I. LITOVSKY

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MINIMAL GENERATORS OF SUBMONOIDS OF A^∞ (*)

by I. LITOVSKY (*)

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Abstract. – *In the monoid A^∞ (unlike the monoid A^*) some submonoids do not have minimal generators with respect to inclusion; here we characterize these submonoids. Next we give algorithms to decide, in the rational case, whether a submonoid has either one smallest generator or minimal generators of finite generators. Finally we prove that every rational submonoid of A^∞ may be obtained from the single submonoid $x^* + (x^*y)^\omega$ through a composition of non-erasing morphisms and non-erasing inverse morphisms.*

Résumé. – *Dans le monoïde A^∞ (à la différence du monoïde A^*) certains sous-monoïdes n'ont pas de générateurs minimaux par rapport à l'inclusion; nous caractérisons ici ces sous-monoïdes. Puis dans le cas rationnel nous proposons des algorithmes pour décider si un sous-monoïde a soit un plus petit générateur, soit des générateurs minimaux, soit des générateurs finis. Pour finir nous montrons que le seul sous-monoïde $x^* + (x^*y)^\omega$ permet d'obtenir tout sous-monoïde rationnel de A^∞ par composition de morphismes et morphismes inverses non effaçants.*

INTRODUCTION

Given an alphabet A , the free monoid A^* is the set of all finite words over A with concatenation. Let M be a submonoid of A^* (i.e. a subset of A^* containing the empty word and closed under the concatenation), a subset G is called a generator of M if and only if $G^* = M$. It is well-known that $\text{Root}(M)$ (i.e. the set of words non-factorizable by using two nonempty words of M) is the smallest generator of M [i.e. each generator of M contains $\text{Root}(M)$].

When we deal furthermore with infinite words, we consider the set, denoted by A^∞ , of all finite or infinite words over A . A^∞ endowed with a natural extension of the concatenation is a monoid and then A^* is a submonoid of

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(¹) C.N.R.S. U.R.A. 1304, Université de Bordeaux-I, L.A.B.R.I., 351, cours de la Libération, 33405 Talence Cedex, France.

A^ω . However the property, $vu = u$ implies v is the empty word, holds in A^* but not in A^ω . We shall see here a few consequences concerning the generators of submonoids of A^ω .

Given M a submonoid of A^ω , the aim of this paper is to look for the “little” generators of M with respect to inclusion. In [3] it is proved that some submonoids do not have a smallest generator and two characterizations are given, one of “Root (M) is the smallest generator” and the other “ M has one smallest generator [possibly not Root (M)]”. In view of these results, it has seemed interesting to study more generally the minimal generators of M . First we note that some submonoids do not have minimal generators. Next by defining three kinds of “minimal” elements for the following transitive relation over M “ u is factorizable in M by v ”, we find again both previous characterizations and we obtain a third one for “ M has minimal generators”.

Then we prove that for the rational case, these three above characterizations are effective, that is to say, assuming that M is a rational submonoid, one can decide whether any one of them is satisfied. That allows us to decide whether M has a finite set as generator.

In a last part we try to generate the rational submonoids no longer with the $\ast$ -operation, but through morphisms and inverse morphisms from the simplest possible submonoid. We start from a result of [5] which states that, for any alphabet A , every rational submonoid of A^* may be obtained, from the single submonoid x^* through a composition of two non-erasing morphisms and one inverse non-erasing morphism. In a same way as in [5, 6], we state that every rational submonoid of A^ω may be obtained through the single submonoid $(x^* + (x^*y)^\omega)$.

I. PRELIMINARIES

Let A be an alphabet, A^* is the set of all (finite) words over A , the empty word is denoted by ε , $A^* - \{\varepsilon\}$ is denoted by A^+ (we use $-$ to denote the difference between two subsets), $|u|$ denotes the length of the word u . A^* with concatenation is a monoid.

A^ω is the set of all infinite words over A (i.e. sequences with value in A), and A^ω denotes $A^* + A^\omega$. Any infinite word is called an ω -word and any subset of A^ω is called a language. Let M be a language of A^ω , $M \cap A^*$ is denoted by M_{fin} and $M \cap A^\omega$ is denoted by M_{inf} .

