Informatique théorique et applications

J. L. Lambert

The local catenativity of DOL-sequences in free commutative monoids is decidable in the binary case

Informatique théorique et applications, tome 26, n° 5 (1992), p. 425-437.

<http://www.numdam.org/item?id=ITA_1992__26_5_425_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 26, n° 5, 1992, p. 425 à 438)

THE LOCAL CATENATIVITY OF DOL-SEQUENCES IN FREE COMMUTATIVE MONOIDS IS DECIDABLE IN THE BINARY CASE (*)

by J. L. LAMBERT [1]

Communicated by C. CHOFFRUT

Abstract. – *Given a matrix* $A \in \mathbb{Z}^{2 \times 2}$ *and a vector* $V_0 \in \mathbb{Z}^2$ *we determine if there exists an integer m and m positive integers* $a_{m-1} \ldots a_0$ *such that* $A^m V_0 = \sum_{i=0}^{m-1} a_i A^i V_0$. *When such an m exists, we compute the smallest one and m positive integers* $a_{m-1} \ldots a_0$ *that satisfy the relation.*

Keywords : DOL-Sequences; Commutative monoids.

Résumé. – *Étant donnés une matrice* $A \in \mathbb{Z}^{2 \times 2}$ *et un vecteur* $V_0 \in \mathbb{Z}^2$ *on détermine l'existence d'un entier m et de m autres entiers positifs* $a_{m-1} \ldots a_0$ *tels que* $A^m V_0 = \sum_{i=0}^{m-1} a_i A^i V_0$. *Quand un tel m existe, on calcule le plus petit ainsi que les entiers* $a_{m-1} \ldots a_0$ *qui satisfont la relation.*

INTRODUCTION

The DOL sequences were introduced by Lindenmayer [3]. They are defined in a free monoid Σ^* (Σ being a finite alphabet) by a morphism $h : \Sigma^* \to \Sigma^*$ and an axiom $w \in \Sigma^*$; the DOL sequences is then the sequence w, $h(w)$, $h^2(w), \ldots h^n(w), \ldots$ Such a sequence in Σ^* is said locally catenative if there exists an integer m and some positive integers $i_0 \ldots i_r$ smaller than m such that:

$$h^m(w) = h^{m-i_0}(w) \ldots h^{m-i_r}(w)$$

(*) Received April 1991, revised October 1991.
A.M.S. 15A24, 15A36, 15A39, 20M14, 68Q45.
C.R. F.4.3.
This work was realised under the auspices of CNRS PRC Mathématiques et Informatique.

([1]) Université de Paris-Nord, Centre Scientifique et Polytechnique, Département de Mathématiques et Informatique, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

C. Choffrut proved in [1] that when $\text{card}(\Sigma)=2$, then the local catenativity of DOL sequences is decidable since it is equivalent to $h^3(w)\in\{w, h(w), h^2(w)\}^*$. The problem of deciding if a given DOL sequence is catenative or not is still open.

In a free commutative monoid, the definition is easily extended. The morphism is given by a matrix $A\in\mathbb{N}^{n\times n}$ and the axiom is a vector $x\in\mathbb{N}^n$. The problem is then to determine if there exist an $m\in\mathbb{N}$ and m positive integers $a_{m-1}, \ldots, a_0$ such that

$$A^m x=\sum_{i=0}^{m-1} a_i A^i x$$

The problem is then a decidability result concerning matrices with entries in $\mathbb{N}$ ([4]).

In this article we will prove the decidability of this property when $n=2$ (the binary case). We will actually solve the more general problem:

Problem 1:

Instance: $V_0\in\mathbb{Z}^2$, $A\in\mathbb{Z}^{2\times 2}$

Question: Do there exist an integer $m\in\mathbb{N}$ and m positive integers $a_{m-1}, \ldots, a_0$ such that

$$A^m V_0=\sum_{i=0}^{m-1} a_i A^i V_0$$

We will prove that this problem is decidable and will compute the smallest m for which some integers $a_{m-1}, \ldots, a_0$ satisfying the property exist.

