INFORMATIQUE THÉORIQUE ET APPLICATIONS

M. ITO
H. J. SHYR
G. THIERRIN

Disjunctive languages and compatible orders

Informatique théorique et applications, tome 23, nº 2 (1989), p. 149-163.
<http://www.numdam.org/item?id=ITA_1989__23_2_149_0>

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 23, n° 2, 1989, p. 149 à 163)

DISJUNCTIVE LANGUAGES AND COMPATIBLE ORDERS (*)

by M. ITO (¹), H. J. SHYR (²) and G. THIERRIN (³)

Communicated by J.-E. PIN

Abstract. – *Let L be a language over an alphabet X. For* $u \in X^*$, *let* $L..u = \{(x, y) \mid xuy \in L\}$. *The language L is called disjunctive if* $L..u = L..v$ *implies* $u = v$. *If L is disjunctive, then the relation* $\leqq_L$ *defined on* X^* *by* $u \leqq_L v$ *if and only if* $L..u \subseteqq L..v$ *is a compatible partial order. In the case that* $\leqq_L$ *is the identity relation, then L is said to be s-disjunctive; otherwise L is called m-disjunctive. Properties of s-disjunctive and m-disjunctive languages as well as non trivial partial orders associated with m-disjunctive languages are investigatted in this paper.*

Résumé. – *Soit L un langage sur l'alphabet X. Pour tout* $u \in X^*$, *soit* $L..u = \{(x, y) \mid xuy \in L\}$. *Le langage L est dit disjonctif si* $L..u = L..v$ *implique* $u = v$. *Si L est disjonctif, la relation* $\leqq_L$ *définie dans* X^* *par* $u \leqq_L v$ *si et seulement si* $L..u \subseteqq L..v$ *est une relation d'ordre partiel compatible. Lorsque* $\leqq_L$ *est l'égalité, le langage L est dit s-disjonctif; dans les autres cas il est dit m-disjonctif. Dans cet article sont étudiées les propriétés des langages s-disjonctifs et m-disjonctifs ainsi que les relations d'ordre partiel qui leur sont associées.*

0. INTRODUCTION

Lest X be a finite alphabet, X^* the free monoid generated by X and $X^+ = X^* - \{1\}$, where 1 denotes the empty word. Elements of X^* are called *words* and subsets of X^* *languages*. With every language L and word u one associates the quotient

$$L..u = \{(x, y) \mid x, y \in X^*, \ xuy \in L\}.$$

(*) Received December 1986, revised November 1987.

This research was supported by the Natural Science and Engineering Research Council of Canada, Grant A7877.

(¹) Faculty of Science, Kyoto Sangyo University, Kyoto 603, Jaman.

(²) Department of Applied Mathematics, National Chung Hsing University, Taichung, Taiwan 400.

(³) Department of Mathematics, The University of Western Ontario, London, Ontario N6A 5B7, Canada.

The equivalence relation P_L defined on X^* by $u \equiv v(P_L)$ if and only if $L..u = L..v$ is the *syntactic congruence* of L. A language L is called *disjunctive* if P_L is the identity relation. For example the set Q of all primitive words is a disjunctive language. Another example is the context-free language $\{ w\tilde{w} \mid w \in X^* \}$.

Every disjunctive language is dense, that is $X^* u X^* \cap L \neq 0$ for every $u \in X^*$. A language that is not dense is said to be *thin*.

A partial order relation $\leqq$ defined on X^* is said to be *right* (*left*) *compatible* if $u \leqq v$ implies $ux \leqq vx$ ($xu \leqq xv$) for every $x \in X^*$; it is called *compatible* if it is both right and left compatible. In such a case $u \leqq v$ and $u' \leqq v'$ imply $uu' \leqq vv'$. Little is known about (right, left) compatible partial orders on X^*. Some results are related to classes of codes like prefix or suffix codes or hypercodes. General results about positive partial orders can be found in Jurgensen, Shyr and Thierrin [3].

In this paper we will show that a large class of compatible partial orders can be associated with disjunctive languages. If L is a disjunctive language, then the relation $\leqq_L$ defined on X^* by $u \leqq_L v$ if and only if $L..u \subseteqq L..v$ is a *compatible partial order.*

This partial order can be the identity relation; in such a case L is called an *s-disjunctive* (strongly disjunctive) language; otherwise L is called an *m-disjunctive* (middle disjunctive) language.

Section 1 contains several examples of s-disjunctive languages as well as a method for constructing a class of m-disjunctive languages. In Section 2, several properties of s-disjunctive and m-disjunctive languages are established, in particular in relation with the language Q of primitive words. The special class of reflective disjunctive languages is considered in Section 3 as well as the reflective closure of disjunctive languages in connection with s-disjunctivity or m-disjunctivity. Section 4 is devoted to the study of the compatible partial orders that can be associated with disjunctive languages; the results from that section show that these orders can be of many different types.

In this paper it is assumed that all the alphabets contain at least two letters.

1. EXAMPLES OF s-DISJUNCTIVE AND m-DISJUNCTIVE LANGUAGES

A disjunctive language $L \subseteqq X^*$ is called s-disjunctive if $L..u \subseteqq L..v$ implies $u = v$ for any $u, v \in X^*$. A disjunctive language which is not s-disjunctive is called m-disjunctive.

Let L be a disjunctive language.

I. The following properties are equivalent:

(1) L is s-disjunctive;

(2) For every $u \neq v$, there exist $x, y \in X^*$ such that $xuy \in L$ and $xvy \notin L$.

II. The following properties are equivalent:

(1) L is m-disjunctive;

(2) There exist $u, v \in X^*$, $u \neq v$, such that $L..u \subseteqq L..v$;

(3) There exist $u, v \in X^*$, $u \neq v$, such that $xuy \in L$ implies $xvy \in L$ for all $x, y \in X^*$.

