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ON DOT-DEPTH TWO (*)

by F. BLANCHET-SADRI

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Abstract – For positive integers m_1, \dots, m_k , congruences $\sim_{(m_1, \dots, m_k)}$ related to a version of the Ehrenfeucht-Fraïssé game are defined which correspond to level k of the Straubing hierarchy of star-free languages. Given any finite alphabet A , a necessary and sufficient condition is given for the monoids $A^*/\sim_{(m_1, m_2, m_3)}$ to be of dot-depth exactly 2.

Résumé – Étant donnés des entiers positifs m_1, \dots, m_k , on définit des congruences $\sim_{(m_1, \dots, m_k)}$ en relation avec une version du jeu de Ehrenfeucht-Fraïssé, et qui correspondent au niveau k de la hiérarchie de concaténation de Straubing. Étant donné un alphabet fini A , une condition nécessaire et suffisante est donnée pour que les monoides définis par ces congruences soient de dot-depth exactement 2.

1. INTRODUCTION

Let A be a given finite alphabet. The regular languages over A are those subsets of A^* , the free monoid generated by A , constructed from the finite languages over A by the boolean operations, the concatenation product and the star. The star-free languages are those regular languages which can be obtained from the finite languages by the boolean operations and the concatenation product only. According to Schützenberger [15], $L \subseteq A^*$ is star-free if and only if its syntactic monoid $M(L)$ is finite and aperiodic. General references on the star-free languages are McNaughton and Papert [10], Eilenberg [6] or Pin [12].

Natural classifications of the star-free languages are obtained based on the alternative use of the boolean operations and the concatenation product. Let $A^+ = A^* \setminus \{1\}$, where 1 denotes the empty word. Let

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$$A^+ \mathcal{B}_0 = \{ L \subseteq A^+ \mid L \text{ is finite or cofinite} \},$$

$$A^+ \mathcal{B}_{k+1} = \{ L \subseteq A^+ \mid L \text{ is a boolean combination of languages of the form } L_1 \dots L_n (n \geq 1) \text{ with } L_1, \dots, L_n \in A^+ \mathcal{B}_k \}.$$

Only nonempty words over A are considered to define this hierarchy; in particular, the complement operation is applied with respect to A^+ . The language classes $A^+ \mathcal{B}_0, A^+ \mathcal{B}_1, \dots$ form the so-called dot-depth hierarchy introduced by Cohen and Brzozowski in [4]. The union of the classes $A^+ \mathcal{B}_0, A^+ \mathcal{B}_1, \dots$ is the class of star-free languages.

Our attention is directed toward a closely related and more fundamental hierarchy, this one in A^* , introduced by Straubing in [18]. Let

$$A^* \mathcal{V}_0 = \{ 0, A^* \},$$

$$A^* \mathcal{V}_{k+1} = \{ L \subseteq A^* \mid L \text{ is a boolean combination of languages of the form } L_0 a_1 L_1 a_2 \dots a_n L_n (n \geq 0) \text{ with } L_0, \dots, L_n \in A^* \mathcal{V}_k \text{ and } a_1, \dots, a_n \in A \}.$$

$L \subseteq A^*$ is star-free if and only if $L \in A^* \mathcal{V}_k$ for some $k \geq 0$. The *dot-depth* of L is the smallest such k .

Using Eilenberg's correspondence, we have that for each $k \geq 0$, there is a variety V_k of finite monoids such that for $L \subseteq A^*$, $L \in A^* \mathcal{V}_k$ if and only if $M(L) \in V_k$. An outstanding open problem is whether one can decide if a language has dot-depth k , *i. e.*, can we effectively characterize the varieties V_k ? The variety V_0 consists of the trivial monoid alone, V_1 of all finite \mathcal{T} -trivial monoids [16]. Straubing [19] conjectured an effective characterization, based on the syntactic monoid of the language, for the case $k=2$. His characterization, formulated in terms of a novel use of categories in semigroup theory recently developed by Tilson [22], is shown to be necessary in general, and sufficient for an alphabet of two elements.

In the framework of semigroup theory, Brzozowski and Knast [1] showed that the dot-depth hierarchy is infinite. Thomas [21] gave a new proof of this result, which shows also that the Straubing hierarchy is infinite, based on a logical characterization of the dot-depth hierarchy that the obtained in [20] (Perrin and Pin gave one for the Straubing hierarchy [11]) and the following version of the Ehrenfeucht-Fraissé game.