1. TURNING THE PROBLEM INTO A PROBLEM CONCERNING POLYNOMIALS

In this section, we will express our initial problem 1 into a more suitable form and study it in any dimension. It is first clear that we can rewrite and generalize the problem under the following form:

Problem 2:

Instance: $V_0 \in \mathbb{Z}^n$, $A \in \mathbb{Z}^{n \times n}$

Question: Does there exist a polynomial $P \in \mathbb{Z}[X]$ of degree m such that $P = X^m - \sum_{i=0}^{m-1} a_i X^i$ where $a_i \in \mathbb{N}$ for $0 \leqq i \leqq m-1$ and $P(A) V_0 = 0$?

We just recall that for a polynomial $P(X) = \sum_{i=0}^{m} a_i X^i$ and a matrix A, $P(A)$ is the matrix given by:

$$P(A) = \sum_{i=0}^{m} a_i A^i$$

A polynomial P of degree m such that $a_m = 1$ is said **monic**. We will denote by $\mathbb{Z}_1[X]$ the set of monic polynomials with coefficients in $\mathbb{Z}$.

We prove here that the monic polynomials of $\mathbb{Z}[X]$ that satisfy $P(A) V_0 = 0$ are the multiples of a computable monic polynomial P_0 of degree at most n. This property is a consequence of the classical Gauss' lemma on integer polynomials and Hamilton-Cayley theorem ([2]).

LEMMA 1 (Gauss' Lemma): *Let P and Q be two polynomials in $\mathbb{Z}[X]$ and denote by $C(P)$ the GCD of the coefficients of P then*

$$C(P.Q) = C(P).C(Q)$$

LEMMA 2 (Hamilton-Cayley Theorem): *Let $A \in \mathbb{Z}^{n \times n}$ and let $K_A(X)$ be the characteristic polynomial of A:*

$$K_A(X) = \mathrm{Det}(A - XI)$$

then $K_A(A) = 0$.

We will use Gauss' lemma under the following more convenient form:

LEMMA 3: *Let $P \in \mathbb{Z}[X]$ a monic polynomial. Then if $P = Q.R$ where Q and R are monic polynomials of $\mathbb{Q}[X]$ then $Q \in \mathbb{Z}[X]$ and $R \in \mathbb{Z}[X]$.*

Proof. – Let $Q' = \lambda Q$ and $R' = \mu R$ where λ and μ are the least positive integers such that Q' and R' have integer coefficients. Then since Q is monic, $C(Q')$ divides λ (λ is the highest degree coefficient of Q') and then $C(Q') = 1$

since λ is minimal. Similarly, $C(R')=1$. Now

$$C(\lambda\mu P)=\lambda\mu=C(Q')\,C(R')=1$$

Thus $\lambda=\mu=1$ and $Q=Q'$, $R=R'$. □

We are in position to prove the first proposition:

PROPOSITION 1: *Let $V_0\in\mathbb{Z}^n$, $A\in\mathbb{Z}^{n\times n}$. Define the set*

$$I=\{P\in\mathbb{Z}_1[X]/P(A)\,V_0=0\}$$

Then we can compute a monic polynomial of degree at most n: $P_0\in\mathbb{Z}[X]$. Such that:

$$I=P_0\,.\,\mathbb{Z}_1[X]$$

Proof: – Let $I'=\{P\in\mathbb{Q}[X]/P(A)\,V_0=0\}$. Then I' is an ideal over $\mathbb{Q}[X]$ which is a principal ring then there exists a monic polynomial $P_0\in\mathbb{Q}[X]$ such that $I'=P_0\,\mathbb{Q}[X]$.

By lemma 2, the monic polynomial $(-1)^n K_A(X)$ is in I' thus

$$(-1)^n K_A(X)=P_0\,Q$$

which implies by lemma 3 that $P_0\in\mathbb{Z}_1[X]$. Since P_0 is a divisor of $K_A(X)$, it is clear its degree is at most n and that there exist only a finite set of values for P_0 which can easily be computed.

Finally, Let $P\in I$, since $I\subset I'$:

$$P=P_0\,Q$$

but since P is monic, $Q\in\mathbb{Z}_1[X]$ and $I=P_0\,\mathbb{Z}_1[X]$. □

With the help of proposition 1, one can see that problem 2 is decidable if the following one is:

Problem 3:

Instance: A polynomial $P_0\in\mathbb{Z}[X]$ of degree at most n

Question: Does there exist a polynomial $Q\in\mathbb{Z}_1[X]$ such that

$$P_0\,.\,Q=X^m-\sum_{i=0}^{m-1}a_i\,X^i \qquad \text{where} \quad a_i\in\mathbb{N}$$

We now solve problem 3 in the case $n=2$.

