an ultrametric on U. In case (2), there are two subcases: (I) y is in $\{a, b, c\}$, (II) y is not in $\{a, b, c\}$. In case (I), the inequality holds by construction. In case (II), there are three subcases: (A) $x=a$, (B) $x=b$, (C) $x=c$. Since y is not in $\{a, b, c\}$, each of these three verifications is straightforward. This concludes the proof of theorem 1.

THEOREM 2: *Let $G=(S, E)$ be a subgraph of the complete graph $K_n=(S, F)$. Then the following are equivalent:*

(*a*) *Every function $v : E \to R^+$ extends to a metric on F;*

(*b*) *G is a forest.*

Proof: Theorem 1 proves that (1 *b*) implies (1 *a*). To show (1 *a*) implies (1 *b*) it suffices to prove that if G is not a forest, then there exists a function v from E to R^+ that does *not* extend to a metric on F. If G is not a forest, then G contains a (simple) cycle $e_1, e_2, \ldots, e_k$, $k>2$. Define v on e_i, $1 \leqq i<k$, to be arbitrary positive numbers. Define v on the edge e_k to be any number greater than the sum of $v(e_i)$, $1 \leqq i<k$. Since v fails to satisfy the triangle inequality on the edge e_k, no extension of v can be a metric on F. This concludes the proof of theorem 2.

We now extend theorem 2 to ultrametrics. We will see that whether a particular function $v : S \to R^+$ has an ultrametric extension depends on the behaviour of v on non-trivial cycles of G. A cycle is any sequence of edge connected vertices $v_0 \ldots v_n$, $v_0=v_n$, allowing repeated vertices and repeated edges. A cycle is trivial, by definition, if it is a cycle with only two edges.

Let p be a (not necessarily simple) path in G. Let max (p) denote the largest value of v on p. Let mult (p) denote the number of times v attains max (p) on p. Clearly, for all paths p, mult $(p) \geqq 1$.

We require two preliminary lemmas.

LEMMA 3: *A symmetric function $u : S \times S - \{(s, s) : s \text{ is in } S\} \to R^+$ is an ultrametric if and only if for each triple x, y, and z of distinct members of S,* mult $(xyzx)>1$.

Proof: If u is an ultrametric, then as remarked at the start of the paper, every triangle is either isosceles or equilateral, that is, mult $(xyzx)>1$. Conversely, to show that u must be an ultrametric when mult $(xyzx)>1$ on all triangles, it suffices to observe that (5) always holds.

LEMMA 4: *Let $G=(S, E)$ be a subgraph of the complete graph $K_n=(S, F)$. Let x and y belong to S. Let v be an arbitrary function from E to R^+. Let Q be the set of all paths from x to y in G. Let P be the set of all paths p in Q such that* mult $(p)=1$. *If all non-trivial cycles p in G satisfy mult $(p)>1$, then*

(1) *For any p_1 and p_2 in P,* max $(p_1)=$max (p_2).

(2) *For each q in Q and each p in P,* max $(q) \geqq$ max (p).

Proof: We prove (1) by contradiction. Suppose there were elements p_1 and p_2 of P with max $(p_1)<$max (p_2). Since $c = p_1 p_2^{-1}$ is a non-trivial cycle in G, we have by hypothesis mult $(c)>1$. Thus, there are at least two places that p_2 takes on its max, contrary to p_2 belonging to P. This proves (1). Similar proof holds for (2).

THEOREM 3: *Let $G=(S, E)$ be a subgraph of the complete graph $K_n=(S, F)$. A function $v : E \rightarrow R^+$ extends to an ultrametric on F if and only if*

($\star$) *for all non-trivial cycles p in G,* mult $(p)>1$.

Proof: First assume that v extends to an ultrametric on F, but that ($\star$) fails for some non-trivial cycle $p=x_0 \ldots x_n$. Of all cycles p with mult $(p)=1$, choose one whose lengyh, n, is minimal. By lemma 3, mult $(p)>1$ on all 3–edged cycles. Therefore, n must be >3. Without loss of generality, let $w=$max $(p)=v\ (x_0, x_1)$. Since mult $(p)=1$, $v\ (x_1, x_2)$ must be strictly less than w. Applying lemma 3 to $x_0\, x_1\, x_2\, x_0$, and knowing that $v\ (x_0, x_1)=w$ and $v\ (x_1, x_2)<w$, we conclude that $v\ (x_0, x_2)$ must also be w. Now form the cycle $q=x_0\, x_2 \ldots x_n$ of length $n-1$. Since mult $(q)=1$ we have obtained a contradiction to the choice of n.

Conversely, suppose that ($\star$) holds. To prove that v extends to an ultrametric, we consider two cases: G is complete, G is not complete. If G is complete, and ($\star$) holds for all triangles of G, then by lemma 3, v must be an ultrametric on S. On the other hand, if G is not complete, then there are x and y in S for which (x, y) is not in E. Let J be the union of E and the edge (x, y) and let $H=(S, J)$. Proceeding by induction on the cardinality of E, it suffices to show that H satisfies ($\star$).

Let Q be the set of paths p from x to y in G. Let P be the set of paths in Q such that mult $(p)=1$. By lemma 4,

(1) for any p_1 and p_2 in P, max $(p_1)=$max (p_2);

(2) for all q in Q and all p in P, max $(q)\geqq$max (p).

Define v on the edge (x, y) to be min {max (q): q in Q}. We need only show that the extension v from J to R^+ still satisfies ($\star$).

Let $s=x_0 \ldots x_n$ be a non-trivial cycle in H. Since G satisfies ($\star$) there is nothing to prove unless the edge (x, y) belongs to the cycle s. Therefore, without loss of generality, we may take $y=x_0$ and $x=x_1$. Thus, $q=x_1 \ldots x_n$, a path x to y, belongs to Q. By the definition of $v\ (x, y)$ and the choice of w, $v\ (x, y)=w\leqq$max (q). There are two possibilities: mult $(q)>1$, mult $(q)=1$. If mult $(q)>1$, then mult $(s)>1$ and we are done. If mult $(q)=1$, then q belongs to P. By (2) and the construction, max (q) must itself be w. Since v (x_0, x_1) is also w, we can conclude in this case also that mult $(s)>1$. This completes the proof of theorem 3.

Theorem 2 and 3 differ significantly in computational requirements. Testing for a forest can be done in a polynomial number of steps; testing $(\star)$ for all cycles may require a factorial number of steps. For example, consider the complete graph on n vertices with a few edges removed. Such a graph has more than $n!$ non-trivial cycles.

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