In this section, several known languages are shown to be s-disjunctive. Also a method is given for constructing some classes of m-disjunctive languages.

PROPOSITION 1.1: *The disjunctive language Q is s-disjunctive.*

Proof: Let $u, v \in X^*$ and suppose that $Q..u \subseteqq Q..v$. Since Q is dense, we can assume without loss of generality, that $u, v \in Q$.

Consider $u^2v^2 \in X^*$. Since $v^4 \notin Q$, we have $u^2v^2 \notin Q$. Therefore $u^2v^2 = f^i$ for some $f \in Q$ and $i > 1$. By Lyndon and Schutzenberger [6], $u = v = f$. This shows that Q is s-disjunctive.

PROPOSITION 1.2: *The languages $\{w\tilde{w} \mid w \in X^*\}$ and $\{w \in X^* \mid w = \tilde{w}\}$ are s-disjunctive.*

Proof: Let $L = \{w\tilde{w} \mid w \in X^*\}$. Suppose that there exist $u, v \in X^*$ such that $u \neq v$ and $L..u \subseteqq L..v$. Let M be a positive integer with $M > 1 + |v|/|u|$. Since $u^M \tilde{u}^M \in L$, we have $uvu^{M-2}\tilde{u}^M \in L$ and $vuu^{M-2}\tilde{u}^M \in L$. Since $|\tilde{u}^M| = M|u| > |uv|$, $uv = vu$. Therefore there exist $p \in Q$ and $i, j \in N\,(i \neq j)$ such that $u = p^i$ and $v = p^j$. Let $p = p'a\,(p' \in X^*, a \in X)$. Since $p^i b^N b^N \tilde{p}^i \in L$, where $b \in X\,(a \neq b)$, $2N > |i-j|$ and $p^j b^N b^N \tilde{p}^i \in L$. However, this in contradiction with $p = p'a$. Hence L must be s-disjunctive. In a similar way, we can prove that $\{w \mid w = \tilde{w}\}$ is s-disjunctive.

Let $X = \{a_1, a_2, \ldots, a_r\}$ and let f be the following mapping of X^* into N:

$$f(1) = 0, \quad f(a_i) = i\,(1 \leqq i \leqq r)$$

and

$$f(a_{i_1} a_{i_2} \ldots a_{i_k}) = f(a_{i_1})(r+1)^{k-1} + f(a_{i_2})(r+1)^{k-2} + \ldots + f(a_{i_{k-1}})(r+1) + f(a_{i_k}).$$

It is clear that f is an injection. This mapping f will be called the *lexicographic function.*

Let

$$L_i = \{ u\bar{a}_i^k \mid u = u' a_i (u' \in X^*), f(u) \leqq k \} \qquad (1 \leqq i \leqq r).$$

Here

$$\bar{a}_i = a_{i+1} (1 \leqq i \leqq r-1) \quad \text{and} \quad \bar{a}_n = a_1.$$

PROPOSITION 1.3: *L_i is an m-disjunctive language for any* $1 \leqq i \leqq r$.

Proof: First we show that L_i is disjunctive. Let $u, v \in X^*$, $u \neq v$.

Without loss of generality, we can assume that $f(u) < f(v)$.

Obviously $f(ua_i) < f(va_i)$. Therefore $ua_i \bar{a}_i^{f(ua_i)} \in L_i$ but $va_i \bar{a}_i^{f(ua_i)} \notin L_i$. This means that L_i is disjunctive. Now we show that $L_i \ldots a_i^2 \subseteq L_i \ldots a_i$. Let $xa_i^2 y \in L_i$. Then

$$y = \bar{a}_i^{f(xa_i^2)+t} \quad \text{or} \quad y = y' a_i \bar{a}_i^{f(xa_i^2 y' a_i)+t}$$

for some $y' \in X^*$ and $t \geqq 0$.

Consequently

$$xa_i y = xa_i \bar{a}_i^{f(xa_i^2)+t} \quad \text{or} \quad xa_i y = xa_i y' a_i \bar{a}_i^{f(xa_i^2 y' a_i)+t}.$$

Note that

$$f(xa_i) < f(xa_i^2) \quad \text{and} \quad f(xa_i y' a_i) < f(xa_i^2 y' a_i).$$

Hence $xa_i y \in L_i$. This means that $L_i \ldots a_i^2 \subseteq L_i \ldots a_i$, i. e. L_i is m-disjunctive.

In a similar way, it can be proved that

$$\{ u\bar{a}_i^k \mid u = u' a_i, u' \in X^*, f(u) \geqq k \}, \qquad \{ ua_i^k \mid u \in X^*, k \geqq f(u) \}$$

and

$$\{ ua_i^k \mid u \in X^*, f(u) \geqq k \}$$

are m-disjunctive.

2. PROPERTIES OF s-DISJUNCTIVE AND m-DISJUNCTIVE LANGUAGES

PROPOSITION 2.1: *Let $L \subseteqq X^*$ be s-disjunctive (m-disjunctive), let $w \in X^*$ and let $A \subseteqq X^*$ be a thin language. Then we have:*

(1) $X^* \setminus L$ *is s-disjunctive (m-disjunctive).*

(2) $L \cap X^* w X^*$ *is s-disjunctive (m-disjunctive).*

(3) $L \cup A$ *and* $L \setminus A$ *are s-disjunctive (m-disjunctive).*

Proof: (1) Obvious from the definition of an s-disjunctive (m-disjunctive) language.

