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## REPETITIONS IN THE FIBONACCI INFINITE WORD (\*)

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*Abstract.* – Let  $\varphi$  be the golden number; we prove that the Fibonacci infinite word contains no fractional power with exponent greater than  $2 + \varphi$  and we prove that for any real number  $\varepsilon > 0$  the Fibonacci infinite word contains a fractional power with exponent greater than  $2 + \varphi - \varepsilon$ .

*Résumé.* – Soit  $\varphi$  le nombre d'or; nous prouvons que le mot infini de Fibonacci ne contient pas la puissance fractionnaire d'exposant supérieur à  $2 + \varphi$ , et nous prouvons qu'il contient des puissances d'exposant supérieur à  $2 + \varphi - \varepsilon$ , quel que soit le nombre réel  $\varepsilon > 0$ .

### INTRODUCTION

Many papers are concerned with the existence of integer powers in “long enough” words or in infinite words; a classical combinatorial property is whether a given infinite word is  $k$  power-free or not, with  $k$  natural number.

No word on a two letters alphabet can avoid a square but it is well known that the Thue infinite word  $t$  on a two letter alphabet does not contain cubes and that the Thue infinite word  $m$  on a three letter alphabet does not contain squares (*see* [9], [10]).

The notion of overlap-free word and more generally the notion of fractional power are considered in many papers (*see* for instance [4], [7], [9], [10]).

In this paper we prove that the Fibonacci infinite word contains no fractional power with exponent greater than  $2 + ((\sqrt{5} + 1)/2)$  and that for any real number  $\varepsilon > 0$  the Fibonacci infinite word contains a fractional power with exponent greater than  $2 + ((\sqrt{5} + 1)/2) - \varepsilon$ .

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To our knowledge this is the first time that this property for a non rational value is looked for in a given infinite word.

## DEFINITIONS AND PRELIMINARY RESULTS

We refer to [6] for the terminology.

Let  $A$  be an alphabet. We denote by  $A^*$  the *free monoid* on  $A$ . The elements of  $A^*$  are called *words* and the elements of  $A$  are called *letters*. We denote by  $1$  the empty word which is the identity of  $A^*$ ; we also denote by  $|v|$  the length of a word  $v$ .

A word  $v$  is a *factor* of a word  $w$  if there exist  $u, u' \in A^*$  such that

$$w = uvu'$$

and we say that  $v$  is a *left factor* of  $w$  if  $u$  is the empty word.

If a word  $w$  is of the form

$$w = v \dots v = v^k$$

with  $u \neq 1$ , we say that  $w$  is a *k-power* of  $v$ ;  $k$  is called the *exponent* of the power and  $v$  is the *base* of the power.

If a word  $w$  is of the form

$$w = v \dots vu = v^k u$$

with  $u \neq 1$ ,  $k \geq 1$  and  $u$  left factor of  $v$ , we say that  $w$  is a *fractional power* of  $u$  of exponent  $e = |w|/|v|$  and  $v$  is the base of the power.

An infinite word  $s$  on an alphabet  $A$  is a map from the set of positive integers into  $A$ ; we denote by  $A^\omega$  the set of all infinite words on the alphabet  $A$ .

A word  $v \in A^*$  is a factor of the infinite word  $s$  if there exist  $u \in A^*$ ,  $s' \in A^\omega$  such that  $s = uv s'$ . If  $u$  is the empty word then  $v$  is a left factor of  $s$ .

The Fibonacci infinite word  $\mathbf{f}$  on the alphabet  $A = \{a, b\}$  is obtained by iterating the morphism  $\psi : \{a, b\} \rightarrow \{a, b\}$  given by

$$\psi(a) = ab, \quad \psi(b) = a$$

starting with the letter  $a$  (*see* [1]). Therefore

$$\mathbf{f} = abaababaabaabab\dots$$

We define the sequence of the finite Fibonacci words by the rule:

$$\begin{aligned} \mathbf{f}_0 &= b, \\ \mathbf{f}_{n+1} &= \psi(\mathbf{f}_n). \end{aligned}$$

It is easy to see that  $\mathbf{f}_{n+2} = \mathbf{f}_{n+1} \mathbf{f}_n$  and, consequently, the sequence  $|\mathbf{f}_n|$ ,  $n \in \mathbb{N}$  is the sequence of Fibonacci numbers; moreover for any  $n \geq 1$ ,  $\mathbf{f}_n$  is a left factor of  $\mathbf{f}_{n+1}$  and of  $\mathbf{f}$ .

For  $n \geq 2$  we denote by  $\mathbf{g}_n$  the word  $\mathbf{f}_{n-2} \mathbf{f}_{n-1}$ . It is easy to see that for each  $n \geq 2$  there exists a word  $\mathbf{v}_n$  such that  $\mathbf{f}_n = \mathbf{v}_n x y$  and  $\mathbf{g}_n = \mathbf{v}_n y x$  with  $x, y \in \{a, b\}$  and  $x \neq y$  and also that  $\mathbf{f}_{n+2} = \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1}$ .