First, one regards a word $w \in A^*$ of length $|w|$ as a word model $w = \langle \{1, \dots, |w|\}, <^w, (Q_a^w)_{a \in A} \rangle$ where the universe $\{1, \dots, |w|\}$ represents the set of positions of letters in w , $<^w$ denotes the $<$ -relation in w , and Q_a^w are unary relations over $\{1, \dots, |w|\}$ containing the positions with letter a , for each $a \in A$. For a sequence $\bar{m} = (m_1, \dots, m_k)$ of positive integers, where $k \geq 0$, the game $\mathcal{G}_{\bar{m}}(u, v)$ is played between two players I and II on the word models u and v . A play of the game consists of k moves. In the

i -th move, player I chooses, in u or in v , a sequence of m_i positions; then player II chooses, in the remaining word, also a sequence of m_i positions. After k moves, by concatenating the sequences chosen from u and v , two sequences $p_1 \dots p_n$ from u and $q_1 \dots q_n$ from v have been formed where $n = m_1 + \dots + m_k$.

Player II has won the play if

$$p_i <^u p_j \quad \text{if and only if} \quad q_i <^v q_j, \tag{1}$$

and

$$Q_a^u p_i \quad \text{if and only if} \quad Q_a^v q_i, \quad a \in A \quad \text{for } 1 \leq i, j \leq n. \tag{2}$$

If there is a winning strategy for II in the game $\mathcal{G}_{\bar{m}}(u, v)$ to win each play we write $u \sim_{\bar{m}} v$. $\sim_{\bar{m}}$ naturally defines a congruence on A^* which we denote also by $\sim_{\bar{m}}$. The standard Ehrenfeucht-Fraissé game [5] is the special case $\mathcal{G}_{(1, \dots, 1)}(u, v)$. Thomas [20], [21] and Perrin and Pin [11] imply that $L \in A^* \mathcal{V}_k$ if and only if L is a $\sim_{\bar{m}}$ -language for some $\bar{m} = (m_1, \dots, m_k)$ (or L is a union of classes of the congruence $\sim_{\bar{m}}$). This congruence characterization implies that the problem of deciding whether a language has dot-depth k is equivalent to the problem of effectively characterizing the monoids $M = A^*/\sim$ with $\sim \supseteq \sim_{\bar{m}}$ for some $\bar{m} = (m_1, \dots, m_k)$, *i. e.*,

$$V_k = \{ A^*/\sim \mid \sim \supseteq \sim_{\bar{m}} \text{ for some } \bar{m} = (m_1, \dots, m_k) \}.$$

This paper is concerned with an application of the above congruence characterization. We show that $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2 = 1$. The proof relies on some properties of the congruences $\sim_{\bar{m}}$ stated in the next section. [2] and [3] include other applications: among them are an answer to a conjecture of Pin [13] concerning tree hierarchies of monoids and also systems of equations satisfied in natural sublevels of level 1 of the Straubing hierarchy. The reader is referred to the books by Eilenberg [6], Lallement [9], Pin [12], Enderton [7] and Fraissé [8] for all the algebraic and logical terms not defined here.

2. SOME PROPERTIES OF THE CHARACTERIZING CONGRUENCES

2.1. An induction lemma

The following lemma is a basic result (similar to one in [14] regarding $\sim_{(1, \dots, 1)}$) which allows to resolve games with $k + 1$ moves into games with

k moves and thereby allows to perform induction arguments. In what follows, $u^{<p}(u_{>p})$ denotes the subword of u to the left (right) of position p and $u_{>p}^{<q}$ the subword of u between positions p and q .

LEMMA 2.1.: *Let $\bar{m} = (m_1, \dots, m_k)$. $u \sim_{(m, m_1, \dots, m_k)} v$ if and only if*

(1) *for every $p_1, \dots, p_m \in u (p_1 \leq \dots \leq p_m)$ there are $q_1, \dots, q_m \in v (q_1 \leq \dots \leq q_m)$ such that*

(i) $Q_a^u p_i$ *if and only if* $Q_a^v q_i, a \in A$ *for* $1 \leq i \leq m$,

(ii) $u^{<p_1} \sim_{\bar{m}} v^{<q_1}$,

(iii) $u_{>p_i}^{<p_{i+1}} \sim_{\bar{m}} v_{>q_i}^{<q_{i+1}}$ *for* $1 \leq i \leq m-1$,