(2) Let $L \subseteqq X^*$ be s-disjunctive. Suppose there exist $u, v \in X^*$, $u \neq v$, such that $L \cap X^* w X^* . . u \subseteqq L \cap X^* w X^* . . v$. Take $uw, vw \in X^*$. Let $xuwy \in L$. Then $xuwy \in L \cap X^* w X^*$. Therefore $xvwy \in L \cap X^* w X^*$ and hence $xvwy \in L$.

This means that $L . . uw \subseteqq L . . vw$, a contradiction. Thus $L \cap X^* w X^*$ must be s-disjunctive. The proof of m-disjunctiveness of $L \cap X^* w X^*$ for an m-disjunctive language L follows from the fact that $L . . u \subseteqq L . . v$ implies $L . . uw \subseteqq L . . vw$ and hence $L \cap X^* w X^* . . uw \subseteqq L \cap X^* . . vw$.

(3) Let $L \subseteqq X^*$ be an s-disjunctive language and $A \subseteqq X^*$ be a thin language.

It is easy to verify that $L \cup A \,(L \setminus A)$ is disjunctive. Now, since A is thin, there exists $w \in X^*$ such that $X^* w X^* \cap A$ is empty. Suppose $L \cup A \,(L \setminus A)$ is not an s-disjunctive language. Then there exist $u, v \in X^*$, $u \neq v$, such that

$$L \cup A . . u \subseteqq L \cup A . . v \,(L \setminus A . . u \subseteqq L \setminus A . . v).$$

Note that

$$L \cup A . . uw \subseteqq L \cup A . . vw \,(L \setminus A . . uw \subseteqq L \setminus A . . vw).$$

Let $xuwy \in L$. Then $xuwy \in L \cup A \,(xuwy \in L \setminus A)$ and $xvwy \in L \cup A \,(xvwy \in L \setminus A)$. Since $X^* w X^* \cap A$ is empty, we have $xvwy \in L$. This means that $L . . uw \subseteqq L . . vw$, a contradiction. The proof for the case for L m-disjunctive can be carried out in a similar way.

COROLLARY: *Let $L \subseteqq X^*$ be s-disjunctive (m-disjunctive) and $F \subseteqq X^*$ be a finite language. Then $L \cup F$ and $L \setminus F$ are s-disjunctive (m-disjunctive).*

A language L is called *semi-discrete* if there exists $k \in N$ such that L contains at most k words of any given length.

PROPOSITION 2.2: *Let $L \subseteqq X^*$ be a semi-discrete disjunctive language. Then L is s-disjunctive.*

Proof: Let $L \subseteq X^*$ be a semi-discrete disjunctive language and let $n = \max\{|C| \mid C \leqq L,\ C \leqq X^i \text{ for some } i \in N\}$. Suppose there exist $u,\ v \in X^*$ such that $L..u \subseteq L..v$. Since L is dense, there exist $x,\ y \in X^*$ such that $xu^{n+1}y \in L$. Consider $x(u^i vu^{n-i})y$ for $0 \leqq i \leqq n$.

From $L..u \subseteq L..v$, we have $x(u^i vu^{n-i})y \in L$. Note that

$$|x(u^i vu^{n-i})y| = |x| + |y| + n|u| + |v| \qquad \text{for any } 0 \leqq i \leqq n.$$

Consequently there exist

$$i, j\ (0 \leqq i \leqq j \leqq n) \qquad \text{such that} \quad xu^i vu^{n-i} y = xu^j vu^{n-j} y.$$

Hence $vu^{j-i} = u^{j-i} v$ and there exist $p \in Q$ and $k,\ r \geqq 1$ such that $u = p^k$ and $v = p^r$. Let $q \in X^*$, $p \neq q$ and $|q| = |p|$. Since L is dense, there exist $w,\ z \in X^*$ such that $w(p^{2k} q)^{n+1} z \in L$. Consider

$$w(p^{2k} q)^i (p^{2r} q)(p^{2k} q)^{n-i} z \qquad \text{for} \quad 0 \leqq i \leqq n.$$

Note that the above words have the same length and belong to L.

Hence there exist $0 \leqq i \leqq j \leqq n$ such that

$$w(p^{2k} q)^i (p^{2r} q)(p^{2k} q)^{n-i} z = w(p^{2k} q)^j (p^{2r} q)(p^{2k} q)^{n-j} z.$$

This yields

$$(p^{2r} q)(p^{2k} q)^{j-i} = (p^{2k} q)^{j-i} (p^{2r} q).$$

There exist $g \in Q$ and $s,\ t \in N$ such that $p^{2k} q = g^s$ and $p^{2r} q = g^t$. It is easy to see that $p = q = g$, a contradiction. Therefore L must be s-disjunctive.

PROPOSITION 2.3: *Let $A \subseteq X^*$ be a prefix code (suffix code) and let $L \subseteq X^*$ be an s-disjunctive language. Then AL (LA) is s-disjunctive.*

Proof: Let $A \subseteq X^*$ be a prefix code and let $L \subseteq X^*$ be an s-disjunctive language. The language AL is disjunctive (*see* Shyr [8]). Now suppose that there exist $u,\ v \in X^*$, $u \neq v$, such that $AL..u \subseteq AL..v$. Let $xuy \in L$. Then $wxuy \in$ AL for $w \in A$. Therefore $wxvy \in AL$. Since A is a prefix code, $xvy \in L$. Consequently $L..u \subseteq L..v$, a contradiction. This means that AL is s-disjunctive.

The proof for the case of a suffix code is similar.

Remark: The preceding proposition is not true for the case of m-disjunctive languages. Let $X = \{a, b\}$ and let $L = \{ua^k \mid k \geqq f(u)\}$ where $a = a_1$, $b = a_2$ and f is the lexicographic function. As it has already been shown, L is an m-disjunctive language. Let $T = \{t_i \mid i \in N\}$ be a subset of positive integers with $t_{i+1}-t_i>i+1$ for any $i\in N$ and let $\{T_i\}_{i\in N}$ be a partition of T with $|T_i|$ infinite for any $i\in N$. Let $A=\bigcup_{i\in N}\{a^m b^i \mid m\in T_i\}$. Obviously A is a prefix code. However AL is not an m-disjunctive language.

