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BI-INFINITARY CODES (*)

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Abstract. – *The notion of bi-infinitary codes is introduced. For this purpose, the monoid ${}^{\infty}A^{\infty}$ of finite, infinite and bi-infinite words over an alphabet A is defined. A necessary and sufficient condition for a set of words to be a bi-infinitary code is formulated. Conditions for a submonoid of ${}^{\infty}A^{\infty}$ to have a minimal generator set are established. Using a specific kind of Thue system, the notion of bi-quasi free sub-monoids is introduced. An “algebraic” characterization of the submonoids generated by bi-infinitary codes is obtained. Finally, a “combinatorial” characterization of bi-quasi free submonoids is studied.*

Résumé. – *On introduit la notion de code biinfini. On définit d’abord le monoïde ${}^{\infty}A^{\infty}$ des mots finis, infinis ou biinfinis sur un alphabet A . On énonce une condition nécessaire et suffisante pour qu’un ensemble de mots soit un code biinfini. On donne également des conditions pour qu’un sous-monoïde de ${}^{\infty}A^{\infty}$ ait un ensemble minimal de générateurs. En utilisant un système de Thue spécifique, on introduit la notion de sous-monoïde bi-quasi libre. Une caractérisation « algébrique » des sous-monoïdes engendrés par des codes bi-infinis est alors obtenue. Finalement, on étudie une caractérisation « combinatoire » des sous-monoïdes bi-quasi libres.*

INTRODUCTION

There has been a systematic study of codes consisting of finite words, initiated by M. P. Schützenberger [16] and developed by many others taking motivation from information theory (see [11-13]).

Recently, infinitary languages consisting of finite and infinite words have served as an adequate tool for studying behaviours of processes. This is the approach of M. Nivat and A. Arnold [14] in some problems of synchronization which stimulated the study of infinite words including bi-infinite words [15].

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Motivated by the theory of codes [1] and the theory of infinitary languages, the notion of infinitary codes has been introduced and examined in [3-10].

This paper is devoted to a study of bi-infinitary codes which are a natural generalization of infinitary codes to bi-infinitary languages *i. e.*, languages of finite, left-infinite, right-infinite and bi-infinite words.

SECTION 1

MONOID ${}^{\infty}A^{\infty}$ AND BI-INFINITARY CODES

Let A be an alphabet. We denote by A^* , the free monoid generated by A . Elements of A^* are called finite words. The length of a word x in A^* is denoted by $|x|$, the empty word by ε and $A^+ = A^* - \{\varepsilon\}$.

We denote by A^N , the set of all right-infinite words, by A^{-N} , the set of all left-infinite words and by A^Z , the set of all bi-infinite words over A . Every (bi) infinite word u has a countable length $|u| = \omega$. For any $X \subseteq A^*$, we denote by $X^{\omega} ({}^{\omega}X, {}^{\omega}X^{\omega})$, the set of all right-infinite (left-infinite, bi-infinite) words of the form $x_1 x_2 \dots (\dots x_2 x_1, \dots x_1 x_2 x_3 \dots)$ for $x_i \in X$. In particular, if $x \in A^*$, then $x^{\omega} = xxx \dots$, ${}^{\omega}x = \dots xxx$ and ${}^{\omega}x^{\omega} = \dots xxx \dots$. We write $A^{\infty} = A^* \cup A^N$, ${}^{\infty}A = A^* \cup A^{-N}$ and ${}^{\infty}A^{\infty} = A^* \cup A^N \cup A^{-N} \cup A^Z$.

We define a product on elements of ${}^{\infty}A^{\infty}$ as follows:

$$\alpha \cdot \beta = \begin{cases} \alpha, & \text{if } \alpha \in A^N \cup A^Z \\ \alpha\beta, & \text{if } \alpha \in A^* \cup A^{-N}, \beta \in A^* \cup A^N \\ \beta, & \text{if } \alpha \in A^* \cup A^{-N}, \beta \in A^{-N} \cup A^Z. \end{cases}$$

It is not difficult to verify that the product is associative and therefore ${}^{\infty}A^{\infty}$ is a monoid. This monoid has A^* , A^{∞} and ${}^{\infty}A$ as its submonoids. For simplicity, instead of $\alpha \cdot \beta$, we write $\alpha\beta$. For any $X \subseteq {}^{\infty}A^{\infty}$, we denote by X^* , the submonoid of ${}^{\infty}A^{\infty}$ generated by X and write $X^+ = X^* - \{\varepsilon\}$. If α is a word, instead of $\{\alpha\}^*$, we write α^* .

For any $X \subseteq {}^{\infty}A^{\infty}$, we write $X_{\text{fin}} = X \cap A^*$,

$$X_{\text{inf}} = X \cap A^N, \quad X_{- \text{inf}} = X \cap A^{-N}, \quad X_{\text{biinf}} = X \cap A^Z,$$

$$X^{\infty} = X_{\text{fin}} \cup X_{\text{inf}}, \quad {}^{\infty}X = X_{\text{fin}} \cup X_{- \text{inf}},$$

$$\bar{X}^{(0)} = X^{(0)} = \{\varepsilon\},$$

$$\bar{X}^{(1)} = X^{(1)} = X,$$

$$X^{(\vec{n})} = \{(x_1, x_2, \dots, x_n) / x_1, x_2, \dots, x_{n-1} \in X_{\text{fin}}, x_n \in X^{\infty}\} \quad \text{for } n \geq 2,$$

$$X^{(\vec{n})} = \{(x_1, x_2, \dots, x_n) / x_1 \in {}^\infty X, x_2, x_3, \dots, x_n \in X_{\text{fin}}\} \quad \text{for } n \geq 2,$$

$$X^{(\vec{n})} = \{(x_1, x_2, \dots, x_n) /$$

$$x_1 \in X_{-\text{inf}}, x_n \in X_{\text{inf}}, x_2, x_3, \dots, x_{n-1} \in X_{\text{fin}}\} \quad \text{for } n \geq 2$$

$$X^{(n)} = X^{(\vec{n})} \cup X^{(\vec{n})} \cup X^{(\vec{n})} \quad \text{for } n \geq 2,$$

$$X^{(*)} = \bigcup_{n \geq 0} X^{(n)}$$

$$\bar{X}^{(\vec{n})} = \{x_1 x_2 \dots x_n / (x_1, x_2, \dots, x_n) \in X^{(\vec{n})}\} \quad \text{for } n \geq 2$$

$$\bar{X}^{(\vec{n})} = \{x_1 x_2 \dots x_n / (x_1, x_2, \dots, x_n) \in X^{(\vec{n})}\} \quad \text{for } n \geq 2,$$

$$\bar{X}^{(\vec{n})} = \{x_1 x_2 \dots x_n / (x_1, x_2, \dots, x_n) \in X^{(\vec{n})}\} \quad \text{for } n \geq 2$$

and

$$\bar{X}^{(n)} = \bar{X}^{(\vec{n})} \cup \bar{X}^{(\vec{n})} \cup \bar{X}^{(\vec{n})} \quad \text{for } n \geq 2.$$

We say that a word $\alpha \in {}^\infty A^\infty$ has a factorization on elements of X if $\alpha = x_1 x_2 \dots x_n$ for some $(x_1, x_2, \dots, x_n) \in X^{(*)}$.

DEFINITION 1.1: A subset X of ${}^\infty A^\infty$ is called a bi-infinitary code if every word $\alpha \in {}^\infty A^\infty$ has atmost one factorization on elements of X . More precisely, X is a bi-infinitary code if for any $n, m \geq 1$ and for any $(x_1, x_2, \dots, x_n) \in X^{(n)}$, $(x'_1, x'_2, \dots, x'_m) \in X^{(m)}$, the equality $x_1 x_2 \dots x_n = x'_1 x'_2 \dots x'_m$ implies $n = m$ and $x_i = x'_i$ ($i = 1, 2, \dots, n$).