The concatenation over A^* is extended over A^ω by:

$$\forall w \in A^\omega, \quad \forall \alpha \in A^\omega : w\alpha = w.$$

$\forall u \in A^*, \forall w \in A^\omega : uw$ is such that

$$\begin{aligned} (uw)(n) &= u(n), & \forall n \leq |u| \\ (uw)(n) &= w(n - |u|), & \forall n > |u|. \end{aligned}$$

So A^ω is a monoid. As usual the concatenation is extended to the languages, and for any language L :

$$\begin{aligned} L^0 &= \{ \varepsilon \} \\ \forall n \geq 1, \quad L^n &= L \cdot L^{n-1} \\ L^* &= \bigcup_{n \geq 0} L^n = (L_{\text{fin}})^* \cup (L_{\text{fin}})^* L_{\text{inf}}. \end{aligned}$$

Let u be a word in A^+ , the ω -word $u \dots u \dots$ is denoted by u^ω and is said to be periodic. Let L be a language in A^+ , as in [2], L^ω denotes the following w -language $\{u^\omega / u \in L\}$. An ω -word w is ultimately periodic if and only if $w = uv^\omega$ for some u in A^* and v in A^+ , then v is called a period of w , and v^ω a periodic right-factor of w . A language L is ultimately periodic if and only if every ω -word of L is ultimately periodic.

A language M is a submonoid of A^* if and only if $M^* = M$. Moreover for any language L , L^* is the smallest submonoid containing L . Clearly M is a submonoid of A^ω if and only if $M_{\text{fin}} = M_{\text{fin}}^*$ and $M_{\text{inf}} = M_{\text{inf}} M_{\text{inf}}$. Let M be a submonoid of A^ω , G is called a generator of M whenever $G^* = M$. Clearly G is a generator of M if and only if $G_{\text{fin}}^* = M_{\text{inf}}$ and $G_{\text{fin}}^* G_{\text{inf}} = M_{\text{inf}}$. The family of all generators of M is denoted by $\text{Gen}(M)$.

In the following we study the minimal languages of this family with respect to the inclusion. Let us recall, in the particular case of the family $\text{Gen}(M)$, the basic following definitions. Let M be submonoid of A^ω , L is the smallest generator of M if and only if $L \in \text{Gen}(M)$ and for each $G \in \text{Gen}(M)$, $L \subset G$. G is a minimal generator of M if and only if $G \in \text{Gen}(M)$ and for each $G' \in \text{Gen}(M)$, $G' \subset G$ implies $G = G'$.

The language $(M - \varepsilon) - (M - \varepsilon)^2$ is denoted by $\text{Root}(M)$. It is well-known that, when M is a submonoid of A^* , $\text{Root}(M)$ is the smallest generator of M . In [3] it is shown that, when M is a submonoid of A^ω , $\text{Root}(M)$ may not be the smallest generator of M and that furthermore some submonoids may have no smallest generator, as shown below.

Example 1: Let M be the submonoid $(a+b)^*(\varepsilon + (ab)^\omega)$.

$G = a + b + (ab)^\omega$ and $G' = a + b + (ba)^\omega$ are two generators of M , but $G \cap G' = a + b$ is not. So M does not have a smallest generator (the smallest generator would be contained in $a + b$!). ■

Hence it is natural to investigate the minimal generators of M .

II. MINIMAL GENERATORS OF SUBMONOIDS OF A^∞

Let M be a submonoid of A^∞ . First let us note that of course for each $G \in \text{Gen}(M)$, $\text{Root}(M)$ is included in G . But unlike A^* , $\text{Root}(M)$ is not always a generator of M (the reason being that the concatenation is a right-regular operation in A^* (i.e. for each $x, y, u \in A^*$, $xu = yu$ implies $x = y$) but it is not a right-regular operation in A^∞). For example, $\text{Root}(A^\infty) = A$ which is not a generator of A^∞ .

We need the three following definitions [3].

DEFINITION 1: Let M be a submonoid of A^∞ .

$\forall w, w' \in M$, $w > w'$ if and only if $w \in (M_{\text{fin}} - \varepsilon)w'$.

We say w is factorizable in M by w' .

As usual ($w > w'$ or $w = w'$) is denoted by $w \geq w'$.

Recall that the previous relation $>$ is only transitive.