Proof: Suppose $AL..u\subseteq AL..v$ for some $u, v\in X^*$, $u\neq v$. If $f(u)<f(v)$, then $(a^m b)uba^{f(ub)}\in AL$, where $m\in T_1$, implies $a^m bvba^{f(ub)}\in AL$.

However this contradicts the fact that A is a prefix code and $f(vb)>f(ub)$. Therefore we have $f(u)>f(v)$.

Case 1: $u=a^k$ for some $k\geqq 1$. Let $m\in T_1$ with $m>t_k$. Consider $a^m ba\in AL$. Since $a^m ba\in AL$, we have $a^{m-k}vba\in AL$. Therefore

$$a^{m-k}vb\in A \quad \text{and} \quad |a^{m-k}v'|\in T \qquad \text{where} \quad v=v'v''\,(v'\in a^*,\ v''\in b^*).$$

However, this contradicts the definition of T.

Case 2: $u=a^k b^r u'$ for some k, $r\geqq 1$ and $u'\in X^*\setminus bX^*$. Let $m\in T_r$ with $m>t_k$. Consider $a^m b^r u' a^{f(u')}\in AL$. Since $a^m b^r u' a^{f(u')}\in AL$, we have $a^{m-k}va^{f(u')}\in AL$. Therefore $v=v'b^s v''$ for some $s\geqq 1$, $v'\in a^*$ and $v''\in X^*\setminus bX^*$. Moreover we have $|a^{m-k}v'|\in T_s$. However this contradicts the definition of T.

Case 3: $u\in bX^*$. Let $u=b^s u'$, where $s\geqq 1$ and $u'\in X^*\setminus bX^*$ and let $m\in T_s$ with $m>t_{|u|}$. Consider $a^m ua^{f(u')}=a^m b^s u' a^{f(u')}\in AL$. Since $a^m ua^{f(u')}\in AL$, we have $a^m va^{f(u')}\in AL$. Therefore $v=v'b^t v''$ for some $t\geqq 1$, $v'\in a^*$ and $v''\in X^*\setminus bX^*$. Hence we have $|a^m v'|\in T_t$ and this contradicts the definition of T. Consequently there exist no pair $(u, v)\in X^*\times X^*$, $u\neq v$, such that $AL...u\subseteq AL...v$, *i. e.* AL is not m-disjunctive.

Note that in the above remark, A is infinite. For the case of finite prefix (suffix) codes, we have the following proposition.

PROPOSITION 2.4: *Let $A\subseteq X^*$ be a finite prefix (suffix) code and let $L\subseteq X^*$ be an m-disjunctive language. Then $AL\,(LA)$ is m-disjunctive.*

Proof: Let A be a finite prefix (suffix) code. Then $AL\,(LA)$ is a disjunctive language. We now show that AL is m-disjunctive. Let $L..u\subseteq L..v$ for some $u\neq v$ and let $n=max\{|x| \mid x\in A\}$. Take $w\in X^+$ such that $|w|>n$. We show that $AL..wu\subseteq AL..wv$. Indeed, if $xwuy\in AL$ for some x, $y\in X^*$, then $xw=zz'$ for some $z\in X^+$, $z'\in X^+$ such that $z'uy\in L$, $z\in A$. Since $L..u\subseteq L..v$, we have $z'vy\in L$ and hence $xwvy\in AL$ holds. This shows that AL is m-disjunctive. Similarly we can show that LA is m-disjunctive if A is a suffix code.

Recall that an infix code C is a code such that $xuy\in C$ and $u\in C$ imply $x=y=1$.

PROPOSITION 2.5: *Let $A \subseteqq X^*$ be an infix code and let $L \subseteqq X^*$ be an s-disjunctive (m-disjunctive) language. Then AL and LA are s-disjunctive (m-disjunctive) languages.*

Proof: Since every infix code is a bifix code, the proposition holds for the case when L is s-disjunctive. Now let $A \subseteqq X^*$ be an infix code and let L be an m-disjunctive language. Let $L..u \subseteqq L..v$ for some $u, v \in X^*$, $u \neq v$. Consider wu, $wv \in X^*$ where $w \in A$. Let $xwuy \in AL$. Since $w \in A$ and A is an infix code, there exist $w' \in A$ and $w'' \in X^*$ such that $xw = w'w''$ and $w''uy \in L$. Therefore $xwuy = w'w''uy$. Since $L..u \subseteqq L..v$, $w''vu \in L$. Consequently, $xwuy = w'w''vy \in AL$, *i.e.* $AL..wu \subseteqq AL..wv$. This means that AL is m-disjunctive. For LA, the proof is similar.

PROPOSITION 2.6: *Let L be a disjunctive language such that $L \subseteqq U_{i \geqq 2} Q^{(i)}$ where $Q^{(i)} = \{p^i \mid p \in Q\}$. Then L is s-disjunctive.*

Proof: Let $L..u \subseteqq L..v$ where $u, v \in X^*$. Let $n > 7 + 5|v|/|u|$.