Unless otherwise stated, from now on code means bi-infinitary code.

Example 1.1 : If $A = \{a, b\}$, the subset

$$X = \{ {}^\omega(ab)^\omega, {}^\omega a, b^\omega, ba \}$$

is a code whereas the subset

$$Y = \{ {}^\omega(ab)^\omega, {}^\omega a, b^\omega, ab \}$$

is not a code, since we have,

$$\begin{aligned} {}^\omega ab^\omega &= {}^\omega a.ab.b^\omega \\ &= {}^\omega a.b^\omega. \end{aligned}$$

SECTION 2

A CHARACTERIZATION OF BI-INFINITARY CODES

In this section, we establish a characterization of codes. We first introduce certain concepts and formulate a fundamental formula.

Let X and Y be two subsets of ${}^\infty A^\infty$. Define the sets

$$\begin{aligned} Y^{-1}X &= \{ \alpha \in {}^\infty A^\infty \mid \exists \beta \in Y: \beta\alpha \in X, \\ &\quad (\beta \in Y_{\text{inf}} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon), \\ &\quad (\beta \in {}^\infty Y \text{ and } \alpha \in A^{-N} \cup A^Z \Rightarrow \beta = \varepsilon) \}, \\ XY^{-1} &= \{ \alpha \in {}^\infty A^\infty \mid \exists \beta \in Y: \alpha\beta \in X, (\alpha \in A^N \cup A^Z \Rightarrow \beta = \varepsilon), \\ &\quad (\alpha \in {}^\infty A \text{ and } \beta \in Y_{\text{-inf}} \cup Y_{\text{biinf}} \Rightarrow \alpha = \varepsilon) \}. \end{aligned}$$

We note that if $u, v \in A^{-N}$ and $u \leq v$, then $u^{-1}v$ is a subset of A^* . For example, if $u = {}^\omega a$ and $v = {}^\omega a = {}^\omega a \cdot a^*$, then $u^{-1}v = a^*$.

We associate with every subset $X \subseteq {}^\infty A^\infty$, a sequence of subsets, denoted by $U_n(X)$ or simply by U_n , defined recursively by

$$\begin{aligned} U_1 &= X^{-1}X - \{ \varepsilon \} \\ U_{n+1} &= X^{-1}U_n \cup U_n^{-1}X, \quad n \geq 1. \end{aligned}$$

LEMMA 2.1: For any subset X of ${}^\infty A^\infty - \{ \varepsilon \}$, (i) if n is the smallest natural number such that $\varepsilon \in U_n$, then $\forall k \in \{1, 2, \dots, n\}$, $\exists u \in U_k$, $\exists i, j \geq 0$:

$$\begin{aligned} u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} &\neq \Phi, \quad i+j+k=n, \\ u \in A^N \cup A^Z &\Rightarrow i=0 \end{aligned} \tag{2.1}$$

(ii) $\forall n \geq 1$, $\forall k \in \{1, 2, \dots, n\}$:

$$\begin{aligned} (\exists u \in U_k, \exists i, j \geq 0: u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} &\neq \Phi, \\ i+j+k &= n, \\ u \in A^N \cup A^Z \Rightarrow i=0) &\Rightarrow \varepsilon \in U_n. \end{aligned}$$

Proof: We prove by recurrence on k .

(i) Let n be the smallest natural number such that $\varepsilon \in U_n$. If $k=n$, then (2.1) holds obviously with $u=\varepsilon$, $i=j=0$. Let $n > k \geq 1$ and suppose the statement is true for $n, n-1, \dots, k+1$. We prove for k . Since the statement

is true for $k+1$, there exist $v \in U_{k+1}$ and integers i', j' such that

$$v(\bar{X}^{(i')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j')} \neq \Phi, \quad i' + j' + k + 1 = n,$$

$v \in A^N \cup A^Z \Rightarrow i' = 0$. Thus we have $x \in \bar{X}^{(i')} - (A^{-N} \cup A^Z)$ and $y \in \bar{X}^{(j')}$ such that $vx = y$. The fact that $v \in U_{k+1}$ gives rise to two cases.

Case (a): $v \in X^{-1} U_k$. Then, there exists $z \in X$, $u \in U_k$ such that

$$zv = u, (z \in X_{\text{inf}} \cup X_{\text{biinf}} \Rightarrow v = \varepsilon)$$

and

$$(z \in {}^\infty X \text{ and } v \in A^{-N} \cup A^Z \Rightarrow z = \varepsilon).$$

If $v \in A^N$, then $i' = 0$, $x = \varepsilon$, $z \in {}^\infty X$ and $u = zy$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j'+1)} \neq \Phi$. Thus (2.1) holds with $i = 0$, $j = j' + 1$.

If $v \in A^{-N} \cup A^Z$, then $z \in {}^\infty X$ and so $z = \varepsilon$. Thus $\varepsilon \in X$ which contradicts the hypothesis that $X \subseteq {}^\infty A^\infty - \{\varepsilon\}$.

If $v \in A^*$ and $z \in X_{\text{inf}} \cup X_{\text{biinf}}$, then $v = \varepsilon$ and $u = z$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(1)} \neq \Phi$ and therefore (2.1) holds with $i = 0$, $j = 1$.

If $v \in A^*$ and $z \in {}^\infty X$, then $ux = zy$ and so

$$u(\bar{X}^{(i')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j'+1)} \neq \Phi.$$

Thus (2.1) holds with $i = i'$, $j = j' + 1$.

Case (b): $v \in U_k^{-1} X$. Then, there exist $u \in U_k$ and $z \in X$ such that $uv = z$, $(u \in A^N \cup A^Z \Rightarrow v = \varepsilon)$ and $(u \in {}^\infty A, v \in A^{-N} \cup A^Z \Rightarrow u = \varepsilon)$.

If $v \in A^N$, then $i' = 0$, $x = \varepsilon$, $v = y$, $u \in {}^\infty A$ and $uy = z$. Hence $u(\bar{X}^{(j')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(1)} \neq \Phi$. So, (2.1) holds with $i = j'$, $j = 1$.

If $v \in A^{-N} \cup A^Z$, then $u \in {}^\infty A$ and therefore $u = \varepsilon$. Thus $\varepsilon = u \in U_k$ with $k < n$, which is contrary to the hypothesis that n is the smallest natural number such that $\varepsilon \in U_n$.

If $v \in A^*$ and $z \in X_{\text{inf}} \cup X_{\text{biinf}}$, then $v = \varepsilon$, $u = z$ and $y = x$. If $i' = j' = 0$, then $k+1 = n$ and the equality $u = z$ implies $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(1)} \neq \Phi$. That is, (2.1) holds with $i = 0$, $j = 1$. Otherwise we have $k+1 < n$ and $v = \varepsilon \in U_{k+1}$ which gives a contradiction.

If $v \in A^*$ and $z \in {}^\infty X$ then $u \in {}^\infty A$. The equation $uy = zx$ gives $u(\bar{X}^{(j')} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i'+1)} \neq \Phi$. Thus (2.1) holds with $i = j'$, $j = i' + 1$.

(ii) Suppose there exist $u \in U_k$ and two integers $i, j \geq 0$ such that $u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} \neq \Phi$, $i + j + k = n$, $u \in A^N \cup A^Z \Rightarrow i = 0$. We have to prove that $\varepsilon \in U_n$. If $k = n$, then $i = j = 0$ and so $u = \varepsilon$. Hence $\varepsilon \in U_n$. Let now

$n > k \geq 1$ and suppose the statement is true for $n, n-1, \dots, k+1$. We prove for k . Suppose $x_1 x_2 \dots x_i \in \bar{X}^{(i)} - (A^{-N} \cup A^Z)$ and $x'_1 x'_2 \dots x'_j \in \bar{X}^{(j)}$ such that $ux_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$. We discuss the following cases:

Case (a): Suppose $u \in A^N \cup A^Z$. Then $i=0, j+k=n, j \geq 1$ and $u = x'_1 x'_2 \dots x'_j$. Let $u' = x'_2 x'_3 \dots x'_j$. Clearly $u' \in U_{k+1}$ and $u'(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi, 0+j-1+k+1=n$. By recurrence hypothesis $\varepsilon \in U_n$.