2. SOLVING PROBLEM 3 FOR $n=2$, THE EASY CASES

Except in one case which is the most interesting one, problem 3 when $n=2$ is easy to solve. In this case the polynomial P_0 of proposition 3 has degree 1 or:

$$P_0(X)=X^2-\mathrm{Tr}(A)X+\mathrm{Det}(A)$$

In this latter case the discussion will concern the signs of $\mathrm{Tr}(A)$ and $\mathrm{Det}(A)$.

In the remainder of the paper a and b are positive integers.

1st case: P_0 has degree 1:

This means that V_0 is an eigenvalue of A.

If $P_0=X-a$, let $Q=1$ else if $P_0=X+a$, then $Q=X-a$ works.

2nd case: $P_0=X^2-a\,X-b$:

$Q=1$ is clearly suitable

3rd case: $P_0=X^2+a\,X+b$, $a\neq 0$:

Then for any large enough $\lambda\in\mathbb{N}$:

$$(X-\lambda)P_0=X^3+(a-\lambda)X^2+(b-a\lambda)X-b\lambda$$

has the convenient form. Thus $Q=X-\lambda$ is suitable.

4th case: $P_0=X^2+a\,X-b$, $a\neq 0$, $b\neq 0$:

Proposition 2: *Let $P_0=X^2+aX-b$ where $(a,b)\in(\mathbb{N}-\{0\})^2$. Then there exists no polynomial $Q\in\mathbb{Z}_1[X]$ such that*

$$P_0 . Q=X^n-\sum_{i=0}^{m-1}\lambda_i X^i \qquad \text{where} \quad \lambda_i\in\mathbb{N}$$

Proof. — Let $Q=X^n+a_{n-1}X^{n-1}+\ldots+a_0$. Then

$$P_0 . Q=(X^2+aX-b)(X^n+a_{n-1}X^{n-1}+\ldots+a_0)$$

$$=X^{n+2}+(a+a_{n-1})X^{n+1}+(a_{n-2}+aa_{n-1}-b)X^n+\sum_{i=0}^{n-3}(a_i+aa_{i+1}-ba_{i+2})X^{i+2}$$

$$+(aa_0-ba_1)X-ba_0$$

Now we want

$$-ba_0\leqq 0$$

$$aa_0-ba_1\leqq 0$$

$$a_i+aa_{i+1}-ba_{i+2}\leqq 0 \qquad \text{for} \quad i=0,\ldots,n-3$$

This implies directly that $a_0 \geqq 0$ (since $b>0$) and then $a_1 \geqq 0$. By induction one has:

$$a_i \geqq 0 \text{ and } a_{i+1} \geqq 0 \Rightarrow a_{i+2} \geqq 0$$

and thus $a_{n-1} \geqq 0$ which gives $a+a_{n-1} \geqq a>0$. A contradiction. □

We now deal with the interesting case $P_0=X^2-aX+b$.

3. PROBLEM 3: THE CASE $P_0=X^2-aX+b$, $b \neq 0$

3.1. The main result

PROPOSITION 3: *Let $P_0=X^2-aX+b$, $a \in \mathbb{N}$, $b \in \mathbb{N}-\{0\}$. Then there exists $Q \in \mathbb{Z}_1[X]$ such that*

$$P_0 \cdot Q = X^n - \sum_{i=0}^{n-1} \lambda_i X^i$$

with $\lambda_i \in \mathbb{N}$ iff P_0 has no root in $\mathbb{R}$.

Proof. — We first prove that the problem is equivalent to the existence of a non fully negative solution to a certain regular system of inequations. To check this existence, we use an easy criteria concerning the signs of the coefficients of a matrix. We compute that matrix and show that those coefficients are given by a linear recursion formula which the discussion is based on.