(iv) $u_{>p_m} \sim_{\bar{m}} v_{>q_m}$ *and*

(2) *for every $q_1, \dots, q_m \in v (q_1 \leq \dots \leq q_m)$ there are $p_1, \dots, p_m \in u (p_1 \leq \dots \leq p_m)$ such that (i), (ii), (iii) and (iv) hold.*

2.2. A lemma for inclusion

Define

$$\mathcal{N}_{(m_1, \dots, m_k)} = m_1 + \dots + m_k + \sum_{1 \leq i_1 < i_2 \leq k} m_{i_1} m_{i_2} + \dots + \sum_{1 \leq i_1 < \dots < i_{k-1} \leq k} m_{i_1} \dots m_{i_{k-1}} + m_1 \dots m_k.$$

One can show that $x^N \sim_{(m_1, \dots, m_k)} x^{N+1}$ ($N = \mathcal{N}_{(m_1, \dots, m_k)}$) and that N is the smallest n such that $x^n \sim_{(m_1, \dots, m_k)} x^{n+1}$ (the proof is similar to the one of a property of $\sim_{(1, \dots, 1)}$ in [21]). We see that if $u, v \in A^*$ and $u \sim_{(m_1, \dots, m_k)} v$, then $|u|_a = |v|_a < \mathcal{N}_{(m_1, \dots, m_k)}$ or $|u|_a, |v|_a \geq \mathcal{N}_{(m_1, \dots, m_k)}$ (here $|w|_a$ denotes the number of occurrences of the letter a in w). The following lemma follows easily from Lemma 2.1 and the above remarks.

LEMME 2.2 :

$$\sim_{(m_1, \dots, m_k)} \subseteq \sim_{(\mathcal{N}_{(m_1, \dots, m_k)})}, \quad \text{and} \quad \sim_{(m_1, \dots, m_k)} \not\subseteq \sim_{(\mathcal{N}_{(m_1, \dots, m_k)} + 1)}.$$

If $k \leq k'$ and $\exists 0 = j_0 < \dots < j_{k-1} < j_k = k'$ such that $m_i \leq \mathcal{N}_{(m'_{j_{i-1}+1}, \dots, m'_{j_i})}$ for $1 \leq i \leq k$, then $\sim_{(m'_1, \dots, m'_k)} \subseteq \sim_{(m_1, \dots, m_k)}$.

3. A SEQUENCE OF MONOIDS OF DOT-DEPTH 2

In this section, we show that for positive integers m_1, m_2 and m_3 , $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2 = 1$. The following lemma shows the necessity of the condition.

LEMMA 3.1: *Let m_1 and m_3 be positive integers. Then $A^*/\sim_{(m_1, 2, m_3)}$ is of dot-depth exactly 3.*

Proof: Let $m > 0$. Consider $u_m = ((xy)^m x (xy)^{2m} y (xy)^m)^m$, $v_m = ((xy)^m y (xy)^{2m} x (xy)^m)^m$. A result of Straubing [17] implies that monoids in V_2 are 2-mutative and hence satisfy $u_m = v_m$ for all sufficiently large m . However, for every $N \geq \mathcal{N}_{(1, 2, 1)}$, $u_N x_{(1, 2, 1)} v_N$. To see this, we illustrate a winning strategy for player I in the game $\mathcal{G}_{(1, 2, 1)}(u_N, v_N)$. (I, i) denotes a position chosen by player I in the i -th move, $i = 1, 2, 3$. Similarly, (II, i) denotes a position chosen by player II in the i -th move. Player I, in the first move, chooses the

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & & & 2N & & N \\
 & & & & \overbrace{\hspace{10em}} & & \overbrace{\hspace{10em}} \\
 u_N = \dots (xy)^N & x & (xy)(xy)\dots(xy)(xy) & y & (xy)(xy)\dots(xy)(xy) \\
 & \uparrow & & \uparrow \uparrow & & & \\
 & (II, 1) & & (I, 2) & & &
 \end{array} \\
 \\
 \begin{array}{ccccccc}
 & & & & 2N & & N \\
 & & & & \overbrace{\hspace{10em}} & & \overbrace{\hspace{10em}} \\
 v_N = \dots (xy)^N & x & (xy)(xy)\dots(xy)(xy) & y & (xy)(xy)\dots(xy)(xy) \\
 & & & & & & \\
 & & & & & & \\
 \overbrace{\hspace{10em}} & & & & \overbrace{\hspace{10em}} & & \\
 (xy)(xy)\dots(xy)(xy) & x & (xy)(xy)\dots(xy)(xy)\dots(xy)(xy) \\
 & \uparrow & & \uparrow \uparrow & & & \\
 & (I, 1) & & (II, 2) & & &
 \end{array}
 \end{array}$$

