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WORST CASE ANALYSIS OF TWO HEURISTICS FOR THE SET PARTITIONING PROBLEM (*)

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Abstract. – *We propose and analyse two simple heuristics for the problem of partioning a set that use few steps of enumeration.*
We show that the new heuristics have a significantly better worst case ratio than previously known heuristics.

Résumé. – *Nous proposons et analysons deux heuristiques simples au sujet du problème de la partition d'un ensemble qui utilisent peu d'énumérations.*
Nous démontrons que les nouveaux algorithmes ont des meilleures performances que les heuristiques connues jusqu'ici.

1. INTRODUCTION

We consider the *Partition* problem, that can be expressed as the well known problem of scheduling tasks on two identical processors to minimize the completion time of the last task completed. The problem can be formulated as follows:

Given a finite set $I=\{a_1, a_2, \ldots, a_n\}$ of positive integers (*items*), partition I into two subsets S, $T=I-S$, such that $\max(\sum_{a_j \in S} a_j, \sum_{a_j \in T} a_j)$ is minimum.

Given an instance $I=\{a_1, a_2, \ldots, a_n\}$ of the Partition problem, (S^*, T^*) will denote an optimum partition and $z^*(I)$ will denote its value,

$$z^*(I)=\max(\sum_{a_j \in S^*} a_j, \sum_{a_j \in T^*} a_j).$$

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Since it has been shown by Karp [K] that this problem is NP-hard, it is unlikely that we can find optimal solutions in polynomial time unless $P = NP$. Therefore several approximate polynomial time algorithms have been proposed and analysed both from a worst case point of view and from a probabilistic one.

Probably the first heuristic analysed is the LPT rule that can be described as follows. Initially the sets S and T are empty; then the algorithm assignes the largest unassigned item to the smallest set, breaking ties arbitrarily until all items have been considered. Graham [G] proved that, if $z^A(I)$ denotes the value of the solution obtained, applying the LPT rule, then $z^A(I)/z^*(I) \leqq 7/6$. He also proved that the bound is tight.

Johnson used partial enumeration to improve the worst case bound of the LPT rule. His algorithm uses a parameter k as follows:

1. find all possible partition of at most k items; choose the partition (S_1, T_1) for which

$$\left| \sum_{a_j \in S_1} a_j - \sum_{a_j \in T_1} a_j \right| \text{ is minimum;}$$

2. complete the partition (S_1, T_1) using the LPT rule.

Johnson [J] proved that

(a) if $z^{A, k}(I)$ is the value of the solution found when enumeration is used on the k largest items, then $z^{A, k}(I)/z^*(I) \leqq 1 + 1/k$;

(b) the running time of the algorithm is $O(n + 2^k)$.

Note that the worst case ratio can be arbitrarily close to 1, but the running time becomes exponential in the accuracy of the approximation obtained.

In this paper we want to investigate the influence of a very limited use of enumeration, that does not increase the running time of the algorithm.

In section 2 we study a modified algorithm A' obtained from the LPT rule, and we show that, if $z^{A'}(I)$ denotes the value of the solution found by algorithm A', then, for any instance I, $z^A(I)/z^*(I) \leqq 9/8$. We also show that the bound is tight.

Afterwards we consider the differencing method recently proposed by Karmakar and Karp [KK]. Fischetti and Martello [FM] analysed the algorithm from a worst case point of view. They proved that, if $z^B(I)$ denotes the value of the solution obtained applying the differencing method, then $z^B(I)/z^*(I) \leqq 7/6$ and that the bound is tight. In section 3 we introduce and analyse a simple modification of the differencing method. If $z^{B'}(I)$ denotes the value of the solution found by the modified differencing method, we show that $z^{B'}(I)/z^*(I) \leqq 10/9$, for any instance I. Again we show that 10/9 is a tight bound.

2. ANALYSIS OF THE MODIFIED ALGORITHM

First of all we give the algorithm obtained using the *LPT* rule.