Case (b): Suppose $u \in A^*$. If $j=0$, then $i=0, u=\varepsilon$ and $k=n$. Thus we have $\varepsilon \in U_n$. Let $j \geq 1$. If $|u| \geq |x'_1|$, that is, $u = x'_1 u'$ for some u' , then $u' \in U_{k+1}$ and

$$u' x_1 x_2 \dots x_i = x'_2 x'_3 \dots x'_j.$$

So $u'(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi, i+j-1+k+1=n$. By recurrence hypothesis, $\varepsilon \in U_n$. If $|u| < |x'_1|$, that is, $x'_1 = uu''$ for some u'' , then $u'' \in U_{k+1}$ and $u'' x'_2 x'_3 \dots x'_j = x_1 x_2 \dots x_i$. Hence

$$u''(\bar{X}^{(j-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \quad j-1+i+k+1=n.$$

This implies $\varepsilon \in U_n$.

Case (c): Suppose $u \in A^{-N}$. Then $j \geq 1$. If $j=1$, then $ux_1 x_2 \dots x_i = x'_1$ which implies $x_1 x_2 \dots x_i \in u^{-1} x'_1$. Let $u' = x_1 x_2 \dots x_i$. We have $u' \in U_{k+1}$ and

$$u'(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \quad 0+i+k+1=n.$$

By recurrence hypothesis $\varepsilon \in U_n$. If $j > 1$, there are two subcases.

If u is a left factor of x'_1 , we have

$$x_1 x_2 \dots x_i = u' x'_2 x'_3 \dots x'_j$$

with $u' \in u^{-1} x'_1$. So, we have $u' \in U_{k+1}$ and

$$u'(\bar{X}^{(j-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i)} \neq \Phi, \quad j-1+i+k+1=n.$$

By recurrence hypothesis $\varepsilon \in U_n$.

If x'_1 is a left factor of u , we have

$$x'_2 x'_3 \dots x'_j = u'' x_1 x_2 \dots x_i$$

with $u'' \in (x'_1)^{-1} u$. Then $u'' \in U_{k+1}$ and

$$u''(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi, \quad i+j-1+k+1=n.$$

By recurrence hypothesis $\varepsilon \in U_n$. This proves lemma 2.1.

We are now in a position to formulate the main result of this section which is a generalization of the result proved by Do Long Van in [5, 10]. The latter is a generalization of Sardinas-Patterson theorem. This in many cases gives us a procedure to check whether or not a given set is a bi-infinity code.

THEOREM 2.1: *A subset X of ${}^\infty A^\infty - \{\varepsilon\}$ is a code iff for all $n \geq 1$, $U_n(X)$ does not contain the empty word ε .*

Proof: Suppose $\varepsilon \notin U_n(X)$, $n \geq 1$. Assume that X is not a code. Then there exists a word $\alpha \in {}^\infty A^\infty$ having two different factorizations on elements of X :

$$\alpha = x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j \quad \text{where } (x_1, x_2, \dots, x_i) \in X^{(i)}$$

and $(x'_1, x'_2, \dots, x'_j) \in X^{(j)}$.

Case (a): Suppose $\alpha \in A^* \cup A^N$. We may assume that $x_1 \neq x'_1$ and $|x_1| > |x'_1|$. Let $x_1 = x'_1 u$ for some $u \neq \varepsilon$. Clearly $u \in U_1$.

If $x_1 \in X_{\text{fin}}$, then $x'_1 \in X_{\text{fin}}$ and $u \in A^+$. So we have

$$u x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j, \quad j \geq 2.$$

Hence

$$u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

By lemma 2.1 (ii), $\varepsilon \in U_{i+j-1}$ which is a contradiction.

If $x_1 \in X_{\text{inf}}$, then $i = 1$, $x'_1 \in X_{\text{fin}}$ and $u \in A^N$. Therefore we have $u = x'_2 x'_3 \dots x'_j$, $j \geq 2$. This implies

$$u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

Again by lemma 2.1 (ii), $\varepsilon \in U_j$ which is a contradiction.

Case (b): Suppose $\alpha \in A^{-N}$. Clearly $x_1, x'_1 \in X_{\text{inf}}$. Since the case $i = j = 1$ is impossible, we may assume that $i \geq 2$. There are two possibilities.

(i) If $x_1 \neq x'_1$ we can assume that $x_1 = x'_1 u$ with $u \in A^+$ such that $u x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$, $j \geq 2$. Then clearly $u \in U_1$ and

$$u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi.$$

Again by lemma 2.1 (ii), $\varepsilon \in U_{i+j-1}$. This is a contradiction.

(ii) Suppose $x_1 = x'_1$. Here, if $j = 1$, then $x_1 x_2 \dots x_i = x'_1$ and so $x_1 = {}^\omega(x_2 x_3 \dots x_i)$. Let $x_2 x_3 \dots x_i = u$. Clearly $u \in x_1^{-1} x'_1 \subseteq U_1$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(i-1)} \neq \Phi$. This implies $\varepsilon \in U_i$ which is a contradiction.

If $j \geq 2$, then $x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$ with

$$x_2 x_3 \dots x_i, x'_2 x'_3 \dots x'_j \in A^*.$$

If $x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$, we may assume that $x_2 \neq x'_2$, and as in case (a), get a contradiction. If $x_2 x_3 \dots x_i \neq x'_2 x'_3 \dots x'_j$, then we have either $x_1 = x'_1 u$ and $ux_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$ or $x'_1 = x_1 u$ and $x_2 x_3 \dots x_i = ux'_2 x'_3 \dots x'_j$ for some $u \in A^+$. By symmetry, we shall discuss one of the two possibilities.

Consider $x_1 = x'_1 u$ and $ux_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$. Now $x_1 = x'_1$ and $x_1 = x'_1 u$ imply $x_1 = x'_1 = {}^0u$ and $u \in (x'_1)^{-1} x_1 \subseteq U_1$. Thus $ux_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$ gives $u(\bar{X}^{(i-1)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$ and so $\varepsilon \in U_{i+j-1}$ which is a contradiction.

Case (c): Suppose $\alpha \in A^Z$. The case $i=j=1$ is impossible. We assume $j \geq 2$. If $i=1$ then $x_1 = x'_1 x'_2 \dots x'_j$ and so we have $u = (x'_1)^{-1} x_1 \in U_1$ with $u = x'_2 x'_3 \dots x'_j$. Hence $u(\bar{X}^{(0)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j-1)} \neq \Phi$ which gives a contradiction.

If $i \geq 2$, then $x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$. Now $x_1, x'_1 \in X_{-\inf}$. There are two possibilities.

- (i) If $x_1 \neq x'_1$, as in case (a), we obtain a contradiction.
- (ii) If $x_1 = x'_1$, then we have either

$$x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j \quad \text{or} \quad x_2 x_3 \dots x_i \neq x'_2 x'_3 \dots x'_j.$$

If $x_2 x_3 \dots x_i = x'_2 x'_3 \dots x'_j$, then we can assume $x_2 \neq x'_2$ and as in case (a), get a contradiction since $x_2 x_3 \dots x_i, x'_2 x'_3 \dots x'_j \in A^N$. If $x_2 x_3 \dots x_i \neq x'_2 x'_3 \dots x'_j$, we can obtain a contradiction as in the last part of Case b (ii). Thus X is a code.