last x followed immediately by an x in v_N . Player II, in the first move, has to choose the last x followed immediately by an x in u_N (if not, player I in the next two moves could win by choosing in the second move the last two consecutive x 's in u_N). Player I, in the second move, chooses the last two consecutive y 's in u_N . Player II, in the second move, cannot choose two consecutive y 's in v_N to the right of the previously chosen position. Hence he is forced to choose two y 's separated by an x . Player I, in the third move, selects that x . But player II loses since he cannot choose an x between the two consecutive y 's chosen in the preceding move by I. The result follows. []

Assume $|u|_a, |v|_a > 0$. Let $u = u_0 a u_1 \dots a u_{|u|_a}$, $v = v_0 a v_1 \dots a v_{|v|_a}$. If $Q_a^u p_i, Q_a^v q_j$ for $i = 1, \dots, |u|_a, j = 1, \dots, |v|_a$, then $u_i = u_{> p_i}^{< p_i + 1}$, $i = 1, \dots, |u|_a - 1, v_j = v_{> q_j}^{< q_j + 1}, j = 1, \dots, |v|_a - 1. u_0 = u_{< p_1}, v_0 = v_{< q_1}, u_{|u|_a} = u_{> p_{|u|_a}}, v_{|v|_a} = v_{> q_{|v|_a}}$.

The next two lemmas will be used in showing that for positive integers m_1 and $m_3, A^*/\sim_{(m_1, 1, m_3)}$ is of dot-depth exactly 2.

LEMMA 3.2.: Assume $u \sim_{(m'_1, m'_2)} v$. Then

$$u^{<p}(s-1)m'_2+i \sim_{(m'_1-s, m'_2)} v^{<q}(s-1)m'_2+i, \tag{1}$$

$$u_{>p|u|_a+1-(s-1)m'_2-i} \sim_{(m'_1-s, m'_2)} v_{>q|v|_a+1-(s-1)m'_2-i}$$

for $i=1, \dots, m'_2$ and $s=1, \dots, m'_1-1$. $\tag{2}$

Proof: (1) Let $1 \leq i \leq m'_2$ and $1 \leq s \leq m'_1-1$. Let $p'_1, \dots, p'_{m'_1-s}$ ($p'_1 \leq \dots \leq p'_{m'_1-s}$) be positions in $u^{<p}(s-1)m'_2+i$. Consider the following play of the game $\mathcal{G}_{(m'_1, m'_2)}(u, v)$. Player I, in the first move, chooses $p_{m'_2}, p_{2m'_2}, \dots, p_{(s-1)m'_2}, p_{(s-1)m'_2+i}, p'_1, \dots, p'_{m'_1-s}$. Hence by the lemma of Induction 2.1, there exist positions $q'_1, \dots, q'_{m'_1-s}$ ($q'_1 \leq \dots \leq q'_{m'_1-s}$) in $v^{<q}(s-1)m'_2+i$ such that player II, by choosing $q_{m'_2}, q_{2m'_2}, \dots, q_{(s-1)m'_2}, q_{(s-1)m'_2+i}, q'_1, \dots, q'_{m'_1-s}$ for the corresponding positions, wins this play of the game. It is clear that

- (i) $u^{<p'_1} \sim_{(m'_2)} v^{<q'_1}$,
- (ii) $u^{<p'_j+1} \sim_{(m'_2)} v^{<q'_j+1}$ for $1 \leq j \leq m'_1-s-1$,
- (iii) $u^{<p_{(s-1)m'_2+i}} \sim_{(m'_2)} v^{<q_{(s-1)m'_2+i}}$.

Note that player II has to choose $q_{m'_2}, q_{2m'_2}, \dots, q_{(s-1)m'_2}, q_{(s-1)m'_2+i}$ because there is a number of $a's < m'_2$ between any two consecutive positions among $p_{m'_2}, p_{2m'_2}, \dots, p_{(s-1)m'_2}, p_{(s-1)m'_2+i}$.