Let us look at a polynomial $Q=X^n+a_{n-1}X^{n-1}+\ldots+a_0$ then

$$\begin{aligned} P_0 \cdot Q &= (X^2-aX+b)(X^n+a_{n-1}X^{n-1}+\ldots+a_0 \\ &= X^{n+2}+(a_{n-1}-a)X^{n+1}+(a_{n-2}-aa_{n-1}+b)+\sum_{i=0}^{n-3}(a_j-aa_{i+1}+ba_{i+2})X^{i+2} \\ &\qquad +(ba_1-aa_0)X+ba_0 \end{aligned}$$

We have to determine if there exists an integer n such that the system of inequation (I):

$$\text{(I)}\qquad \begin{aligned} & ba_0 \leqq 0 \\ & -aa_0 + ba_1 \leqq 0 \\ & a_0 - aa_1 + ba_2 \leqq 0 \\ & \ddots \\ & a_i - aa_{i+1} + ba_{i+2} \leqq 0 \\ & \ddots \\ & a_{n-2} - aa_{n-1} + b \leqq 0 \\ & a_{n-1} - a \leqq 0 \end{aligned}$$

has a solution $a_0, \ldots, a_{n-1}$ in $\mathbb{Z}$.

We claim that this is equivalent to the existence of an integer n such that the system of equation (II):

$$\text{(II)}\qquad \begin{aligned} & bx_0 \leqq 0 \\ & -ax_0 + bx_1 \leqq 0 \\ & x_0 - ax_1 + bx_2 \leqq 0 \\ & \ddots \\ & x_i - ax_{i+1} + bx_{i+2} \leqq 0 \\ & \ddots \\ & x_{n-2} - ax_{n-1} + bx_n \leqq 0 \end{aligned}$$

has a solution $(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ satisfying:

$$\exists\, i / x_i > 0.$$

First it is clear that if $(a_0, \ldots, a_{n-1})$ is a solution of (I), then $x = (a_0, \ldots, a_{n-1}, 1)$ is a solution of (II) with $x_n > 1$.

Conversely, let $(x_0, \ldots, x_n)$ be a solution of system (II) that satisfies $x_i > 0$. Since $bx_0 \leqq 0$ and $b > 0$ one has $x_0 \leqq 0$. Let x_{n_0} be the first x_i such that $x_{n_0} > 0$. By the previous remark, $n_0 > 0$.

Now we just have to check that $a_0 = x_0, \ldots a_{n_0-1} = x_{n_0-1}$ is a solution of system (I). The $n_0 - 1$ first inequations are satisfied and since

$$a_{n_0-2} - aa_{n_0-1} + b \leqq x_{n_0-2} - ax_{n_0-1} + bx_{n_0} \leqq 0$$

and

$$a_{n_0-1}-a \leqq x_{n_0-1} \leqq 0$$

all inequations of (I) are satisfied. This states the claim.

We will thus solve the latter problem. Let us define

$$A_n=\begin{pmatrix} b & & & & \\ -a & b & & & \\ 1 & -a & b & & \\ & & \ddots & & \\ & & 1 & -a & b \end{pmatrix} \in \mathbb{Z}^{(n+1)\times(n+1)}$$

and for two vectors x, y in $\mathbb{Z}^{n+1}$:

$$x \leqq y \Leftrightarrow \forall i, \qquad 0 \leqq i \leqq n, \quad x_i \leqq y_i$$

We have to determine if

$$\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{Z}^{n+1}, \qquad A_n x \leqq 0 \Rightarrow x \leqq 0$$

But this is clearly equivalent to

$$\forall n \in \mathbb{N}, A_n^{-1} \text{ is positive}$$

Now we easily compute that

$$A_n^{-1}=\begin{pmatrix} \frac{\alpha_0}{b} & & & \\ \frac{\alpha_1}{b^2} & \frac{\alpha_0}{b} & & \\ & & \ddots & \\ \frac{\alpha_n}{b^{n+1}} & \dots & \frac{\alpha_1}{b^2} & \frac{\alpha_0}{b} \end{pmatrix}$$

where $\alpha_0, \ldots, \alpha_n$ is an integer sequence given by $\alpha_0=1$, $\alpha_1=a$ and the recursion formula:

$$\alpha_n=a\,\alpha_{n-1}-b\,\alpha_{n-2}$$

The characteristic equation of the recursion is $P_0(X)=0$.

We now have three cases.