We shall prove the converse. Suppose X is a code. Assume that there are some sets $U_i(X)$ containing ε . Let $U_n(X)$ be one among these, with the smallest index. By lemma 2.1 (i), there exists a word $u \in U_1$ with two integers $i, j \geq 0$ such that

$$u(\bar{X}^{(i)} - (A^{-N} \cup A^Z)) \cap \bar{X}^{(j)} \neq \Phi, \quad i+j+1=n,$$

$u \in A^N \cup A^Z \Rightarrow i=0$. So, we have $ux_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$ for some $x_1 x_2 \dots x_i \in \bar{X}^{(i)} - (A^{-N} \cup A^Z)$ and $x'_1 x'_2 \dots x'_j \in \bar{X}^{(j)}$. Since $u \in U_1$, there exist words $x, x' \in X$ with either $x \neq x'$ and $x = x' u$ or $x = x'$ and $x = x' u$.

If $u \in A^+$, then both x, x' are either in X_{fin} or in $X_{-\inf}$. Let $x, x' \in X_{\text{fin}}$. Then we have $x \neq x'$ and $x = x' u$. So $xx_1 x_2 \dots x_i = x' x'_1 \dots x'_j$ and therefore

X is not a code, a contradiction. Let $x, x' \in X_{-\text{inf}}$. If $x \neq x'$ and $x = x' u$, then as before, we get a contradiction. If $x = x'$ and $x = x' u$, then $x = x' = {}^\omega u$ and either

$$x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j \quad \text{or} \quad x_1 x_2 \dots x_i \neq x'_1 x'_2 \dots x'_j.$$

If $x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$, then $i = j$ and $x_k = x'_k$ ($k = 1, 2, \dots, i$) since X is a code. Then the equation $u x_1 x_2 \dots x_i = x'_1 x'_2 \dots x'_j$ implies $u = \varepsilon$, a contradiction. If $x_1 x_2 \dots x_i \neq x'_1 x'_2 \dots x'_j$ then the equation $x' x_1 x_2 \dots x_i = x' x'_1 x'_2 \dots x'_j$ shows that X is not a code, a contradiction.

If $u \in A^N$, then $i = 0$ and either $x \in X_{\text{inf}}$, $x' \in X_{\text{fin}}$ or $x \in X_{\text{biinf}}$, $x' \in X_{-\text{inf}}$. In both cases, we have $x = x' x'_1 \dots x'_j$ which shows X is not a code, a contradiction.

If $u \in A^Z$, then $i = 0$, $x \in X_{\text{biinf}}$ and $x' = \varepsilon$. Since $x' \in X \subseteq {}^\omega A^\omega - \{\varepsilon\}$, this case is not possible.

If $u \in A^{-N}$, then $x = u$ and $x' = \varepsilon$. As before, this case is also not possible. Thus $\varepsilon \notin U_n(X)$, $\forall n \geq 1$.

Example 2.1: (i) Let $X = \{(ab)^\omega, {}^\omega a, b^\omega, ab\}$. $U_1(X) = \{a^+\}$, $U_2(X) = \{b\}$, $U_3(X) = \{b^\omega\}$ and $U_4(X) = \{\varepsilon\}$. So X is not a code.

(ii) Let $X = \{(ab)^\omega, {}^\omega a, b^\omega, ba\}$. $U_1(X) = \{a^+\}$, $U_2(X) = \Phi$. So, X is a code.

SECTION 3

MINIMAL GENERATOR SET OF A SUBMONOID OF ${}^\omega A^\omega$.

We recall that a generator set X of a monoid M is minimal if X is contained in any generator set of M . Such a set, if it exists, is unique and called the base of M , denoted as $\text{BASE}(M)$. Every submonoid of A^* has a minimal generator set whereas there are submonoids of ${}^\omega A^\omega$ which have no minimal generator sets. We illustrate this in the following example.

Example 3.1: Let $A = \{a, b\}$ and let M be the submonoid of ${}^\omega A^\omega$ given by $M = \{\alpha \in {}^\omega A^\omega \mid |\alpha|_a = |\alpha|_b\}$ where $|\alpha|_a$ stands for the number of occurrences of a in α . This monoid has no minimal generator set.

DEFINITION 3.1: Let M be a submonoid of ${}^\omega A^\omega$ and u, v , two elements of M_{inf} . We say that u precedes v , denoted by $u < v$, if there exists $f \in M_{\text{fin}} - \varepsilon$ such that $u = fv$. An element $u \in M_{\text{inf}}$ is called stable if $\forall v \in M_{\text{inf}}$: $(u < v) \Rightarrow (u = v)$. The set of all stable elements of M_{inf} is denoted by $\text{STAB}(M_{\text{inf}})$.

Let x, y be two elements of $M_{-\text{inf}}$. Here also we say that x precedes y , denoted by $x < y$ if there exists $g \in M_{\text{fin}} - \varepsilon$ such that $x = yg$. As before, $x \in M_{-\text{inf}}$ is called stable if $\forall y \in M_{-\text{inf}} : (x < y) \Rightarrow (x = y)$. The set of all stable elements of $M_{-\text{inf}}$ is denoted by $\text{STAB}(M_{-\text{inf}})$.

We say that a submonoid M satisfies the stability condition if every unstable element of M_{inf} (resp. $M_{-\text{inf}}$) precedes a stable element of M_{inf} (resp. $M_{-\text{inf}}$). We introduce the following two sets:

$$\begin{aligned}\text{BASE}(M_{\text{fin}}) &= (M_{\text{fin}} - \varepsilon) - (M_{\text{fin}} - \varepsilon)^2 \\ \text{UNFAC}(M_{\text{biinf}}) &= M_{\text{biinf}} - (M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}).\end{aligned}$$

THEOREM 3.1: *A submonoid M of ${}^\infty A^\infty$ has a minimal generator set iff M satisfies the stability condition and in that case, the minimal generator set of M is*

$$\begin{aligned}X &= \text{BASE}(M) \\ &= \text{BASE}(M_{\text{fin}}) \cup \text{STAB}(M_{\text{inf}}) \cup \text{STAB}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}).\end{aligned}$$

Proof: Assume X satisfies the stability condition. Let

$$\begin{aligned}X_{\text{fin}} &= \text{BASE}(M_{\text{fin}}), & X_{\text{inf}} &= \text{STAB}(M_{\text{inf}}), & X_{-\text{inf}} &= \text{STAB}(M_{-\text{inf}}), \\ X_{\text{biinf}} &= \text{UNFAC}(M_{\text{biinf}}) & \text{and} & & X &= X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}.\end{aligned}$$

Since

$$\begin{aligned}X_{\text{fin}}^* &= M_{\text{fin}}, & M_{\text{inf}} &= \text{STAB}(M_{\text{inf}}) \cup (M_{\text{fin}} - \varepsilon) \text{STAB}(M_{\text{inf}}) \\ & & &= M_{\text{fin}} \text{STAB}(M_{\text{inf}}) = X_{\text{fin}}^* X_{\text{inf}}.\end{aligned}$$

Similarly,

$$\begin{aligned}M_{-\text{inf}} &= X_{-\text{inf}} X_{\text{fin}}^* & \text{and} & & M_{\text{biinf}} &= \text{UNFAC}(M_{\text{biinf}}) \\ & & & & & \cup M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}} = X_{\text{biinf}} \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}}.\end{aligned}$$

Therefore,

$$\begin{aligned}M &= M_{\text{fin}} \cup M_{\text{inf}} \cup M_{-\text{inf}} \cup M_{\text{biinf}} \\ &= X_{\text{fin}}^* \cup X_{\text{fin}}^* X_{\text{inf}} \cup X_{-\text{inf}} X_{\text{fin}}^* \cup X_{\text{biinf}} \\ & \quad \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} = X^*.\end{aligned}$$

Thus X is a generator set of M . We shall prove that X is minimal. Let Y be an arbitrary generator set of M . We can assume that $\varepsilon \notin Y$. It is enough if

we prove that

$$\begin{aligned} X_{\text{fin}} &\subseteq Y_{\text{fin}}, & X_{\text{inf}} &\subseteq Y_{\text{inf}}, \\ X_{-\text{inf}} &\subseteq Y_{-\text{inf}} & \text{and} & & X_{\text{biinf}} &\subseteq Y_{\text{biinf}}. \end{aligned}$$