The following fact is straightforward

Fact. — If  $u$  is a left factor of  $\mathbf{f}_n$  and also of  $\mathbf{g}_{n-1}$  then  $u$  is a left factor of  $\mathbf{v}_{n-1}$  and, consequently

$$|u| \leq |\mathbf{v}_{n-1}| = |\mathbf{g}_{n-1}| - 2 = |\mathbf{f}_{n-1}| - 2.$$

In the sequel we will use the following results.

PROPOSITION 1 (Karhumäki [4]): *The Fibonacci infinite word  $\mathbf{f}$  contains no 4-power.*

PROPOSITION 2 (Séebold [8]): *Let  $v \neq 1$ ; if  $v^2$  is a factor of the Fibonacci infinite word  $\mathbf{f}$  then there exists  $n$  such that  $|v| = |\mathbf{f}_n|$ ; more precisely  $v = wz$  with  $zw = \mathbf{f}_n$  for some words  $z$  and  $w$ ,  $|w| > 0$ , i.e.  $v$  is a conjugate of  $\mathbf{f}_n$ .*

Now let  $u \neq 1$ ,  $u \in A^*$  and let  $u = x_1 \dots x_n$ ,  $x_i \in A$ ; we denote by  $\hat{u}$  the mirror image of  $u$ , that is  $x_n \dots x_1$ .

We say that a factor  $u$  of  $\mathbf{f}$  is special if  $ua$  and  $ub$  are both factors of  $\mathbf{f}$ .

PROPOSITION 3 (Berstel [1]): *If  $u$  is a special factor of the Fibonacci infinite word  $\mathbf{f}$  then  $\hat{u}$  is a left factor of  $\mathbf{f}$ .*

Since the sequence  $|\mathbf{f}_n|$ ,  $n \in \mathbb{N}$ , is the sequence of Fibonacci numbers, we have the following proposition.

PROPOSITION 4 (Hardy and Wright [5]): *For any  $n > 1$*

$$\frac{|\mathbf{f}_{n+1}| - 2}{|\mathbf{f}_n|} = \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} < \frac{\sqrt{5} + 1}{2}$$

and

$$\lim_{n \rightarrow \infty} \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} = \frac{\sqrt{5} + 1}{2}.$$

**PROPOSITION 5** (de Luca [2]): *For each  $i$  the word  $\mathbf{f}_i$  is primitive; therefore for each  $i$  the conjugates of  $\mathbf{f}_i$  are distinct.*

**RESULTS AND PROOFS**

Let us prove the following lemma.

**LEMMA:** *No fractional power with exponent greater than  $1 + (\sqrt{5} + 1)/2$  can be a left factor of the Fibonacci infinite word  $\mathbf{f}$ . More precisely, if  $vvu$  is a fractional power which is a left factor of  $\mathbf{f}$  then  $v = \mathbf{f}_n$  for some  $n$  and  $|vvu| \leq |\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2$ .*

*Proof:* Let  $vvu$  be a fractional power which is a left factor of  $\mathbf{f}$ .

By using Proposition 2 we have that  $|v| = |\mathbf{f}_n|$  for some  $n$ , and, consequently  $vv$  is a left factor of  $\mathbf{f}$  with length  $2|\mathbf{f}_n|$ . By inspection one can easily see that  $n$  is greater than or equal to 3.

As  $\mathbf{f}_n$  is a left factor of  $\mathbf{f}$  we have that  $v = \mathbf{f}_n$  for some  $n \geq 3$ . Thus  $vvu = \mathbf{f}_n \mathbf{f}_n u$  and either  $u$  is a left factor of  $\mathbf{f}_n$  or  $\mathbf{f}_n$  is a left factor of  $u$ .

But for  $n \geq 3$   $\mathbf{f}_{n+2} = \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1}$  is a left factor of  $\mathbf{f}$ .

Hence, since  $\mathbf{g}_{n-1}$  is not a left factor of  $\mathbf{f}_n$ , we have that  $u$  is necessarily a left factor of  $\mathbf{g}_{n-1}$ ; by the fact

$$|u| \leq |\mathbf{f}_{n-1}| - 2.$$

Thus  $|vvu| \leq |\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2$  and, by Proposition 4,

$$\frac{|vvu|}{|v|} \leq \frac{|\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} < 1 + \frac{\sqrt{5} + 1}{2}, \quad \square$$

We are now ready to prove our main result.