1st case: P_0 has two distinct real roots. Let $\lambda_1 > \lambda_2 > 0$ be these roots. An elementary computation leads to the formula:

$$\alpha_n = \frac{(\lambda_1)^{n+1} - (\lambda_2)^{n+1}}{\sqrt{\Delta}} \quad (\Delta = a^2 - 4b)$$

and $\alpha_n > 0$ for every n, the problem has no solution.

2nd case: P_0 has an unique double root $\lambda = a/2$. The new formula is

$$\alpha_n = (n+1)(a/2)^n$$

then $\alpha_n > 0$ for every $n \in \mathbb{N}$, there is no solution either.

3rd case: P_0 has no root in $\mathbb{R}$ then P_0 has two roots in $\mathbb{C}: \lambda$ and $\bar{\lambda}$, we get the new formula:

$$\alpha_n = \frac{(\lambda)^{n+1} - (\bar{\lambda})^{n+1}}{i\sqrt{-\Delta}}$$

Let $\lambda = \rho e^{i\theta}$ then $\rho = \sqrt{b}$ and $\theta = \text{Arctan}\,(\sqrt{(4b/a^2) - 1})$ (if $a = 0$ then $\theta = \pi/2$) and

$$\alpha_n = \frac{2\sqrt{b}^{n+1}}{\sqrt{-\Delta}} \sin(n+1)\theta$$

and $\alpha_n < 0$ as soon as $(n+1)\theta > \pi$ then for $n = [\pi/\text{Arctan}\,(\sqrt{(4b/a^2) - 1})]$ (where $[x]$ is the integer part of x) the problem has a solution of degree n. □

3.2. Some examples

Let A be $a\,2 \times 2$ integer matrix such that $\text{Det}(A) > 0$ and $\text{Tr}(A) > 0$. When the matrix is positive, there is no solution since the characteristic equation has real roots. If we do not restrict the matrix A to be positive it is notable that even for matrices with coefficients of small size then the polynomial Q of proposition 3 can have a relatively high degree and surprisingly large size coefficients.

Before introducing some explicit examples, we note that it is easy to compute a polynomial Q satisfying the conclusion of proposition 3. Let X

be such that:

$$A_n X = \begin{pmatrix} -b^{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

since one has

$$A_n^{-1} = \begin{pmatrix} \frac{\alpha_0}{b} & & & \\ \frac{\alpha_1}{b^2} & \frac{\alpha_0}{b} & & \\ & & \ddots & \\ \frac{\alpha_n}{b^{n+1}} & \cdots & \frac{\alpha_1}{b^2} & \frac{\alpha_0}{b} \end{pmatrix}$$

we get

$$X = \begin{pmatrix} -\alpha_0 b^n \\ -\alpha_1 b^{n-1} \\ \vdots \\ -\alpha_i b^{n-i} \\ \vdots \\ -\alpha_n \end{pmatrix} \in \mathbb{N}^n \times (-\mathbb{N})$$

Now following the argument of proposition 3, the vector:

$$X' = \begin{pmatrix} -\alpha_0 b^n \\ -\alpha_1 b^{n-1} \\ \vdots \\ -\alpha_i b^{n-i} \\ \vdots \\ -\alpha_{n-1} b \\ 1 \end{pmatrix} = X + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1+\alpha_n \end{pmatrix}$$

is a solution of system (I), the coefficients of $P_0 \cdot Q$ are then given by:

$$A_n X' = \begin{pmatrix} -b^{n+1} \\ 0 \\ \vdots \\ 0 \\ (1+\alpha_n) b \end{pmatrix}$$

and $a_{n-1} - a = -\alpha_{n-1} b - a$. Thus we get:

$$P_0 \cdot Q = X^{n+2} - (a + b\,\alpha_{n-1})\,X^{n+1} + (1 + \alpha_n)\,b\,X^n - b^{n+1}$$

Note that the computation of that polynomial is reduced to the computation of α_n and α_{n-1} by a simple linear recursion formula!