As $Y_{\text{fin}}^* = M_{\text{fin}}$ and X_{fin} is the minimal generator set of M_{fin} , we have $X_{\text{fin}} \subseteq Y_{\text{fin}}$. Let $u \in X_{\text{inf}}$. Then $u = y_1 y_2 \dots y_n$ for some $(y_1, y_2, \dots, y_n) \in Y^{(n)}$, $n \geq 1$. If $n = 1$, then $u = y_n \in Y_{\text{inf}}$. If $n > 1$, we have $u = f y_n$ with $f = y_1 y_2 \dots y_{n-1} \in M_{\text{fin}} - \varepsilon$ i.e., $u < y_n$. Since u is stable $u = y_n \in Y_{\text{inf}}$. Thus $X_{\text{inf}} \subseteq Y_{\text{inf}}$. Similarly we can show that $X_{-\text{inf}} \subseteq Y_{-\text{inf}}$. Let $u \in X_{\text{biinf}}$. Then $u = w_1 w_2 \dots w_n$ for some $(w_1, w_2, \dots, w_n) \in Y^{(n)}$, $n \geq 1$. If $n = 1$, $u = w_1$ where $w_1 \in Y_{\text{biinf}}$. If $n \geq 2$, $u = w_1 w_2 \dots w_n$ is an element of $M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}$ since $w_1 \in Y_{-\text{inf}}$, $Y_{\text{fin}}^* = M_{-\text{inf}}$, $w_n \in Y_{\text{fin}}^* Y_{\text{inf}} = M_{\text{inf}}$ and $w_2 w_3 \dots w_{n-1} \in Y_{\text{fin}}^*$. This contradicts the choice of u since $u \in \text{UNFAC}(M_{\text{biinf}})$. Hence $u \in Y_{\text{biinf}}$ and so $X_{\text{biinf}} \subseteq Y_{\text{biinf}}$.

We prove the converse part now. Let Y be a minimal generator set of M . Suppose M does not satisfy the stability condition. Then, there exists an unstable element of M_{inf} (resp. $M_{-\text{inf}}$), which does not precede any stable element of M_{inf} (resp. $M_{-\text{inf}}$). Let u be an unstable element of M_{inf} and v any element of M_{inf} such that $v \neq u$ and $u < v$. If $u \in Y_{\text{inf}}$, then since $Y_{\text{fin}}^* = M_{\text{fin}}$, the set $Y' = (Y - \{u\}) \cup \{v\}$ is a generator set of M . Since Y' does not contain Y , we get a contradiction to the minimality of Y . If $u \notin Y_{\text{inf}}$, then $u = y_1 y_2 \dots y_n$ for some $(y_1, y_2, \dots, y_n) \in Y^{(n)}$ with $n > 1$. Therefore $u < y_n$. By hypothesis, y_n is unstable. Therefore there exists $w \in M_{\text{inf}}$ such that $w \neq y_n$ and $y_n < w$. Thus, the set $Y'' = (Y - \{y_n\}) \cup \{w\}$ is a generator set of M . Since Y'' does not contain Y , we have a contradiction. Hence M satisfies the stability condition.

Example 3.2: Let $A = \{a, b\}$. Let M be the submonoid of ${}^\infty A^\infty$ given by

$$M = \{{}^\omega a(ab)^\omega\} \cup A^* \cup {}^\omega b A^* \cup A^* a^\omega \cup {}^\omega b A^* a^\omega.$$

Every element of M_{inf} precedes the unique stable element a^ω . Every element of $M_{-\text{inf}}$ precedes the unique stable element ${}^\omega b$. M satisfies the stability condition. By theorem 3.1, M has a minimal generator set which is $A \cup \{a^\omega, {}^\omega b, {}^\omega a(ab)^\omega\}$.

DEFINITION 3.2: Let M be a submonoid of ${}^\infty A^\infty$. Any increasing sequence $u_1 < u_2 < \dots$ of elements of M_{inf} or $M_{-\text{inf}}$ is called a chain. An infinite chain is called stationary if there exists $n \geq 1$, such that $u_m = u_n$, for all $m \geq n$. We say that M satisfies the stationary chain condition if every infinite chain of M_{inf} as well as $M_{-\text{inf}}$ is stationary.

We note that stationary chain condition implies the stability condition but the converse is not true.

DEFINITION 3.3: A submonoid M of ${}^\infty A^\infty$ is freeable if $M^{-1}M \cap MM^{-1} \subseteq M$.

The next theorem explains the existence of the minimal generator set for a freeable monoid M .

THEOREM 3.2: *For any freeable submonoid M , the following conditions are equivalent.*

- (i) M has a minimal generator set.
- (ii) M satisfies the stationary chain condition.
- (iii) M satisfies the stability condition.

Proof is similar to that of theorem 2.4 of Chapter II in [10] and is therefore omitted. The main difference is to consider infinite chains of elements of $M_{-\text{inf}}$.

DEFINITION 3.4: Let M be a submonoid of ${}^\infty A^\infty$. An element u of M_{inf} (resp. $M_{-\text{inf}}$) is maximal if there is no element v of M_{inf} (resp. $M_{-\text{inf}}$) such that $u < v$. The set of all maximal elements of M_{inf} (resp. $M_{-\text{inf}}$) is denoted by $\text{MAX}(M_{\text{inf}})$ [resp. $\text{MAX}(M_{-\text{inf}})$]. It is evident that $\text{MAX}(M_{\text{inf}}) \subseteq \text{STAB}(M_{\text{inf}})$ and $\text{MAX}(M_{-\text{inf}}) \subseteq \text{STAB}(M_{-\text{inf}})$. We say that M satisfies the maximality condition if every non maximal element of M_{inf} (resp. $M_{-\text{inf}}$) precedes a maximal element of M_{inf} (resp. $M_{-\text{inf}}$). Clearly, maximality condition implies stability condition but not the converse.

DEFINITION 3.5: Any subset X of ${}^\infty A^\infty$ is called distinguished if $X_{\text{inf}} \cap X_{\text{fin}}^+ X_{\text{inf}} = \Phi$, $X_{-\text{inf}} \cap X_{-\text{inf}} X_{\text{fin}}^+ = \Phi$ and $X_{\text{biinf}} \cap X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} = \Phi$.

The following theorem gives the connection between maximality condition and the distinguished minimal generator set of a monoid M .

THEOREM 3.3: *For any submonoid M , the following conditions are equivalent.*

- (i) M has a distinguished minimal generator set which is

$$\begin{aligned} & \text{BASE}(M_{\text{fin}}) \cup \text{MAX}(M_{\text{inf}}) \cup \text{MAX}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}) \\ &= (M - \varepsilon) - [(M_{\text{fin}} - \varepsilon)^2 \cup (M_{\text{fin}} - \varepsilon) M_{\text{inf}} \cup M_{-\text{inf}} (M_{\text{fin}} - \varepsilon) \\ & \quad \cup M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}] \end{aligned}$$

- (ii) M has a distinguished generator set
- (iii) M satisfies the maximality condition

Proof: It is clear that (i) implies (ii). We show that (ii) implies (iii). Let Y be a distinguished generator set of M . Since Y is a generator set, it is easy to see that every element of $M_{\text{inf}} - Y_{\text{inf}}$ (resp. $M_{-\text{inf}} - Y_{-\text{inf}}$) precedes an element of Y_{inf} (resp. $Y_{-\text{inf}}$) and so it is enough to prove that

$$Y_{\text{inf}} \subseteq \text{MAX}(M_{\text{inf}}) [\text{resp. } Y_{-\text{inf}} \subseteq \text{MAX}(M_{-\text{inf}})].$$

We shall prove that $Y_{\text{inf}} \subseteq \text{MAX}(M_{\text{inf}})$. Suppose this is not true. Then, there exists $y \in Y_{\text{inf}}$ which is not maximal. So, for some $v \in M_{\text{inf}}$, we have $y < v$. Let $y = gv$ where $g \in M_{\text{fin}} - \varepsilon$ and $v = y_1 y_2 \dots y_n$ for some $(y_1, y_2, \dots, y_n) \in Y^{(n)}$, $n \geq 1$. Since $gy_1 y_2 \dots y_{n-1} \in Y_{\text{fin}}^+$, we have $y \in Y_{\text{inf}} \cap Y_{\text{fin}}^+ Y_{\text{inf}}$. This is a contradiction since Y is distinguished. Hence (ii) implies (iii).