Since L is dense, there exist $x, y \in X^*$, $f \in Q$ and $i \geqq 2$ such that $xu^n y = f^i \in L$. Since $L..u \subseteqq L..v$, we have

$$xu^{n-2}uvy = g^j \in L \qquad \text{and} \qquad xu^{n-2}vuy = h^k \in L$$

for some $g, h \in Q$ and $j, k \geqq 2$. Therefore there exist $g', h' \in Q$ such that $yxu^{n-2}uv = g'^j$ and $yxu^{n-2}vu = h'^k$. If $j = k = 2$, then $uv = vu$. Otherwise, since

$$\begin{aligned} |g'| + |h'| &= (1/j + 1/k)(|y| + |x| + (n-2)|u| + |u| + |v|) \\ &\leqq 5/6(|y| + |x| + (n-2)|u| + |u| + |v|) < |yxu^{n-2}|, \end{aligned}$$

we have $g' = h'$ and $uv = vu$, too (this result follows from Lothaire [5] or Shyr [8]). Consequently there exist $p \in Q$, $r, m \in N$ such that $u = p^r$ and $v = p^m$. Suppose $r \neq m$, *i.e.* $u \neq v$.

Let q be an element in X^* such that $q \neq p$ and $|q| = |p|$.

Since L is dense, there exist $z, w \in X^*$, $f_1 \in Q$ and $s \geqq 2$ such that $zqp^{2r}w = f_1^s \in L$. Since $L..p^r \subseteqq L..p^m$, there exist $g_1 \in Q$, $t \geqq 2$ such that $zqp^{2m}w = g_1^t \in L$. Therefore there exist $f'_1, g'_1 \in Q$ such that

$$wzqr^{2r} = f'^s_1 \qquad \text{and} \qquad wzqp^{2m} = g'^t_1.$$

If $r > m$ ($r < m$), then $g'^t_1 p^{2(r-m)} = f'^s_1$ ($f'^s_1 p^{2(m-r)} = g'^t_1$). By Lyndon and Schutzenberger (1962), $f'_1 = g'_1 = p$. However, since $q \neq p$ and $|q| = |p|$, this yields a contradiction. Hence $r = m$, *i.e.* $u = v$.

This completes the proof of the proposition.

COROLLARY: *For any disjunctive language L, the language* $L^{(i)}=\{x^i \mid x\in L\}$, $i\geqq 2$, *is s-disjunctive.*

PROPOSITION 2.7: *Let* $L\subseteq X^*$ *be an m-disjunctive language. Then* $L\cap Q$ *is dense. Moreover, one of the following two properties holds:*

(1) $L\cap Q$ *is m-disjunctive;*

(2) $L\setminus Q$ *is s-disjunctive.*

Proof: Let $L\subseteq X^*$ be an m-disjunctive language. First we prove that $L\cap Q$ is dense. Suppose that $L\cap Q$ is thin. Then by Proposition 2.1, $L\setminus(L\cap Q)$ is m-disjunctive. On the other hand, since $L\setminus(L\cap Q)\subseteq U_{i\geqq 2}Q^{(i)}$, by Proposition 2.6, $L\setminus(L\cap Q)$ is s-disjunctive, a contradiction. Hence $L\cap Q$ must be dense. Now we prove the second part of the proposition. First consider the case where $L\setminus Q$ is dense. By dense. By Ito, Jurgensen, Shyr and Thierrin [2], $L\setminus Q$ is disjunctive. By Proposition 2.6, $L\setminus Q$ must be s-disjunctive. Now, suppose that $L\setminus Q$ is thin. Then since $L\cap Q=L\setminus(L\setminus Q)$, by Proposition 2.1, $L\cap Q$ is m-disjunctive.

3. Reflective s-disjunctive and m-disjunctive languages

A language $L\subseteq X^*$ is called *reflective* if $xy\in L$ implies $yx\in L$ for any $x, y\in X^*$. The *reflective closure* $\tilde{L}$ of a language L is the smallest reflective language containing L. In this section, we consider languages that are reflective and also s-disjunctive or m-disjunctive. An example of a s-disjunctive language that is also reflective is the set Q of all the primitive words over X. The next proposition shows that a reflective language can be m-disjunctive.

PROPOSITION 3.1: *For every alphabet X there exists an m-disjunctive language which is reflective.*

Proof: Let $X=\{a, b, \ldots\}$, $a_1=a$, $a_2=b$, ... and let

$$L=\{ua^k \mid u\in X^+,\ k\geqq f(u)\}$$

where f is the lexicographic function. Let $\tilde{L}$ be the reflective closure of L. First, we prove that $\tilde{L}$ is disjunctive. Let $u, v\in X^*$, $u\neq v$. Without loss of generality, we can assume that $f(u)<f(v)$. Therefore for any $x, y\in X^*$ we have $f(xuy)<f(xvy)$. Let n be a positive integer such that $|v|<f(b^n)$. Consider $b^n ub^n$, $b^n vb^n\in X^*$. Since $b^n ub^n a^{f(b^n ub^n)}\in L$, $b^n ub^n a^{f(b^n ub^n)}\in\tilde{L}$. We will show that $b^n vb^n a^{f(b^n ub^n)}\notin\tilde{L}$. Suppose $b^n vb^n a^{f(b^n ub^n)}\in\tilde{L}$. Since $b^n vb^n a^{f(b^n ub^n)}\notin L$ [because $f(b^n vb^n)>f(b^n ub^n)$], there exist $w, w'\in X^+$ such that $b^n vb^n a^{f(b^n ub^n)}=ww'$ and $w'w\in L$. However this is impossible by the fact that $|v|<f(b^n)$. This completes the proof of disjunctivity of $\tilde{L}$. We prove now

that $\tilde{L}$ is m-disjunctive. To this end, it is enough to show that $\tilde{L}..b \subseteq \tilde{L}..1$. Let $xby \in \tilde{L}$. This means that there exist $w, w' \in X^*$ such that $xby = ww'$ and $w'w \in L$.

Obviously $|w| \leqq |x|$ or $|w'| < |y|$.