DEFINITION 2: Let M be a submonoid of A^∞ . Let $w \in M$.

w is non-factorizable (in M) if and only if

$$\forall w' \in M, \quad w \not> w'.$$

The set of all non-factorizable words of M is denoted by $\text{nf}(M)$.

Remark: $\text{nf}(M) = \text{Root}(M)$ [notation $\text{nf}(M)$ is here convenient, see both following definitions].

DEFINITION 3: Let M be a submonoid of A^∞ . Let $w \in M$.

w is self-factorizable (in M) if and only if

$$\forall w' \in M, \quad w > w' \Rightarrow w' = w.$$

The set of all self-factorizable words of M is denoted by $\text{sf}(M)$.

For our study, we give another definition.

DEFINITION 4: Let M be a submonoid of A^∞ . Let $w \in M$.

w is weakly-factorizable (in M) if and only if

$$\forall w' \in M, \quad w > w' \Rightarrow w' > w.$$

The set of all weakly-factorizable words of M is denoted by $\text{wf}(M)$.

In A^* where $w > w'$ implies $w' \nrightarrow w$, we have $\text{nf}(M) = \text{sf}(M) = \text{wf}(M) = \{w/w \mid w \text{ is minimal with respect to } >\}$. But in A^∞ , we have generally: $\text{nf}(M) \subset \text{sf}(M) \subset \text{wf}(M)$.

Exemple 2: Let M be the submonoid

$$(aaba + ab)^* [\varepsilon + (ab)^\omega + (ba)^\omega + (aba)^\omega + a(aba)^\omega].$$

$$\text{nf}(M) = aaba + ab + (ba)^\omega$$

$$\text{sf}(M) = \text{nf}(M) + (ab)^\omega$$

$$\text{wf}(M) = \text{sf}(M) + (aba)^\omega + a(aba)^\omega$$

(indeed $(aba)^\omega = ab(a(aba)^\omega)$ and $a(aba)^\omega = aaba(aba)^\omega$ furthermore there are not other factorizations). ■

However $\text{nf}(M_{\text{inf}}) = \text{nf}(M)_{\text{fin}} = \text{sf}(M)_{\text{fin}} = \text{wf}(M)_{\text{fin}}$.

LEMMA 1: Let M be a submonoid of A^∞ .

Let G be a minimal generator of M , then we have: $\text{sf}(M) \subset G \subset \text{wf}(M)$ (and a fortiori $\text{Root}(M_{\text{fin}}) = (G_{\text{fin}})$).

Proof: The first inclusion holds for any generator.

Let us assume that g is in $G - \text{wf}(M)$.

For some $w \in M$, we have: $g > w$ and $w \nrightarrow g$.

As G is a generator of M , $\exists g' \in G/w \geq g'$.

Hence $g > g$ and $g \neq g'$, it follows that $(G - g)^* = G^*$. ■

But let us note that $\text{wf}(M)$ is not necessarily a generator of M as shown by the following example.

Exemple 3: Let M be the submonoid $(a + b)^* (\varepsilon + \bigcup_{i \geq 0} a^i b a^{i+1} b \dots)$

$$\text{wf}(M) = a + b, \quad \text{which is not a generator of } M. \quad \blacksquare$$

Notation: For $x \in \{n, s, w\}$, we say that a submonoid M satisfies the condition C_x iff $M_{\text{inf}} \subset M_{\text{fin}} x f(M)$.

PROPOSITION 2: Let M be a submonoid of A^∞ .

(1) The smallest generator of M is $\text{Root}(M)$ iff M satisfies C_n .

(2) M has one smallest generator iff M satisfies C_s .

(3) M has minimal generators iff M satisfies C_w .

Both first equivalences are proved in [3]. For the third one, we take:

DEFINITION 5: Let (u_n) be a sequence of ω -words in M_{inf} .

(u_n) is strictly decreasing (with respect to $>$) iff (u_n) is an injective sequence (i. e. $i \neq j \Rightarrow u_i \neq u_j$) such that for each $i \geq 0$, $u_i > u_{i+1}$.

LEMMA 3: Let M a submonoid of A^ω .

M does not satisfy C_w implies: $\forall G \in \text{Gen}(M)$, there exists a strictly decreasing sequence in G_{inf} .

Proof: As M does not satisfy C_w , the set $M_{\text{inf}} - M_{\text{fin}} \text{ wf}(M)$ denoted by L is nonempty.

