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NON RECURSIVE FUNCTIONS HAVE TRANSCENDENTAL GENERATING SERIES (*)

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Abstract. – *In this note we shall prove that non primitive recursive functions have transcendental generating series. This result translates a certain measure of the complexity of a function, the fact of not being primitive recursive, into another measure of the complexity of the generating series associated to the function, the fact of being transcendental.*

Résumé. – *On démontre que les fonctions qui ne sont pas recursives primitives ont des séries génératrices transcendantes. Ce résultat traduit une certaine mesure de complexité d'une fonction, le fait de ne pas être récursive primitive, dans une autre mesure de la complexité de la série génératrice associée à cette fonction, le fait d'être transcendante.*

A relation between the generating series of a function and its algebraic character was established in [2], proving the following result: *Let L be an unambiguous context-free language. Then the generating series of the language* $\zeta(z)=\sum \zeta_n z^n$ *where* $\zeta_n=\mathrm{Card}\{\omega\in L/|\omega|=n\}$, *is algebraic.*

This theorem was employed to prove that some languages are inherently ambiguous in [1] and [3].

Our main result is the following

THEOREM 1: *Let* $f:\mathbb{N}\to\mathbf{Q}$ *be a total function and consider the generating series*

$$\varphi(z)=\sum_{n=0}^{\infty} f(n)\,z^n.$$

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If $\varphi(z)$ is algebraic over $\mathbf{Q}(z)$, then f is primitive recursive.

Remarks: 1. The converse of this theorem is not true. For instance, the function $f(n) = 1/n!$ is clearly primitive recursive but its associated generating series is

$$\sum_{n=0}^{\infty} 1/n!\, z^n$$

i. e. e^z, a very well known transcendental series.

2. We finally remember some notions coming from algebra.

If K is any field, the field $K((t))^*$ of Puiseux series in t is defined as the union for $d \in \mathbb{N}$ of the fields $K((t^{1/d}))$ of meromorphic formal power series in the variable $t^{1/d}$. The main feature of this field is that if K is algebraically closed, so is $K((t))^*$. Hence, if we denote by $\bar{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$, we have that the algebraic closure of $\mathbf{Q}(z)$ is the field $\bar{\mathbf{Q}}((z))^*_{\text{alg}} = \{ \varphi \in \bar{\mathbf{Q}}((z))^* / \varphi$ is algebraic over $\mathbf{Q}(z) \}$ *i. e.* the series of the form $\Sigma\, c_n z^{n/d}$ where $d \in \mathbb{N}$ and the c_n are algebraic numbers. (The reader wishing to read a more detailed exposition of this material will find a very good one in Walker [5]).

Proof of theorem 1: The general idea of the proof is that the fact of being algebraic characterizes a series with a finite amount of data from which it is possible to compute the sequence of coefficients.

We first observe that since $\varphi(z) \in \mathbf{Q}[[z]]_{\text{alg}}$, there is a polynomial $P \in \mathbf{Z}[z][Y]$ such that $P(z, \varphi(z)) = 0$. Moreover, we can suppose that P is irreducible. Now, if r is the degree in Y of P we know that P has exactly r different roots in $\bar{\mathbf{Q}}((z))^*_{\text{alg}}$, $\varphi_1(z), \ldots, \varphi_r(z)$, and $\varphi(z)$ is one of them.

To distinguish $\varphi(z)$ between the others roots of P in $\bar{\mathbf{Q}}[[z]]_{\text{alg}}$ we can consider a natural number k and rational numbers $c_0, \ldots, c_k$ such that $\varphi(z)$ is the only root of P begining with $c_0 + \ldots + c_k z^k$.

We will see now how to compute $f(n)$ for every n from the *finite* collection of data $\{ P, k, c_0, \ldots, c_k \}$. This is done using the *indeterminate coefficients method*; *i. e.* once we know $c_0, \ldots, c_h$ such that $\varphi(z)$ is the only root of P in $\mathbf{Q}[[z]]_{\text{alg}}$ begining with $c_0 + \ldots + c_h z^h$, we consider the equality $P(z, c_0 + \ldots + c_h z^h + \lambda z^{h+1} + \ldots) = 0$. This translates into an infinite system of equations corresponding to the coefficients of the resulting series which must be all zero. But the equation corresponding to the term of least degree is an equation in λ having only one solution because there is only one possible root of P beggining with $c_0 + \ldots + c_h z^h$. In fact, it is well known that when computing the solutions of P in $\bar{\mathbf{Q}}[[z]]_{alg}$ after this point the equations giving the coefficients of the solutions are linear in this coefficient, *i. e.* the indeterminate λ turns out to be a polynomial function of the previously computed numbers $c_0, \ldots, c_h$ (*see* [5]). We then compute c_{h+1}, the only solution of this equation in λ.

We have then seen that knowing the first $h+1$ coefficients $c_0, \ldots, c_h$ we can compute the following c_{h+1}, *i. e.* we can recursively compute $f(n)$ for every n.

An almost immediate consequence of theorem 1 is

COROLLARY: *Let $L \subset \Sigma^*$ be a recursively enumerable language, where Σ is a finite alphabet, such that its generating series*

$$\zeta(z) = \sum_{n=0}^{\infty} \zeta_n z^n$$

mentioned above is algebraic. Then L is recursive.

Proof: Let us consider $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \zeta_n$. Following theorem 1, we have that f is recursive. Then, we can compute the number of words in L having length n. Since there are only a finite number of words of length n in Σ^* we compute in parallel their membership to L until we get ζ_n of them. At this moment we know we have all of them and we stop the computation.

From this we conclude that every recursively enumerable but non recursive language in Σ^* has a transcendental generating series.

In a classic paper about computability of real numbers (*see* [4]) Rice has shown the following result: *the field of recursive real numbers $\mathscr{E}$ is real closed* (*i. e. $\mathscr{E}[i]$ is algebraically* closed). As a trivial consequence of that we get that *every real algebraic number is recursive*. In some sense our theorem 1 is an analog of this last result. In the last part of this note we will push this analogy a bit further.

DEFINITION: Let $Comp = \{\varphi \in \mathbf{Q}[[z]] / \exists f: \mathbb{N} \to \mathbf{Q}$ recursive such that $\varphi = \Sigma f(n) z^n\}$. We will say that *Comp* is the ring of computable power series over $\mathbf{Q}$.

Our last result is the following

THEOREM 2: Comp *is algebraically closed in* $\mathbf{Q}[[z]]$.

Proof: Let $\varphi \in \mathbf{Q}[[z]]$ and $\psi_0, \ldots, \psi_m \in Comp$ such that φ is a root of $P = \psi_0 + \ldots + \psi_m Y^m$.

The same proof given in theorem 1 applies here. It suffices to note that in the linear equation giving us $f(n+1)$ the coefficients are recursive functions of some (finite number of) coefficients of the $\psi_i' s$ (as well as some previously computed $f(j)' s$ for $j \leqq n$), and these last numbers are computable by hypothesis.

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