**PROPOSITION 6:** *The Fibonacci infinite word  $\mathbf{f}$  contains no fractional power with exponent greater than  $2 + ((\sqrt{5} + 1)/2)$  and, for any real number  $\varepsilon > 0$ , it contains a fractional power with exponent greater than  $2 + ((\sqrt{5} + 1)/2) - \varepsilon$ .*

*Proof:* Let  $vvvu$  be a fractional power factor of  $\mathbf{f}$ . As in  $\mathbf{f}$  there are no 4 powers (Proposition 1) one can find in  $\mathbf{f}$  a factor

$$u'xu''u'xu''u'xu''u'y$$

where  $u'xu''=v$ ,  $u$  is a left factor of  $u'$ ,  $u' \in \{a, b\}^*$  and  $x, y \in \{a, b\}$  with  $x \neq y$ .

It follows that  $u'xu''u'xu''u'$  is a special factor of  $\mathbf{f}$ . By Proposition 3,  $\hat{u}'\hat{u}''x\hat{u}'\hat{u}''x\hat{u}'$  is a left factor of  $\mathbf{f}$ . From the Lemma

$$\frac{|\hat{u}'\hat{u}''x\hat{u}'\hat{u}''x\hat{u}'|}{|\hat{u}'\hat{u}''x|} = \frac{|vvu'|}{|v|} < 1 + \frac{\sqrt{5}+1}{2},$$

and, consequently,

$$\frac{|vvvu|}{|v|} \leq \frac{|vvvu'|}{|v|} < 2 + \frac{\sqrt{5}+1}{2}.$$

At last, for  $n \geq 3$ ,  $\mathbf{f}_{n+4} = \mathbf{f}_{n+1} \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1} \mathbf{f}_{n-1} \mathbf{f}_n$ .

Hence, for  $n \geq 3$ ,  $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}$  is always a factor of  $\mathbf{f}$ .

Since

$$\frac{|\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}|}{|\mathbf{f}_n|} = 2 + \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|},$$

the second part of the proposition follows from Proposition 4.  $\square$

In the proof of the above proposition we used the fact that for  $n \geq 3$ ,  $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}$  is a factor of  $\mathbf{f}$ . As a consequence all words of the form  $wzwz$  with  $zw = \mathbf{f}_n$  and  $|z| \leq |\mathbf{v}_{n-1}|$  are factors of  $\mathbf{f}$ ; by Proposition 5 all these words are distinct. Since  $0 \leq |z| \leq |\mathbf{v}_{n-1}|$ , the number of these words is  $|\mathbf{v}_{n-1}| + 1$ .

Let us suppose that  $vvv$  is a factor of  $\mathbf{f}$  and that  $|v| = |\mathbf{f}_n|$  for some  $n \geq 3$ . By proposition 2,  $v = wz$ ,  $|w| > 0$ , and  $zw = \mathbf{f}_n$ .

Suppose that  $|z| > |\mathbf{v}_{n-1}|$ ; since  $\mathbf{f}_n = \mathbf{f}_{n-1} \mathbf{f}_{n-2} = \mathbf{v}_{n-1} yx \mathbf{f}_{n-2}$  with  $x, y \in \{a, b\}$  and  $x \neq y$ , we can write  $\mathbf{f}_n = \mathbf{v}_{n-1} yuw$  with  $z = \mathbf{v}_{n-1} yu$  and, consequently,  $vvv = w \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yu$ .

We know that  $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1} = \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} xy$  is a factor of  $\mathbf{f}$ ; thus  $w \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} = w \mathbf{v}_{n-1} (yuw \mathbf{v}_{n-1})^2$  is a special factor and by Proposition 3 its mirror image must be a prefix of  $\mathbf{f}$ . This is impossible by the Lemma because  $|w| > 0$ .

Hence we have proved the following proposition.

PROPOSITION 7: For  $n \geq 3$  the number of distinct factors  $v$  of  $\mathbf{f}$  with length  $|\mathbf{f}_n|$  such that  $vvv$  is also a factor of  $\mathbf{f}$  is exactly  $|\mathbf{v}_{n-1}| + 1$ . More precisely they are all the words of the form  $wz$  with  $zw = \mathbf{f}_n$  and  $|z| \leq |\mathbf{v}_{n-1}|$ .

OBSERVATION: As  $2 + ((\sqrt{5} + 1)/2)$  is an irrational number it cannot exist a fractional power with exponent equal to it.

In the Thue infinite word  $\mathbf{t}$  on a two letters alphabet  $A$  there are clearly squares but there are no overlaps (that is factors like  $xvxvy$ ,  $x \in A$ ,  $v \in A^*$ ). On the contrary it is easy to see that, for any  $\varepsilon > 0$ , in the Thue infinite word  $\mathbf{m}$  on a three letters alphabet there exists a fractional power with exponent greater than  $2 - \varepsilon$  but it is a classical result that  $\mathbf{m}$  is square free.

*Remark:* Proposition 6 and 7 were firstly proved by using techniques of Sturmian words. Following the suggestion of P. Séébold we tried to find a simpler proof; actually our proof is simpler than the previous one and use only elementary properties of the Fibonacci infinite word.

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