We now prove (iii) $\Rightarrow$ (i). Let M satisfy the maximality condition. This means M satisfies the stability condition. By theorem 3.1, M has a minimal generator set X , namely,

$$X = \text{BASE}(M_{\text{fin}}) \cup \text{STAB}(M_{\text{inf}}) \cup \text{STAB}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}}).$$

Since a non maximal stable element cannot precede a maximal element,

$$\text{STAB}(M_{\text{inf}}) = \text{MAX}(M_{\text{inf}}) = M_{\text{inf}} - (M_{\text{fin}} - \varepsilon) M_{\text{inf}}$$

and

$$\text{STAB}(M_{-\text{inf}}) = \text{MAX}(M_{-\text{inf}}) = M_{-\text{inf}} - M_{-\text{inf}}(M_{\text{fin}} - \varepsilon).$$

Since

$$\text{UNFAC}(M_{\text{biinf}}) = M_{\text{biinf}} - (M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}})$$

and

$$\text{BASE}(M_{\text{fin}}) = (M_{\text{fin}} - \varepsilon) - (M_{\text{fin}} - \varepsilon)^2,$$

we have

$$X = (M - \varepsilon) - [(M_{\text{fin}} - \varepsilon)^2 \cup (M_{\text{fin}} - \varepsilon) M_{\text{inf}} \cup M_{-\text{inf}}(M_{\text{fin}} - \varepsilon) \cup M_{-\text{inf}} M_{\text{fin}} M_{\text{inf}}].$$

Since $X = X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}$, let $X_{\text{fin}} = \text{BASE}(M_{\text{fin}})$, $X_{\text{inf}} = \text{MAX}(M_{\text{inf}})$, $X_{-\text{inf}} = \text{MAX}(M_{-\text{inf}})$ and $X_{\text{biinf}} = \text{UNFAC}(M_{\text{biinf}})$. Thus $X_{\text{inf}} \cap X_{\text{fin}}^+ X_{\text{inf}} = \Phi$, $X_{-\text{inf}} \cap X_{-\text{inf}} X_{\text{fin}}^+ = \Phi$ and

$$X_{\text{biinf}} \cap X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} = \Phi.$$

Hence X is distinguished.

SECTION 4

SUBMONOID GENERATED BY CODES AND A THUE SYSTEM

In this section we introduce a bi-quasi free monoid whose underlying set is the set of all normal forms with respect to a specific Church-Rosser Thue system. We establish a characterisation of codes in terms of morphisms of monoids. We show the relation between bi-quasi free monoids, minimal generator sets and codes.

Let B be any finite alphabet. Let R be a binary relation on B^* . Elements of R are written as equations, i.e., $R = \{(u=v) \mid u, v \in B^*\}$. Let $T(B) = \langle B; R \rangle$. We call $T(B)$ as a Thue system associated with B . We say $(u=v)$ is in $T(B)$ iff $(u=v)$ is in R .

Define the relation $=_{T(B)}$ on elements of B^* as follows: For any $(u=v)$ in $T(B)$ and any $x, y \in B^*$, we write $xuy =_{T(B)} xvy$. The reflexive transitive closure of the symmetric relation $=_{T(B)}$ is denoted as $\equiv_{T(B)}$. Clearly $\equiv_{T(B)}$ is a congruence relation on B^* . If $x \equiv_{T(B)} y$, for any $x, y \in B^*$, we say that x is congruent to y . The congruence class of x is denoted by $[x]$.

If $(u=v)$ is in $T(B)$, we write $u \rightarrow_{T(B)} v$ if the length of u is greater than the length of v . $\rightarrow_{T(B)}^*$ is the reflexive, transitive closure of the relation $\rightarrow_{T(B)}$.

When $x \rightarrow_{T(B)}^* y$, we say that x is an ancestor of y and y is a descendant of x . x is said to be irreducible if it has no descendant except itself. For any $x, y \in B^*$, if $x \equiv_{T(B)} y$ and y is irreducible, then y is called a normal form of x .

$T(B)$ is Church-Rosser if for all $x, y \in B^*$, if $x \equiv_{T(B)} y$, then for some $z \in B^*$, $x \rightarrow_{T(B)}^* z$ and $y \rightarrow_{T(B)}^* z$. This means that every two congruent words have a common descendant. It is known that if $T(B)$ is Church-Rosser, then every congruence class has a unique normal form [2]. We make use of this result in the following discussion.

We partition B into four mutually disjoint subsets $B_1, B_2, \bar{B}_2, B_3$ and call B as a quadruple alphabet $(B_1, B_2, \bar{B}_2, B_3)$. With B , we associate a Thue system defined by $T(B) = \langle B; R \rangle$ where

$$R = \{(bb' = b) \mid b \in B_2 \cup B_3, b' \in B\} \cup \{(bb' = b') \mid b \in B_1 \cup \bar{B}_2, b' \in \bar{B}_2 \cup B_3\}.$$

Now, $\equiv_{T(B)}$ is a congruence relation on B^* . Consider the quotient monoid $B^*/\equiv_{T(B)}$ and denote this by $B^{[*]}$. It is easy to see that $T(B)$ is Church-Rosser. Hence every congruence class has a unique normal form. It is interesting to note that the set of all normal forms of elements of B^* is

$$B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3.$$

By a mild abuse of language, we write

$$B^{[*]} = B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3.$$

Define a product on $B^{[*]}$ as follows: For $x, y \in B^{[*]}$,

$$x \cdot y = \begin{cases} xy & \text{if } x \in B_1^*, \quad y \in B_1^* B_2 \cup B_1^* \\ & \text{or} \\ & x \in \bar{B}_2 B_1^*, \quad y \in B_1^* \\ x & \text{if } x \in B_1^* B_2 \cup B_3 \cup \bar{B}_2 B_1^* B_2 \\ y & \text{if } x \in B_1^* \cup \bar{B}_2 B_1^*, \\ & y \in \bar{B}_2 B_1^* \cup B_3 \cup \bar{B}_2 B_1^* B_2. \end{cases}$$

Clearly $B^{[*]}$ is a monoid which we shall call as a bi-quasi free monoid generated by B .

LEMMA 4.1: *If $\varphi : B^{[*]} \rightarrow {}^\infty A^\infty$ is an injective morphism and $\varphi(B) = X$, then $\varphi(B_1) = X_{\text{fin}}$, $\varphi(B_2) = X_{\text{inf}}$, $\varphi(\bar{B}_2) = X_{-\text{inf}}$ and $\varphi(B_3) = X_{\text{biinf}}$.*

Proof: We first show that $\varphi(B_1) \subseteq X_{\text{fin}}$. Suppose it is not true. Then there exists $b \in B_1$ such that

$$\varphi(b) \in X_{\text{inf}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}.$$

If $\varphi(b) \in X_{\text{inf}} \cup X_{\text{biinf}}$, for $b' \in B$,

$$\varphi(bb') = \varphi(b) \varphi(b') = \varphi(b).$$

Since φ is injective, $bb' = b$ which is impossible. If $\varphi(b) \in X_{-\text{inf}}$, for $b' \in \bar{B}_2 \cup B_3$, $bb' = b'$. So, $\varphi(b) \varphi(b') = \varphi(b')$ which is impossible since $\varphi(b) \neq \varepsilon$. Hence $\varphi(B_1) \subseteq X_{\text{fin}}$.