The proof is similar, when starting with positions in $v^{<q}(s-1)m'_2+i$.

For (2), we consider $p_{|u|_a+1-m'_2}, p_{|u|_a+1-2m'_2}, \dots, p_{|u|_a+1-(s-1)m'_2}, p_{|u|_a+1-(s-1)m'_2-i}, p'_1, \dots, p'_{m'_1-s}$. []

LEMMA 3.3: Assume $u \sim_{(m'_1, m'_2)} v$. Then

- (1) $u_{>p_{(s-1)m'_2+i}} \sim_{(m'_1-s, m'_2)} v_{>q_{(s-1)m'_2+i}}$
- (2) $u^{<p}|u|_a+1-(s-1)m'_2-i \sim_{(m'_1-s, m'_2)} v^{<q}|v|_a+1-(s-1)m'_2-i$ for $i=1, \dots, m'_2$ and $s=1, \dots, m'_1-1$.

Proof: Similar to Lemma 3.2. []

In the following theorem we talk about positions spelling the first and last occurrences of every subword of length $\leq m$ of a word w . We illustrate what we mean by this with the following example. Let $A = \{a, b, c\}$ and

$$u = abccccabbabbaccabababccaaaabbaa \dots$$

$$\begin{matrix} \uparrow \uparrow \uparrow \uparrow \uparrow & \uparrow & & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow \uparrow \\ & & & & p \end{matrix}$$

The six arrows on the left point to the positions which spell the first occurrences of every subword of length ≤ 2 in $u^{<p}$ and the eight arrows on the right (before the one pointing to p) to the positions which spell the last occurrences of every subword of length ≤ 2 in $u^{<p}$.

THEOREM 3.4: *Let m_1, m_2 and m_3 be positive integers. Then $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2 if and only if $m_2 = 1$.*

Proof: If $A^*/\sim_{(m_1, m_2, m_3)}$ is of dot-depth exactly 2, then $m_2 < 2$ by

LEMMA 3.1.: *Conversely, for $|A|=r>1$, we show that for any positive integers $m'_1, m'_2, \sim_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)} \cong \sim_{(m'_1, 1, m'_2)}$.*

To see this, suppose $u \sim_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)} v$. Then there is a winning strategy for player II in the game $\mathcal{G}_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)}(u, v)$ to win each play. A winning strategy for player II in the game $\mathcal{G}_{(m'_1, 1, m'_2)}(u, v)$ to win each play is described as follows. Let $p'_1, \dots, p'_{m'_1}$ ($p'_1 \leq \dots \leq p'_{m'_1}$) be positions in u chosen by player I in the first move. Player II chooses positions $q'_1, \dots, q'_{m'_1}$ ($q'_1 \leq \dots \leq q'_{m'_1}$) by considering the following play of the game $\mathcal{G}_{(m'_1+(m'_1+1)2m'_2(r+1)^{m'_2}, m'_2)}(u, v)$. In the first move, player I chooses $p'_1, \dots, p'_{m'_1}$ and the positions which spell the first and last occurrences of every subword of length $\leq m'_2$ in $u^{<p'_1}, u^{>p'_1}, \dots, u^{<p'_{m'_1-1}}$ and $u^{>p'_{m'_1}}$ for a total of no more than $m'_1 + (m'_1 + 1)2m'_2(r+1)^{m'_2}$ positions (there are $r^{m'_2}$ possible words of length m'_2 for a total of no more than $m'_2(r+1)^{m'_2}$ positions to spell the first (last) occurrences of every subword of length $\leq m'_2$). More details follow for the special case $u \sim_{(1+4m'_2(r+1)^{m'_2}, m'_2)} v$. We have a winning strategy for player II in the game $\mathcal{G}_{(1+4m'_2(r+1)^{m'_2}, m'_2)}(u, v)$ to win each play. Let us describe a winning strategy for player II in the game $\mathcal{G}_{(1, 1, m'_2)}(u, v)$ to win each play. Let p be a position in u chosen by player I in the first move. Suppose $Q_a^u p$ for some $a \in A$. If p is the i -th occurrence of a in u ($1 \leq i \leq \mathcal{N}_{(1, m'_2)} = 2m'_2 + 1$), then player II chooses the same occurrence of a in v , say position q . The fact that $u^{<p} \sim_{(1, m'_2)} v^{<q}$ and $u^{>p} \sim_{(1, m'_2)} v^{>q}$ follows from Lemmas 3.2 and 3.3 ($\mathcal{N}_{(1, m'_2)} \leq (4m'_2(r+1)^{m'_2})m'_2$). If p is the $|u|_a + 1 - i$ -th occurrence of a in u ($1 \leq i \leq \mathcal{N}_{(1, m'_2)}$), player II chooses the $|v|_a + 1 - i$ -th occurrence of a in v . If p is among $p_{2m'_2+2}, \dots, p_{|u|_a-2m'_2-1}$, then player II chooses position q , an a , among $q_{2m'_2+2}, \dots, q_{|v|_a-2m'_2-1}$ by considering the following play of the game $\mathcal{G}_{(1+4m'_2(r+1)^{m'_2}, m'_2)}(u, v)$. In the first move, player I chooses p , the positions which spell the first and last occurrences of every subword of length $\leq m'_2$ in $u^{<p}$ and in $u^{>p}$. Hence there exists a position q in v such that player II, by choosing q , the positions which spell the first and last occurrences of every subword of length $\leq m'_2$ in $v^{<q}$ and in $v^{>q}$, wins the play of the game. Let us show that $u^{<p} \sim_{(1, m'_2)} v^{<q}$ (the proof that $u^{>p} \sim_{(1, m'_2)} v^{>q}$ is similar). Let p' be