We have for each w in L :

(a) $\forall w' \in M_{\text{inf}}, w > w' \Rightarrow w' \in L$,

(b) $w' \in L / w > w'$ and $w' \not> w$.

We are going to construct a strictly decreasing sequence in G_{inf} by induction.

– Let w_1 be in $L \cap G$ [according to (a), w_1 exists].

– Let us assume that $w_1, \dots, w_n$ are constructed.

As $w_n \in L$, there exists $w' \in L$ such that $w_n > w'$ and $w' \not> w_n$ (hence $w_n \neq w'$). As for each $i < n$, $w_i > w_n$, we have $w_i \neq w'$.

As $w' \geq g$ for some g in $G \cap L$, according to (a), by keeping $w_{n+1} = g$, we obtain the $(n+1)$ th term of a strictly decreasing sequence in $G \cap L$. ■

Now to prove that not C_w implies that M does not have minimal generators, let us note that $(G - w_1)^* = G^*$.

Suppose now that $M_{\text{inf}} = M_{\text{fin}} \text{ wf}(M)$ (i. e. M satisfies the condition C_w). Let $\sim$ be the equivalence associated with the preorder $\geq$, i. e. $\sim$ is defined over M by $u \sim v$ if and only if $(u \geq v \text{ or } v \geq u)$. It is easy to verify that $\sim$ saturates $\text{wf}(M)$. For each w in M_{inf} , the $\sim$ -class of w is denoted by $\text{cl}(w)$.

Hence, for each w in $\text{wf}(M)$, $\text{cl}(w)$ is equal to $\{w' \in \text{wf}(M) / w \geq w'\}$ and $\text{cl}(w)$ is a finite language (indeed $w > w'$ and $w' \geq w$ imply w is a periodic ω -word). Let us remark that in $\text{wf}(M)$ the words w of $\text{sf}(M)$ are characterized by $\text{cl}(w) = \{w\}$ [that holds in particular for w in $\text{Root}(M_{\text{inf}})$]. Concerning the generators of M , we can state both following results:

LEMMA 4: $\forall G \in \text{Gen}(M)$, $\forall w \in \text{wf}(M)$, $\text{card}(\text{cl}(w) \cap G) \geq 1$.

LEMMA 5: Let M be a submonoid of A^ω satisfying the condition C_w .

$\forall G \in \text{Gen}(M)$, G is a minimal generator if and only if

(a) $G \subset \text{wf}(M)$ and

(b) $\forall w \in \text{wf}(M)$, $\text{card}(G \cap \text{cl}(w)) = 1$.

Proof: Let G be a minimal generator of M .

Conditions (a) is given by lemma 1.

For condition (b), in view of lemma 4, it remains to consider every w in $G_{\text{inf}} \cap (\text{wf}(M) - \text{sf}(M))$.

Let w' be an ω -word in $\text{cl}(w) \cap G_{\text{inf}}$.

$\forall w'' \in M_{\text{inf}} / w'' \geq'$, we have $w'' \geq w$, hence $w' = w$ otherwise G is not a minimal generator (this implication holds even if M does not satisfy C_w).

Reciprocally, conditions (a) and (b) imply that G_{fin} is the smallest generator of M_{fin} .

Conditions (b) implies that $M_{\text{fin}} \text{wf}(M) = M_{\text{fin}} G_{\text{inf}}$, hence in view of condition C_w , G is a generator of M .

Now conditions (a) and (b) imply that G is a minimal generator of M . ■

The previous lemma closes the proof of the third equivalence of Proposition 2.

COROLLARY 6 : *Let M be a submonoid of A^∞ satisfying the condition C_w . Each generator of M contains at least one minimal generator of M .*

Remark: We find again:

– a proof of equivalence (2) of proposition 2, indeed M has one smallest generator if and only if condition C_w is satisfied and for each w in $\text{wf}(M)$, $\text{cl}(w) = \{w\}$;

– a proof of equivalence (1) of proposition 2, indeed $\text{Root}(M)$ is the smallest generator of M if and only if condition C_s is satisfied and for each w in $\text{wf}(M)$, $w \succ w$.