Case 1: $|w| \leqq |x|$. In this case,

$$x = x' a^{|x| - |x'| - |x''|} x'' \qquad (x' \in X^+ X^* a, x'' \in X^*)$$

and

$$|x| - |x'| - |x''| \geqq f(x''byx'), \qquad \text{or} \qquad x = a^{|x| - |x'|} x' \quad (x' \in X^*),$$
$$y = y' a^{|y| - |y'|} \qquad (y' \notin X^* a)$$

and

$$|x| + y| - |x'| - y'| \geqq f(x' by').$$

Consider $xy \in X^*$. For the first case, $xy = x' a^{|x| - |x'| - |x''|} x'' y$. Then $x'' yx' a^{|x| - |x'| - |x''|} \in L$, because $|x| - |x'| - |x''| > f(x'' yx')$. For the second case, $xy = a^{|x| - |x'|} x' y' a^{|y| - |y'|}$. Then $x' y' a^{|x| + |y| - |x'| - |y'|} \in L$, because $|x| + |y| - |x'| - |y'| > f(x' y')$.

Therefore in either case $xy \in \tilde{L}$.

Case 2: $|w'| < |y|$. In this case

$$y = y' a^{|y| - |y'| - |y''|} y'' \qquad (y' \notin X^* a, y'' \in X^*)$$

and

$$|y| - |y'| - |y''| \geqq f(y'' xby').$$

Consider $xy = xy' a^{|y| - |y'| - |y''|} y''$. Then $y'' xy' a^{|y| - |y'| - |y''|} \in L$, because $|y| - |y'| - |y''| > f(y'' xy')$. Therefore $xy \in \tilde{L}$. Consequently $\tilde{L}...b \subseteq \tilde{L}...1$, *i.e.* $\tilde{L}$ is m-disjunctive.

PROPOSITION 3.2: *There exists an s-disjunctive language whose reflective closure is not s-disjunctive.*

Proof: Let $x = \{a_1, a_2, \ldots a_r\}$ and let $M = \{u_i a_i \mid u_i \in X^+, a_i \in X, i \in N\}$ be a discrete dense language. Let

$$T = \{(ua_i \bar{a}_i^{|ua_i|})^2 \mid i \in N\}$$

where

$$\bar{a}_i = a_{i+1} \, (1 \leqq i < r) \qquad \text{and} \qquad \bar{a}_r = a_1.$$

Since T is dense and $T \subseteq Q^{(2)}$, by Proposition 2.6, T is s-disjunctive. Therefore $L = X^*/T$ is s-disjunctive.

We prove now that $\tilde{L}$ is not s-disjunctive. Consider $(\bar{a}_i^{|u_i a_i|} u_i a_i)^2 \in X^*$. Note that T is discrete. Consequently $(\bar{a}_i^{|u_i a_i|} u_i a_i)^2 \notin T$, *i.e.* $(\bar{a}_i^{|u_i a_i|} u_i a_i)^2 \in L$. Therefore $(u_i a_i \bar{a}_i^{|u_i a_i|})^2 \in \tilde{L}$ and $T \subseteq \tilde{L}$. Hence $\tilde{L} = X^*$, *i.e.* $\tilde{L}$ is not s-disjunctive. In fact, $\tilde{L}$ is not even disjunctive.

COROLLARY: *There exists a disjunctive language whose reflective closure is not disjunctive.*

PROPOSITION 3.3: *There exists an m-disjunctive language whose reflective closure is not m-disjunctive.*

Proof: Let $x = \{a_1, a_2, \ldots, a_r\}$ and let

$$L_i = \{u \bar{a}_i^k \mid u = u' a_i \, (u' \in X^*), \, f(u) \leqq k\} \, (1 \leqq i \leqq r).$$

As we have shown in proposition 1.3, L_i is m-disjunctive. Consider, $M_i = X^*/L_i$. Then M_i is m-disjunctive. Let $u \bar{a}_i^k \in L_i$. Since $u = u' a_i$, $\bar{a}_i^k u \notin L_i$, i.e. $\bar{a}_i^k u \in M_i$. Therefore $u \bar{a}_i^k \in \tilde{M}_i$. Consequently $\tilde{M}_i = X^*$, *i.e.* $\tilde{M}_i$ is not m-disjunctive. In fact $\tilde{M}_i$ is not even disjunctive.

4. Compatible partial orders

Let $L \subseteq X^*$ and let $u, v \in X^*$. The relation $\leqq_L$ is defined by $u \leqq_L v$ if and only if $L \,.. u \subseteq L \,.. v$. If L is a disjunctive language, then the relation $\leqq_L$ becomes a compatible partial order in X^*, *i.e.* $(X^*, \leqq_L)$ is a p. o. monoid (partially ordered monoid). Furthermore the relation $\leqq_L$ is a nontrivial partial order if and only if L is m-disjunctive.

PROPOSITION 4.1: *Let $L \subseteq X^*$ be an m-disjunctive language and let $v \in X^*$. Then v is a maximal (minimal) element in $(X^*, \leqq_L)$ if and only if every factor v' of v is a maximal (minimal) element in $(X^*, \leqq_L)$.*

Proof: Obvious.

COROLLARY: *If there exists a maximal (minimal) element $u \in X^*$ in $(X^*, \leqq_L)$, then* 1 *and, if the number of occurrences of the letter a in the word u is not zero, the letter a are maximal (minimal) elements in $(X^*, \leqq_L)$.*

An element $u \in X^*$ is said to be *isolated* if it is a maximal element and at the same time a minimal element in $(X^*, \leqq_L)$.

COROLLARY: *There exists a maximal element and a minimal element in* $(X^*, \leqq_L)$ *if and only if* 1 *is isolated.*

We consider now the following problem: Given a compatible partial order $\leqq$ on X^*, is it possible to find an m-disjunctive language L such that $\leqq = \leqq_L$?