a position in $u^{<p}$ (the proof is similar when starting with a position in $v^{<q}$). Assume $Q_{a_i}^u p'$.

Case 1: p' is among the first m'_2 occurrences of a_i in $u^{<p}$.

Let q' be the same occurrence among the first m'_2 occurrences of a_i in $v^{<q}$. It is clear that $u^{<p}_{>p'} \sim_{(m'_2)} v^{<q}_{>q'}$ and $u^{<p'} \sim_{(m'_2)} v^{<q'}$.

Case 2: p' is among the last m'_2 occurrences of a_i in $u^{<p}$. Similar to case 1.

Case 3: p' is not among the first m'_2 nor the last m'_2 occurrences of a_i in $u^{<p}$.

Let p'' and p''' ($p'' < p'''$) be the closest positions to p' in $u^{<p'}$ and $u^{<p}_{>p'}$, respectively among the chosen positions by player I. Let q'' and q''' ($q'' < q'''$) be the corresponding positions chosen by player II.

Since $u^{<p}_{>p'''} \sim_{(m'_2)} v^{<q}_{>q'''}$, there is q' in $v^{<q}_{>q'''}$ such that $Q_{a_i}^v q'$.

Let us show that $u^{<p}_{>p'} \sim_{(m'_2)} v^{<q}_{>q'}$. $u^{<p'} \sim_{(m'_2)} v^{<q'}$ follows similarly.

Let $w = w_1 \dots w_{|w|}$, $|w| \leq m'_2$ in $v^{<q}$. The proof is similar when starting with w in $u^{<p}_{>p'}$. If $w \in v^{<q}_{>q'''}$, it is clear that $w \in u^{<p}_{>p'''}$, hence in $u^{<p}_{>p'}$. So let us assume $w \notin v^{<q}_{>q'''}$. Let $p_{w_1}, \dots, p_{w_{|w|}}$ in $v^{<q}$, at least p_{w_1} being in $v^{<q}_{>q'''}$, which spell $w_1 \dots w_{|w|}$. $p_{w_1}, \dots, p_{w_{|w|}}$ are hence positions which spell an occurrence of a subword of length $\leq m'_2$ in $v^{<q}$. Hence they are smaller than or equal to those positions which spell the last occurrence of w in $v^{<q}$ which are in $v^{<q}_{\geq q'''}$. Hence $w \in u^{<p}_{>p'}$. []

The following corollary gives another result for inclusion (one was Lemma 2.2).

COROLLARY 3.5: Let $|A| = r$. Then

$$\sim_{(m'_1 + (m'_1 + 1) 2m'_2 (r + 1)^{m'_2}, m'_2)} \cong \sim_{(m'_1, \mathcal{N}_{(1, m'_2)})}$$

Proof: From Theorem 3.4 and Lemma 2.2. []

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