Example 4: Let M be the monoid A^∞ .

$$\text{nf}(M) = a + b$$

$$\text{sf}(M) = a + b + a^\omega + b^\omega$$

$$\text{wf}(M) = a + b + (A^+)^\Omega.$$

Since A^ω is not included in $A^*(A^+)^\Omega$, A^∞ does not have minimal generators. ■

We end this part with an example where M has infinitely many minimal generators (which is not possible whenever M is a rational submonoid, as shown in the following part).

Example 5: Let M be the submonoid $(a+b)^*[\varepsilon + \bigcup_{i \geq 0} (a^i b)^\omega]$.

$$\begin{aligned} \text{nf}(M) &= a + b \\ \text{sf}(M) &= a + b + b^\omega \\ \text{wf}(M) &= \bigcup_{i \geq 0} \{a^j b (a^i b)^\omega / 0 \leq j \leq i\}. \end{aligned}$$

There are infinitely many $\sim$ -classes:

$$\forall i \geq 0, \quad \text{cl}_i = \{a^j b (a^i b)^\omega / 0 \leq j \leq i\}.$$

Hence M has infinitely many minimal generators. ■

III. RATIONAL CASE

Now we assume that M is a rational submonoid of A^ω (i.e. M_{fin} is a rational language of A^* and M_{inf} is a rational language of A^ω). Let us recall that a ω -language is rational if and only if it is a finite union of ω -languages such as XY^ω where X and Y are rational languages of A^* . We also know [1] that rational ω -languages are characterized as ω -languages recognized by a Büchi automaton.

We are going to prove that one can decide, given a rational submonoid M , whether M satisfies or not a condition C_x . But we first recall the definition of ifl-codes [9] and give two preliminary results.

DEFINITION: Let C be a language, C is an ifl-code if and only if for each u, v in C , $u C^\omega \cap v C^\omega \neq \emptyset$ or $u = v$.

LEMMA 7: Let u, v be two words in A^+ .

If the language $(u+v)$ is a code, then it is an ifl-code.

Proof: We can assume that $|u| \leq |v|$.

So we can write $v = u^n u'$ for some integer $n \geq 0$ and some word u' which is not a prefix of u .

– If u' is a proper prefix of u (i.e. $u = u' u''$ for some u'' in A^+) and $u(u+v)^\omega \cap v(u+v)^\omega \neq \emptyset$ (i.e. $u+v$ is not an ifl-code), we have necessarily: $u^n u' u' u'' = u u^{n-1} u' u'' u'$.

Hence $u'u'' = u''u'$, it follows that $u + v$ is not a code.

– If u' is not a prefix of u , then $u + v$ is an ifl-code. ■

LEMMA 8: *Let L be a language of A^+ .*

If L^ω is an ultimately periodic ω -language then any word m in L satisfies $\{m^\omega\} = L^\omega$.

Proof: Let u be a fixed word in L and let v be any word in L .

The w -word $w = uv \dots u^n v^n \dots$ being ultimately periodic, it is easy to see that $w = m' m^\omega$ for some m, m' in $(u + v)^+$.

Hence $u + v$ is not an ifl-code.

By using the previous lemma, $u + v$ is not a code, the result follows. ■

To decide whether a rational submonoid M satisfies C_n raises no problem since $\text{nf}(M)$ [i.e. $\text{Root}(M)$] is a rational language. But neither $\text{sf}(M)$ nor $\text{wf}(M)$ are rational languages as shown by the following example.

Example 6: Let M be the submonoid $(a^* b)^* (\varepsilon + (a^* b)^\omega)$.

$$\text{nf}(M) = a^* b$$

$$\text{sf}(M) = a^* b + (a^* b)^\omega$$

$[(a^* b)^\omega]$ is not a rational ω -language]

$$\text{wf}(M) = \text{sf}(M) + ((a^* b) +)^\omega - ((a^* b)^\omega). \quad \blacksquare$$

Now we are going to propose a way for deciding, given a rational submonoid, M , whether M satisfies the condition C_s .

Notation: An ω -word w is properly self-factorizable if and only if $w \in \text{sf}(M) - \text{nf}(M)$. The set $\text{sf}(M) - \text{nf}(M)$ is denoted by $\text{Psf}(M)$.