Now we can give some examples. We note that for matrices of the form:

$$A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}$$

The characteristic polynomial is $X^2 - 2aX + a^2 + 1$ which discriminant is $\Delta = -4$. Thus the degree of the polynomial Q of lowest degree is:

$$n = \left[\frac{\pi}{\operatorname{Arctan}(\sqrt{1/a^2})} \right] \geqq [\pi\, a]$$

For example, if $a = 1$, the degree of Q is 4 since:

$$\alpha_0 = 1, \alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 0, \alpha_4 = -4$$

The polynomial $Q = X^4 - 8X^2 - 16X - 16$ and

$$(X^4 - 8X^2 - 16X - 16)(X^2 - 2X + 2) = X^6 - 2X^5 - 6X^4 - 32$$

The size of the polynomial grows rapidly with a. For $a = 3$ for example, the formula gives $n \geqq 9$. In fact $n = 9$ is the lowest degree as one can see by computing the sequence:

$$\alpha_0 = 1$$
$$\alpha_1 = 6$$
$$\alpha_2 = 26$$
$$\alpha_3 = 96$$
$$\alpha_4 = 316$$
$$\alpha_5 = 936$$
$$\alpha_6 = 2\,456$$
$$\alpha_7 = 5\,376$$
$$\alpha_8 = 7\,696$$
$$\alpha_9 = -7\,584$$

Q is then a polynomial of surprisingly large size while $P_0 = X^2 - 6X + 10$:

$$Q = X^9 - 76\,960\,X^8 - 537\,600\,X^7 - 2\,456\,000\,X^6 - 9\,360\,000\,X^5 - 31\,600\,000\,X^4 - 96\,000\,000\,X^3 - 260\,000\,000\,X^2 - 600\,000\,000\,X - 1\,000\,000\,000$$

and we get by the proved formula:

$$(X^2 - 6X + 10) . Q = X^{11} - 76\,966\,X^{10} - 75\,830\,X^9 - 10\,000\,000\,000$$

The size of the obtained polynomials is the most surprising fact. If $a = 5$ then the degree of the polynomial is at least 15 with very big coefficients, for the following polynomial

$$X^2 - 200\,X + 10\,001\ (a = 100)$$

We must search for a polynomial of degree at least 314!

4. CONCLUSIONS

We now easily express the results of section 3 in terms of problem 1.

THEOREM 1: *Problem* 1 *has a solution except if* V_0 *is not an eigenvector of* A *and*
i) Tr $(A) < 0$, Det $(A) > 0$
or
ii) Tr $(A) > 0$, Det $(A) > 0$ *and* A *has eigenvalues in* $\mathbb{R}$.

Moreover the smallest value m satisfying the property can be determined in each case. If V_0 *is an eigenvector of A then it is* 1 *if the eigenvalue is positive,* 2 *otherwise. If* V_0 *is not an eigenvector then the results are summed up in the following tableau:*

	Det $(A) > 0$	Det $(A) < 0$	Det $(A) = 0$
Tr $(A) > 0$	n or imp.	2	2
Tr $(A) < 0$	3	imp.	3
Tr $(A) = 0$	4	2	2

$$\textit{where } n = \left[\frac{\pi}{\operatorname{Arctan}\left(\sqrt{(4 \det(A)/\operatorname{Tr}(A)^2) - 1}\right)} \right] + 2$$

In the case where A and V_0 are positive, the result is greatly simplified:

THEOREM 2: *In a commutative free monoid, a* Dol *sequence given by a matrix A and an axiom V_0 is locally contenative iff V_0 is an eigenvector of A or* $\mathrm{Det}(A) \leqq 0$. *In both cases one has:*

$$A^2 V_0 \in \{ A V_0, V_0 \}^*$$

CONCLUSION

The technique involved to solve the problem in the binary case can be in part generalized to solve the other cases. But the analysis of the sequence α_n will not be possible in the same way (but perhaps the decidability will remain true) and principally the particularity of sign that permits to deduce from a suitable non monic polynomial another monic polynomial with the same quality will not remain valid. The generalization is then not so clear.

ACKNOWLEDGEMENTS

I thank the anonymous referees for their help in improving the presentation and C. Choffrut who made me know about this problem.

REFERENCES

1 C. CHOFFRUT, Iterated substitutions and locally catenative systems: a decidability result in the binary case, *private communication.*

2 S. LANG, Algebra, Addison Wesley 1965.

3 A. LINDENMAYER, G. ROZENBERG, Developmental systems with locally catenative formulas, *Acta Informatica, 2*, 1973, pp. 214-248.

4 WEBER, SEIDL, On finite generated monoids of matrices with entries in ℕ, *RAIRO, Inf. Theor. Appl., 25*, 1991, pp. 19-38.