To prove that $\varphi(B_2) \subseteq X_{\text{inf}}$, we suppose that it is not true. Then there exists $b \in B_2$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{-\text{inf}} \cup X_{\text{biinf}}$. For $b' \in B$, $bb' = b$. So, $\varphi(b) \varphi(b') = \varphi(b)$ and this is not possible since $\varphi(b')$ need not be ε . Hence $\varphi(B_2) \subseteq X_{\text{inf}}$.

We now show that $\varphi(\bar{B}_2) \subseteq X_{-\text{inf}}$. If it were not so, there would exist $b \in \bar{B}_2$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{\text{biinf}}$. Now, for $b' \in \bar{B}_2 \cup B_3$, $bb' = b'$ and so $\varphi(b)\varphi(b') = \varphi(b')$. This is not possible since $\varphi(b) \neq \varepsilon$.

Finally, in order to prove that $\varphi(B_3) \subseteq X_{\text{biinf}}$, assume that it is not true. Then there exists $b \in B_3$ such that $\varphi(b) \in X_{\text{fin}} \cup X_{\text{inf}} \cup X_{-\text{inf}}$. For $b' \in B$, $bb' = b$. Therefore $\varphi(b)\varphi(b') = \varphi(b)$ which is not possible since $\varphi(b') \neq \varepsilon$.

Since $\varphi(B) = X$, we have, $\varphi(B_1) = X_{\text{fin}}$, $\varphi(B_2) = X_{\text{inf}}$, $\varphi(\bar{B}_2) = X_{-\text{inf}}$ and $\varphi(B_3) = X_{\text{biinf}}$. This proves the lemma.

Given a quadruple alphabet $B = (B_1, B_2, \bar{B}_2, B_3)$ we denote $B^{(1)} = B$ and

$$B^{(n)} = \left\{ (b_1, b_2, \dots, b_n) / \begin{array}{l} b_1, b_2, \dots, b_{n-1} \in B_1, b_n \in B_1 \cup B_2 \\ \text{or } b_1 \in B_1 \cup \bar{B}_2, \quad b_2, b_3, \dots, b_n \in B_1 \\ \text{or } b_1 \in \bar{B}_2, b_n \in B_2, b_2, b_3, \dots, b_{n-1} \in B_1 \end{array} \right\}$$

LEMMA 4.2: (i) If a subset X of ${}^\infty A^\infty$ is a code, then every morphism $\varphi: B^{[*]} \rightarrow {}^\infty A^\infty$ which induces a bijection from B onto X with $\varphi(B_1) \subseteq X_{\text{fin}}$, $\varphi(B_2) \subseteq X_{\text{inf}}$ and $\varphi(\bar{B}_2) \subseteq X_{-\text{inf}}$ is injective.

(ii) If $\varphi: B^{[*]} \rightarrow {}^\infty A^\infty$ is an injective morphism, then $X = \varphi(B)$ is a code.

Proof is on lines close to that of lemma 1.3 of Chapter III in [10] and is therefore omitted.

We now give a necessary and sufficient condition for a subset of ${}^\infty A^\infty$ to be a code.

THEOREM 4.1: A subset X of ${}^\infty A^\infty$ is a code iff there exists a bi-quasi free monoid $B^{[*]}$ and an injective morphism $\varphi: B^{[*]} \rightarrow {}^\infty A^\infty$ such that $\varphi(B) = X$.

Proof: Let X be a code. Let $B = (B_1, B_2, \bar{B}_2, B_3)$ be a quadruple alphabet chosen so that $B_1, B_2, \bar{B}_2$ and B_3 are in one to one correspondence with $X_{\text{fin}}, X_{\text{inf}}, X_{-\text{inf}}$ and X_{biinf} respectively. This correspondence shows the existence of an isomorphism

$$\begin{aligned} \varphi: B^{[*]} &\rightarrow X^* && \text{with } \varphi(B_1) = X_{\text{fin}}, \\ & && \varphi(B_2) = X_{\text{inf}}, \\ \varphi(\bar{B}_2) &= X_{-\text{inf}} && \text{and } \varphi(B_3) = X_{\text{biinf}}. \end{aligned}$$

By lemma 4.2, the theorem holds.

DEFINITION 4.1: A submonoid M of ${}^\infty A^\infty$ is said to be bi-quasi free if it is isomorphic to a bi-quasi free monoid $B^{[*]}$.

The following theorem exhibits that the class of submonoids generated by codes coincides with the class of biquasi free submonoids.

THEOREM 4.2: (i) *Every bi-quasi free submonoid M has a minimal generator set X which is a code.*

(ii) *If X is a code, then X^* is a bi-quasi free submonoid having X as its minimal generator set.*

Proof: (i) Suppose M is a bi-quasi free submonoid. Then there is an isomorphism $\varphi: B^{[*]} \rightarrow M$ from a bi-quasi free monoid onto M . By theorem 4.1, $X = \varphi(B)$ is a code. By lemma 4.2, $\varphi(B_1) = X_{\text{fin}}$, $\varphi(B_2) = X_{\text{inf}}$, $\varphi(\bar{B}_2) = X_{-\text{inf}}$ and $\varphi(B_3) = X_{\text{biinf}}$. We have

$$\begin{aligned} M = \varphi(B^{[*]}) &= \varphi(B_1^* \cup B_1^* B_2 \cup \bar{B}_2 B_1^* \cup \bar{B}_2 B_1^* B_2 \cup B_3) \\ &= [\varphi(B_1)]^* \cup [\varphi(B_1)]^* \varphi(B_2) \cup \varphi(\bar{B}_2) [\varphi(B_1)]^* \\ &\quad \cup \varphi(\bar{B}_2) [\varphi(B_1)]^* \varphi(B_2) \cup \varphi(B_3). \\ &= X_{\text{fin}}^* \cup X_{\text{fin}}^* X_{\text{inf}} \cup X_{-\text{inf}} X_{\text{fin}}^* \cup X_{-\text{inf}} X_{\text{fin}}^* X_{\text{inf}} \cup X_{\text{biinf}} = X^*. \end{aligned}$$

Hence X generates M . To prove the minimality of X , let Y be any generator set of M and $x \in X$. Then $x = y_1 y_2 \dots y_n$ for some $(y_1, y_2, \dots, y_n) \in Y^{(n)}$, $n \geq 0$. Since $x \neq \varepsilon$, $n \geq 1$. Since X is a code, $n = 1$ and so $x = y_1$. Hence $X \subseteq Y$. Thus X is minimal.

(ii) Suppose X is a code. By theorem 4.1, there exists a bi-quasi free monoid $B^{[*]}$ and an injective morphism $\varphi: B^{[*]} \rightarrow {}^\infty A^\infty$ such that $\varphi(B) = X$. Now φ is indeed an isomorphism from $B^{[*]}$ onto $\varphi(B^{[*]}) = X^*$. Thus X^* is a bi-quasi free submonoid. By the similar argument as in (i), X is a minimal generator set of X^* .