Then the condition C_s can be reformulated by:

LEMMA 9: *Let M be a submonoid of A^ω .*

M satisfies the condition C_s if and only if $M_{\text{inf}} - M_{\text{fin}} \text{nf}(M)$ is included in $M_{\text{fin}} \text{Psf}(M)$.

Now we note that $\text{Psf}(M)$ is a periodic language included in $(M_{\text{fin}})^\omega$, so we have:

LEMMA 10: *Let M be a submonoid of A^ω .*

If M satisfies the condition C_s then $M_{\text{inf}} - M_{\text{fin}} \text{nf}(M)$ is an ultimately periodic language (note that the converse does not hold).

On the other hand:

LEMMA 11: *Let M be a rational language of A^ω .*

On can decide whether M is an ultimately periodic language.

Proof: Let M be a rational language of A^ω given by a rational expression such as $\bigcup_{1 \leq i \leq n} A_i B_i^\omega$, where all A_i and B_i are rational languages of A^* .

If M is an ultimately periodic language, then B_i^ω is also one. By using lemma 8, we obtain: $\forall b_i \in B_i, B_i^\omega = b_i^\omega$.

Hence M is an ultimately periodic language if and only if for each $i \in \{1, \dots, n\}$, $B_i^\omega = b_i^\omega$ for any word b_i in B_i (the sense "if" is trivial).

Consequently one can decide whether M is an ultimately periodic language. ■

COROLLARY 12: *Each rational and ultimately periodic language has a finite number of periodic right-factors. Furthermore everyone is a constructible ω -word (a periodic ω -word is constructible means that one can construct a (finite) period of this ω -word).*

LEMMA 13: *Let M be a submonoid of A^ω .*

Given a periodic ω -word (by a period), one can construct all ω -words w' in $(M_{\text{fin}})^\Omega$ satisfying $w > w'$.

Proof: Let $w = u^\omega$ be a periodic ω -word.

First the number of w' such that $w > w'$ is less than $|u|$.

Let $w' = \hat{u}^\omega$ be a periodic ω -word in $(M_{\text{fin}})^\Omega$ such that $w > w'$. So there exists $v \in M_{\text{fin}} - \varepsilon$ such that $w = vw'$.

Let Q be the set of states of the minimal automaton recognizing M_{fin} .

One can check that $u^\omega = v\hat{u}^\omega$ for some v and $\hat{u}$ in M_{fin} if and only if $u^\omega = \alpha\beta^\omega$ for some α and β in $M_{\text{fin}} \cap \{m \in A^* / |m| \leq 1 + |u| \cdot \text{Card}(Q)\}$.

That closes the proof. ■

COROLLARY 14: *Let M be a rational submonoid of A^ω .*

Given a periodic ω -word (by a period), one can decide whether w belongs to $\text{Psf}(M)$.

Proof:

algorithm:

. decide whether w belongs to M_{inf}

. if yes then

.. construct the set E of all w' in $(M_{\text{fin}})^\Omega$ such that $w > w'$

- .. check whether $E \cap M_{\text{inf}} = \{w\}$
- .. if yes then w belongs to $\text{Psf}(M)$
- else w does not belong to $\text{Psf}(M)$. ■

Now we can state:

PROPOSITION 15: *Given M a rational submonoid of A^∞ , one can decide whether M has a smallest generator.*

Proof:

algorithm:

- .. decide whether $M_{\text{inf}} - M_{\text{fin}} \text{nf}(M)$ is an ultimately periodic language { lemma 11 }
- .. if yes then
 - .. construct the set E of all periodic factors of $M_{\text{inf}} - M_{\text{fin}} \text{nf}(M)$ { corollary 12 }
 - .. construct $E \cap \text{Psf}(M)$ { corollary 14 }
 - .. decide whether $M_{\text{inf}} - M_{\text{fin}} \text{nf}(M)$ is included in $M_{\text{fin}}(E \cap \text{Psf}(M))$
 - .. if yes then M satisfies C_s
 - else M does not satisfy C_s { lemma 9 }
 - else M does not satisfy C_s { lemma 10 }. ■

As $\text{Psf}(M)$ is included in $M_{\text{inf}} - M_{\text{fin}} \text{nf}(M)$, in the previous algorithm, $E \cap \text{Psf}(M)$ is equal to $\text{Psf}(M)$, hence we obtain:

COROLLARY 16: *Let M be a rational submonoid of A^∞ , the smallest generator (if any) is equal to $\text{sf}(M)$ which is a rational and constructible language.*

In the same way, one can prove that:

PROPOSITION 17: *Given M a rational submonoid of A^∞ , one can decide whether M has minimal generators. Furthermore these minimal generators are in finite number, rational and constructible languages.*

Remark: Example 5 shows that, when M is not a rational language, it may have infinitely many minimal generators.