PROPOSITION 4.2: *There exists a compatible partial order* $\leqq$ *on* X^* *such that* $\leqq \neq \leqq_L$ *for every disjunctive* $L \subseteq X^*$.

Proof: Let $u, v \in X^*$. We define the relation $u \leqq v$ as follows: $u \leqq v$ if and only if $u = v$ or $|u| < |v|$.

Then clearly $\leqq$ is a compatible partial order. We prove now that $\leqq \neq \leqq_L$ for every m-disjunctive language $L \subseteq X^*$. Suppose that there exists an m-disjunctive language $L \subseteq X^*$ such that $\leqq = \leqq_L$. Since $L \neq \varnothing$, there exists $u \in L$. By definition, $X^* \setminus \bigcup_{0 \leqq i \leqq |u|} X^i \subseteq L$. On the other hand, since $\bigcup_{0 \leqq i \leqq |u|} X^i$ is a finite set, L must be a regular language, a contradiction. Therefore $(X^*, \leqq) \neq (X^* \leqq_L)$ for any m-disjunctive language $L \subseteq X^*$.

LEMMA 4.3: *Let* $L \subseteq X^*$ *be an* m-*disjunctive language such that* $\leqq_L$ *is a total order. If there exist* $u, v \in X^*$ *such that* $u <_L 1 <_L v$, *then one of the sets* $\{w \in X^* \mid u <_L w <_L 1\}$, $\{w \in X^* \mid 1 <_L w <_L v\}$ *is infinite.*

Proof: Suppose that the above assertion is not true. Then there exist

$$u', v' \in X^* \quad \text{such that} \quad u' <_L 1 <_L v',$$
$$\{w' \in X^* \mid u' <_L w' <_L 1\} = \varnothing \quad \text{and} \quad \{w' \in X^* \mid 1 <_L w' <_L v'\} = \varnothing.$$

Since the order $\leqq_L$ is compatible, $u' v' \leqq_L v'$ and $u' \leqq_L u' v'$, *i. e.* $u' \leqq_L u' v' \leqq_L v'$. Obviously $u' \neq u' v' \neq v'$.

Therefore $u' v' = 1$, a contradiction. This completes the proof of the lemma.

Let $(S, \leqq)$ be a p. o. set (partially ordered set). Then $(S, \leqq)$ is said to be *discrete* if $\{r \in S \mid s \leqq r \leqq t\}$ is finite for any $s, t \in S$. If $(S, \leqq)$ is an infinite discrete t. o. set (totally ordered set), then the structure of $(S, \leqq)$ is one the following three types:

(i) $(S, \leqq) = \{s_0 \leqq s_1 \leqq s_2 \leqq \ldots\}$;

(ii) $(S, \leqq) = \{\ldots \leqq s_2 \leqq s_1 \leqq s_0\}$;

(iii) $(S, \leqq) = \{\ldots \leqq s_5 \leqq s_3 \leqq s_1 \leqq s_0 \leqq s_2 \leqq s_4 \leqq s_6 \ldots\}$,

PROPOSITION 4.4: *Let* $L \subseteq X^*$ *be an* m-*disjunctive language. Then the partial order* $\leqq_L$ *is not a discrete total order.*

Proof: Let $L \subseteqq X^*$ be an m-disjunctive language. Suppose that $\leqq_L$ is a discrete total order. We prove firtst that $(X^*, \leqq_L)$ is either of type (i) or (ii). If $(X^*, \leqq_L)$ is of type (iii), then there exist $u, v \in X^*$ such that $u <_L 1 <_L v$. Since $(X^*, \leqq_L)$ is discrete, $|\{w \in X^* \mid u <_L w <_L 1\}|$ and $|\{w \in X^* \mid 1 <_L w <_L v\}|$ are finite. However this contradicts Lemma 4.3. Therefore $(X^*, \leqq_L)$ must be of type (i) or (ii).

Case of type (i): Note that, by Lemma 4.3, 1 is a minimum element.

Since $L \neq 0$, there exists $u \in L$. By definition $\{u \in X^* \mid u \leqq_L v\} \subseteqq L$. On the other hand, $\{w' \in X^* \mid w' \leqq_L u\} = \{w' \in X^* \mid 1 \leqq_L w' \leqq_L u\}$ is finite.

Therefore $L = X^* \setminus F$ where $F \subseteqq \{w' \in X^* \mid 1 \leqq_L w' \leqq_L u\}$. This means that L is regular, a contradiction.

Case of type (ii): Since L is infinite, for any $v \in X^*$ there exists $u \in L$ such that $u \leqq_L v$. Therefore $v \in L$. This means that $L = X^*$, a contradiction. Therefore in either case we have a contradiction. This completes the proof of the proposition.

At present, it is not known if the assertion of the above proposition is true or not for other kinds of t. o. sets.

Let $\leqq$ be a partial order in X^*. A subset K of X^* is called a $<$*-antichain* if u and v are not comparable for any $u, v \in K$, $u \neq v$.

In order to prove the next proposition, we need the following lemma.

LEMMA 4.5: *Let $X = \{a, b, \ldots\}$. If $K \subseteqq X^*$ is thin, then $K' = \bigcup_{u \in K} a^+ buba^+$ is thin.*

Proof: Since K is thin, there exists $w \in X^*$ such that $X^* wX^* \cap K = \varnothing$.

Suppose $X^* bwbX^* \cap K' \neq \varnothing$. Then there exist $x, y \in X^*$, $m, n \geqq 1$ and $u \in K$ such that $a^m buba^n = xbwby$. Since $|xb| \geqq |a^m b|$ and $|by| \geqq |ba^n|$, we have $x' wy' = u$ for some $x', y' \in X^*$. This contradicts $X^* wX^* \cap K = \varnothing$.