SECTION 5

A COMBINATORIAL CHARACTERIZATION OF BI-QUASI FREE SUBMONOIDS

LEMMA 5.1: *Every bi-quasi free submonoid is freeable.*

Proof: Let M be a bi-quasi free submonoid with the minimal generator set X . By theorem 4.2, X is a code. Let $\alpha \in M^{-1} M \cap M M^{-1}$. Since $\alpha \in M^{-1} M$, there exists $\beta \in M$ such that $\beta \alpha \in M$, $(\beta \in M_{\text{inf}} \cup M_{\text{biinf}} \Rightarrow \alpha = \varepsilon)$ and $(\beta \in {}^\infty M \text{ and } \alpha \in A^{-N} \cup A^Z \Rightarrow \beta = \varepsilon)$. Since $\alpha \in M M^{-1}$, there exists $\mathcal{V} \in M$ such that $\alpha \mathcal{V} \in M$,

$$(\alpha \in {}^\infty A \text{ and } \mathcal{V} \in M_{\nwarrow \text{inf}} \cup M_{\text{biinf}} \Rightarrow \alpha = \varepsilon)$$

and

$$(\alpha \in A^N \cup A^Z \Rightarrow \mathcal{V} = \varepsilon).$$

Let

$$\begin{aligned} \beta &= x_1 x_2 \dots x_k && \text{with } (x_1, x_2, \dots, x_k) \in X^{(k)}; \\ \alpha \mathcal{V} &= x_{k+1} \dots x_n && \text{with } (x_{k+1}, x_{k+2}, \dots, x_n) \in X^{(n-k)}; \\ \beta \alpha &= x'_1 x'_2 \dots x'_l && \text{with } (x'_1, x'_2, \dots, x'_l) \in X^{(l)}; \\ \mathcal{V} &= x'_{l+1} x'_{l+2} \dots x'_m && \text{with } (x'_{l+1}, \dots, x'_m) \in X^{(m-l)}. \end{aligned}$$

If $\beta \in M_{\text{inf}} \cup M_{\text{biinf}}$, then $\alpha = \varepsilon \in M$. If $\beta \in {}^\infty M$ and $\alpha \in A^{-N} \cup A^Z$, then $\beta = \varepsilon$. Therefore $\beta \alpha \in M$ implies $\alpha \in M$. If $\alpha \in A^N$, then we have $\mathcal{V} = \varepsilon$ and so $\alpha \mathcal{V} \in M$ implies $\alpha \in M$. When $\alpha \in {}^\infty A$ and $\mathcal{V} \in M_{-\text{inf}} \cup M_{\text{biinf}}$, then $\alpha = \varepsilon \in M$. We have to consider the only case when $\beta \in {}^\infty M$, $\alpha \in A^*$ and $\mathcal{V} \in M^\infty$. Since $\beta(\alpha \mathcal{V}) = (\beta \alpha) \mathcal{V}$, we get

$$x_1 x_2 \dots x_k x_{k+1} \dots x_n = x'_1 x'_2 \dots x'_l x'_{l+1} \dots x'_m.$$

Since X is a code, $n = m$ and $x_i = x'_i$, $i = 1, 2, \dots, n$. Since β is a left factor of $\beta \alpha$, we have $l \geq k$. Therefore

$$\beta \alpha = x'_1 x'_2 \dots x'_l = x_1 x_2 \dots x_k x_{k+1} \dots x_l = \beta x_{k+1} \dots x_l.$$

This implies $\alpha = x_{k+1} \dots x_l \in M$. Thus $M^{-1}M \cap MM^{-1} \subseteq M$ and so M is freeable.

DEFINITION 5.1: We say that a submonoid M satisfies finite chain condition if all the chains in M_{inf} and $M_{-\text{inf}}$ are finite.

The finite chain condition implies the maximality condition.

LEMMA 5.2: Every bi-quasi free submonoid satisfies the finite chain condition.

Proof is similar to that of proposition 3.3 of Chapter III in [10] and is omitted. The difference is to consider infinite chains in $M_{-\text{inf}}$.

THEOREM 5.1: For any submonoid M , the following conditions are equivalent.

- (i) M is bi-quasi free i.e., generated by a code.
- (ii) M is freeable and satisfies the finite chain condition.
- (iii) M is freeable and satisfies the maximality condition
- (iv) M is freeable and has a distinguished (minimal) generator set.

Proof: It is clear that (iii) $\Leftrightarrow$ (iv) by theorem 3.3. (i) $\Rightarrow$ (ii) is by lemmas 5.1 and 5.2. (ii) $\Rightarrow$ (iii) is evident. We have to show that (iii) $\Rightarrow$ (i).

Suppose M is freeable and satisfies the maximality condition. By theorem 3.3, M has a distinguished minimal generator set X which is $\text{BASE}(M_{\text{fin}}) \cup \text{MAX}(M_{\text{inf}}) \cup \text{MAX}(M_{-\text{inf}}) \cup \text{UNFAC}(M_{\text{biinf}})$. By theorem 4.2, it is enough if we prove that X is a code. Suppose X is not a code. Then there exists a word α such that it has two different factorizations on elements of X . i. e.,

$$\alpha = x_1 x_2 \dots x_n = x'_1 x'_2 \dots x'_m$$

where $n, m \geq 1$, $(x_1, x_2, \dots, x_n) \in X^{(n)}$ and $(x'_1, x'_2, \dots, x'_m) \in X^{(m)}$. Clearly either n , or, m should be greater than 1. Let $m > 1$.

Case (a): Suppose $\alpha \in A^*$. We may assume that $x_1 \neq x'_1$. Let $|x_1| > |x'_1|$. Then there exists a word $f \neq \varepsilon$ such that $x_1 = x'_1 f$ and $f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$. From the freeability of M , it follows that $f \in M_{\text{fin}} - \varepsilon$. This contradicts the hypothesis that $x_1 \in \text{BASE}(M_{\text{fin}})$.

Case (b): Suppose $\alpha \in A^N$. Then $x_n, x'_m \in \text{MAX}(M_{\text{inf}})$. If $n=1$, then $x_n = x'_1 x'_2 \dots x'_m$ and so $x_n < x'_m$ which is a contradiction to the maximality of x_n . Suppose $n \geq 2$. If

$$|x_1 x_2 \dots x_{n-1}| = |x'_1 x'_2 \dots x'_{m-1}|,$$

then

$$x_1 x_2 \dots x_{n-1} = x'_1 x'_2 \dots x'_{m-1} \in A^*.$$

As in case (a), we get a contradiction. If not, we assume that $|x_1 x_2 \dots x_{n-1}| > |x'_1 x'_2 \dots x'_{m-1}|$. This implies that there exists $f \neq \varepsilon$ with

$$x_1 x_2 \dots x_{n-1} = x'_1 x'_2 \dots x'_{m-1} f \quad \text{and} \quad f x_n = x'_m.$$

Again, by freeability of M , $f \in M_{\text{fin}} - \varepsilon$ and so we have $x'_m < x_n$ which contradicts the maximality of x'_m .

Case (c): Suppose $\alpha \in A^{-N}$. Then $x_1, x'_1 \in \text{MAX}(M_{-\text{inf}})$. We can discuss as in case (b) and obtain a contradiction.

Case (d): Suppose $\alpha \in A^Z$. If $n=1$, then $x_1 = x'_1 x'_2 \dots x'_m$ which is a contradiction since X is a distinguished minimal generator set. Suppose $n \geq 2$. Then $x_1, x'_1 \in \text{MAX}(M_{-\text{inf}})$. There are two possibilities.

(i) If $x_1 \neq x'_1$, we assume $x_1 = x'_1 f$ and so we have

$$f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m, \quad m \geq 2.$$

This implies $f \in M_{\text{fin}} - \varepsilon$ since M is freeable. Hence $x_1 < x'_1$ which contradicts the maximality of x_1 .

(ii) Suppose $x_1 = x'_1$. We have either

$$x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$$

or

$$x_2 x_3 \dots x_n \neq x'_2 x'_3 \dots x'_m.$$

If $x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$, then we assume $x_2 \neq x'_2$ and proceed as in case (b) and get a contradiction. If $x_2 x_3 \dots x_n \neq x'_2 x'_3 \dots x'_m$ since α has two factorizations, we have either $x_1 = x'_1 f$ and

$$f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m \quad \text{or} \quad x'_1 = x_1 f$$

and

$$x_2 x_3 \dots x_n = f x'_2 x'_3 \dots x'_m.$$

Since the two cases are similar, it is enough to consider any one of the possibilities, say $x_1 = x'_1 f$ and $f x_2 x_3 \dots x_n = x'_2 x'_3 \dots x'_m$. Clearly $f \in M_{\text{fin}} - \varepsilon$ as M is freeable and hence $x_1 < x'_1$ which contradicts the maximality of x_1 .

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