Finally we are interested in the submonoids having a finite set for generator.

DEFINITION: Let M be a submonoid of A^∞ , M is finitely generated if and only if M has a finite generator.

PROPOSITION 18: *Let M be a submonoid of A^∞ .*

M is finitely generated if and only if

(a) $\text{wf}(M)$ is a finite language and (b) M satisfies condition C_w .

Proof: If M is finitely generated, we have:

– condition (a) since, for each w in $\text{wf}(M)$, $\text{cl}(w)$ is a finite set and $\text{wf}(M)$ is then a finite union of finite sets.

– condition (b) indeed M having a finite generator has *a fortiori* minimal generators (but not necessarily one smallest generator, see example 1).

The converse is immediate. ■

COROLLARY 19: *Let M be a rational submonoid of A^ω .*

One can decide whether M is finitely generated.

If so, then M has a finite number of finite generators and furthermore all minimal generators are finite and have the same cardinality.

IV. CHARACTERIZATION OF RATIONAL SUBMONOIDS OF A^ω WITH NON-ERASING MORPHISMS

In this last part we prove that the submonoid $x^* + (x^*y)^\omega$ enable us to obtain every rational submonoid over some alphabet A through a composition of two non-erasing morphisms and one inverse non-erasing morphism.

DEFINITION [5]: Let A, B be two alphabets, a morphism h mapping A^* to B^* is said to be non-erasing if and only if $h(A) \subset B^+$.

We first give a characterization of rational languages of A^ω which is similar to the ones of rational languages either of A^* or of A^ω [5, 6].

PROPOSITION 20: *Let M be a language of A^ω .*

M is a rational language of A^ω if and only if

$$M = h_1 \circ h_2 \circ h_3^1 (x^* z + (x^* y)^\omega)$$

for some non-erasing morphisms h_1, h_2, h_3 .

Proof: The “if”-part is clear since $x^* z + (x^* y)^\omega$ is a rational language.

The “only if”-part is adapted from the proof of proposition 3.1 in [6].

Let $@ = (A, Q, q_0, T, \delta)$ be an automaton recognizing M_{fin} (where A is an alphabet, Q is a finite set of states, q_0 is the initial state, δ is the transition relation and T is the set of recognizing states).

We can assume that $q_0 \notin \delta(Q, A)$.

Let $@' = (A, Q', q'_0, T', \delta')$ be a Büchi automaton recognizing M_{inf} and having a single initial state q'_0 .

We can assume that $q'_0 \notin \delta'(Q', A)$.

We consider the automaton $@ \cup @'$ where q_0 and q'_0 are merged.

In the automaton $@ \cup @'$, the states of $@$ range in $0, \dots, k$ and the states of $@'$ range in $0, k+1, \dots, n$.

Let $\check{A}$ be the alphabet $\{\check{a}/a \in A\}$, $\mathring{A}$ be the alphabet $\{\mathring{a}/a \in A\}$ and t be a new letter.

Let F be the following set

$$F = \{t^i a t^{n-j}/a \in A, q_j \in \delta(q_i, a)\} \cup \{t^i \check{a} t^n/a \in A, \delta(q_i, a) \in T'\} \cup \\ \{t^i a t^{n-j}/a \in A, q_j \in \delta(q_i, a) - T'\} \cup \{t^i \mathring{a} t^{n-j}/a \in A, q_j \in \delta(q_i, a) \cap T'\}.$$

Let h be the morphism defined by:

$$\forall a \in A, \quad h(a) = h(\check{a}) = h(\mathring{a}) = a \quad \text{and} \quad h(t) = \varepsilon$$

So we have:

$$M_{\text{fin}} = h(F^* \cap (A t^n)^* \check{A} t^n) \quad \text{and} \quad M_{\text{inf}} = h(F^\omega \cap [(A t^n)^* \mathring{A} t^n]^\omega)$$

(the assumption $q_0 \notin \delta(Q, A)$ and $q'_0 \notin \delta'(Q', A)$ is here necessary).