Hence K' is thin.

PROPOSITION 4.6: *If $K \subseteqq X^*$ is thin, then there exists an m-disjunctive language $L \subseteqq X^*$ such that K is a $\leqq_L$-antichain.*

Proof: Let $X = \{a, b, \ldots\}$ and let $K = \{u_1, u_2, \ldots\}$ where $|u_i| \leqq |u_{i+1}|$ for any $i \geqq 1$. Let $L_0 \subseteqq X^*$ be any m-disjunctive language. Consider

$$L = (L_0 \setminus \bigcup_{i \geqq 1} a^+ bu_i ba^+) \cup (\bigcup_{i \geqq 1} a^{|u_i| + i} bu_i ba^{|u_i| + i}).$$

By Lemma 4.5 and Proposition 2.1, L is m-disjunctive. Now suppose $u_i \leqq_L u_j$ for some $i \neq j$. Since $a^{|u_i|+i} b u_i b a^{|u_i|+i} \in L$, we have $a^{|u_i|+i} b u_j a^{|u_i|+i} \in L$. By the definition of L, there exists

$$k \geqq 1 \quad \text{such that} \quad a^{|u_i|+i} b u_j b a^{|u_i|+i} = a^{|u_k|+k} b u_k b a^{|u_k|+k}.$$

It can easily be seen that $i=j=k$, a contradiction. Therefore K is a $\leqq_L$-antichain.

COROLLARY: *Let $L(X)=\{L_i \mid i \in I\}$ be the set of all m-disjunctive languages over X. For every $i \in I$, choose $u_i, v_i \in L_i$ such that $u_i \leqq_{L_i} v_i$.* Let $M = \bigcup_{i \in I} \{u_i, v_i\}$. *Then M is dense.*

Proof: Suppose that M is thin. By Proposition 4.6, there exists $L \in L(X)$ such that M is a $\leqq_L$-antichain. Let $L=L_j$ where $j \in I$. Note that $\{u_j, v_j\} \subseteq M$. However $u_j \leqq_L v_j$. This contradicts the fact that M is a $\leqq_L$-antichain. Hence M is dense.

Let $u, v \in X^*$, $u \neq v$. We consider now an m-disjunctive language such that $u \leqq_L v$.

PROPOSITION 4.7: *Let $u, v \in X^*$, $u \neq v$. Then there exists an m-disjunctive language $L \subseteq X^*$ such that $u \leqq_L v$.*

Proof: Let $X=\{a, b, \ldots\}$ with $a=a_1$, $b=a_2, \ldots$ and let f be the lexicographic function. We can assume without loss of generality $u \notin X^* a$.

Case 1: $f(u) < f(v)$. Let $L=\{xba^k \mid k \leqq f(x)\}$. Then L is m-disjunctive and $u \leqq_L v$.

Case 2: $f(u) > f(v)$. Let $L=\{xba^k \mid k \geqq f(x)\}$. Then L is m-disjunctive and $u \leqq_L v$.

It has been shown before that there is no discrete total order that coincides with the order $\leqq_L$ defined by an m-disjunctive language. (If the total order is not discrete, the corresponding problem is open.) The following proposition shows that an m-disjunctive language can define an "almost" total order on X^*.

PROPOSITION 4.8: *Let X be an alphabet. Then there exist an m-disjunctive language L, a subalphabet Y of X with $|Y| = |X| - 1$ such that the restriction to Y^* of the partial order $\leqq_L$ defined by L is total.*

Proof: Let $X=\{a_1, a_2, \ldots, a_{r-1}, a_r\}$ and let $Y=a_1, a_2, \ldots, a_{r-1}\}$.

Define the order $\leqq$ on Y^* by $u \leqq v$ if $f(u) \leqq f(v)$ where f is the lexicographic function; this is clearly a total order. Let $L = \{ua_r^k \mid u \notin X^* a_r, f(u) \geqq k\}$. Suppose that $u \leqq v$ where $u, v, \in Y^*$ $u \neq v$.

Let $xuy \in L$ for some $x, y \in X^*$. Then $y = y' a_r^m$ where $m \leqq f(xuy')$.

Since $f(u) < f(v)$, $f(xuy') < f(xvy')$. Therefore $m \leqq f(xvy')$ and $xvy \in L$, *i. e.* $u \leqq_L v$. Therefore $\leqq_L$ coincides with $\leqq$ on Y^*.

REFERENCES

1. J. Berstel and D. Perrin, *Theory of Codes*, Academic Press, Orlando, 1985.
2. M. Ito, H. Jurgensen, H. J. Shyr and G. Thierrin, Anti-commutative Languages and n-codes (to appear in Applied Discrete Mathematics).
3. H. Jurgensen, H. J. Shyr and G. Thierrin, Codes and compatible partial orders on free monoids, *Proceedings of the First International Symposium on Ordered Algebraic Structures*, Marseille 1984, Research and Exposition in Mathematics, Vol. 14, Helolermann Verlag Berlin, 1986.
4. G. Lallement, *Semigroups and Combinatorial Applications*, John Wiley, New York, 1979.
5. M. Lothaire, Combinatorics on Words, *Encyclopedia of Mathematics and its Applications*, Addison-Wesley, Reading, 1983.
6. R. C. Lyndon and M. P. Schützenberger, *The equation* $a^M = b^N c^P$ *in a free group*, Mich. J., Vol. 9, 1962, pp. 289-298.
7. C. M. Reis and H. J. Shyr, *Some Properties of Disjunctive Languages on a Free Monoid*, Information and Control, Vol. 34, 1978, pp. 334-344.
8. H. J. Shyr, *Free Monoids and Languages*, Lecture Notes, Dept. Math., Soochow Univ., Taiwan, 1979.