We denote by $f_1, \dots, f_p$ the elements of F and let Y be a new alphabet $\{y_1, \dots, y_p\}$.

Let k_1 be the non-erasing morphism defined by:

$$\forall i \in \{1, \dots, p\}, \quad k_1(y_i) = f_i$$

then we have:

$$\forall L \subset A^\infty, \quad L \cap (F^* \cup F^\omega) = k_1 \circ k_1^{-1}(L).$$

So it follows:

$$M = h \circ k_1 \circ k_1^{-1}((A t^n)^* \check{A} t^n + [(A t^n)^* \mathring{A} t^n]^\omega)$$

where $(h \circ k_1)$ is a strictly alphabetic morphism.

On the other hand:

$$(A t^n)^* \check{A} t^n + [(A t^n)^* \mathring{A} t^n]^\omega = k_2^{-1} \circ h_3(x^* y + (x^* z)^\omega)$$

where k_2 is a strictly alphabetic morphism defined by:

$$k_2(t) = t \quad \text{and} \quad \forall a \in A : k_2(a) = x, k_2(\ddot{a}) = z, k_2(\overset{\circ}{a}) = y$$

and h_1 is a non-erasing morphism defined by:

$$h_3(x) = xt^n, h_3(y) = yt^n, h_3(z) = zt^n.$$

Now by denoting $h_2 = k_2 \circ k_1$ and $h_1 = k \circ k_1$, we have the result. ■

Note that $(x^*y + (x^*y)^\omega)$ does not enable us to obtain all rational languages of A^∞ , indeed: if m belongs to $(h_1 \circ h_2^{-1} \circ h_3)(x^*y)$ then m^ω belongs to $(h_1 \circ h_2^{-1} \circ h_3)((x^*y)^\omega)$. That is, $(M_{\text{fin}})^\omega$ is included in M_{inf} !

Now in the same way, we characterize the rational submonoids of A^∞ .

PROPOSITION 21: *Let M be a language of A^∞ .*

M is a rational submonoid of A^∞ if and only if

$$M = h_1 \circ h_2^{-1} \circ h_3(x^* + (x^*y)^\omega)$$

for some non-erasing morphisms h_1, h_2, h_3 .

Proof: The “if”-part holds since the family $\text{Rat}(A^\omega)$ and the family of all submonoids of A^∞ are closed under morphisms and inverse morphisms.

For the “only if”-part, let $@ = (A, Q, q_0, T, \delta)$ be the minimal automaton recognizing $\text{Root}(M_{\text{fin}})$.

Let $@ = (A, Q', q'_0, T', \delta)$ be a Büchi automaton recognizing M_{inf} and such that q'_0 is the single initial state and $q'_0 \notin \delta(Q', A)$.

Replacing letter $\ddot{a}$ by a and hence removing the letter z in the above construction, we obtain the result. ■

Finally we note that none of the families of submonoids satisfying some condition C_x is closed under either morphism, inverse morphism or intersection as shown by the three following examples.

Example 7: Let M be the submonoid $(a+b)^* + [(a+b)^*(c+d)]^\omega$.

M satisfies the condition C_n , but with the morphism h defined by:

$$h(a) = h(c) = a$$

$$h(b) = h(d) = b$$

$h(M) = (a+b)^\infty$ which does not satisfy C_w . ■

Example 8: Let M be the submonoid $(a+b+bc)^*[\varepsilon + ca^*(bca^*)^\omega]$.

M satisfies the condition C_n , but with the morphism h defined by:

$$h(x) = a$$

$$h(y) = bc$$

$h^{-1}(M) = (x+y)^*(\varepsilon + (yx^*)^\omega)$ which does not satisfy C_w . ■

Example 9: Let M be the submonoid $(a+b+bcd)^*[\varepsilon + cda^*(bcd a^*)^\omega]$.

Let M' be the submonoid $(a+bc+bcd)^*[\varepsilon + da^*(bcd a^*)^\omega]$.

M and M' satisfy the condition C_n , but the submonoid

$$M \cap M' = (a+bcd)^*[\varepsilon + (bcd a^*)^\omega]$$

does not satisfy C_w . ■

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