Algorithm A

Input: a set of numbers $I=\{a_1, a_2, \ldots, a_n\}$ (ordered in non increasing order);
begin
  $S:=\emptyset$; $T:=\emptyset$;
  while there are items in I
  do begin
    choose largest $a_j \in I$;
    if $(\sum_{a_j \in S} a_j) < (\sum_{a_j \in T} a_j)$
      then $S:=S \cup \{a_j\}$
      else $T:=T \cup \{a_j\}$;
    $I:=I-\{a_j\}$
    end
  end
Output: $z^A(I)=\max(\sum_{a_j \in S} a_j, \sum_{a_j \in T} a_j)$.

THEOREM 1 [G]: *Given any instance $I=\{a_1, a_2, \ldots, a_n\}$ of the Partition problem then $z^A(I)/z^*(I) \leqq 7/6$. Furthermore, the bound is tight.*

Now we consider a simple modification of algorithm A.

Algorithm A'

input: a set of numbers $I=\{a_1, a_2, \ldots, a_n\}$ (ordered in non increasing order);
begin
/first phase/
if $n>6$ then let J be the set of the six largest items of I;
if $n<7$ then let J be equal to I;
apply algorithm A to J obtaining $z_1(J)$;
/second phase/
  $a_1:=a_1+a_2$;
  $I':=\{a^1\}$;
  for $i:=2$ **to** $n-1$
  do begin
    $a_i:=a_{i+1}$;
    $I':=I' \cup \{a_i\}$
    end;
if $n>6$ let J' be the set of the five largest items of I';
if $n<7$ let J' be equal to I';
apply algorithm A to J' obtaining $z_2(J')$;
if $z_1(J)<z_2(J')$
  then apply algorithm A to I obtaining $z^{A'}(I)$

else apply algorithm A to I' obtaining $z^{A'}(I)$
end
Output: $z^{A'}(I)$

THEOREM 2: *Given any instance I of the Partition problem then $z^{A'}(I)/z^*(I) \leqq 9/8$. Furthermore, the bound is tight.*

Proof: Let us consider the following instance I of the partition problem:

$$I = \{5, 4, 3, 2, 2\}.$$

It is easy to see that $z^*(I) = 8$ and that $z^{A'}(I) = 9$. Hence the ratio $z^{A'}(I)/z^*(I)$ cannot be better than 9/8.

In order to prove that $z^{A'}(I)/z^*(I) \leqq 9/8$ we will first obtain some information on the structure of the smallest counterexample.

LEMMA 1: *Let $I = \{a_1, a_2, \ldots, a_n\}$ be an istance of the Partition problem of minimum size that does not satisfy $z^{A'}(I)/z^*(I) \leqq 9/8$. Then $a_n > z^*(I)/4$.*

Proof: If the lemma does not hold, let us consider an instance $I = \{a_1, a_2, \ldots, a_n\}$ of minimum size, such that $z^A/z^*(I) > 9/8$. Without loss of generality suppose that the six largest items are $a_1, a_2, \ldots, a_6$. Since I is a minimum size counterexample then $z^{A'}(I) \leqq 1/2(\sum_{i<n} a_i) + a_n$. This implies that

$$\frac{9}{8} < \frac{z^{A'}(I)}{z^*(I)} \leqq \frac{1/2(\sum_{i<n} a_i) + a_n}{z^*(I)} \leqq \frac{1/2(\sum_{i \leq n} a_i) + 1/2\, a_n}{z^*(I)} \leqq 1 + \frac{a_n}{2\, z^*(I)}$$

hence

$$\frac{9}{8} < 1 + \frac{a_n}{2\, z^*(I)} \quad \text{and} \quad a_n > \frac{z^*(I)}{4}.$$

this completes the proof of the lemma.

Lemma 1 implies that it is sufficient to prove the theorem for instances of the problem having at most six items.

In fact, as soon as there are seven or more items in $I = \{a_1, a_2, \ldots, a_n\}$, then in the optimal partition (S^*, T^*) of I, either S^* or T^* must have four or more items not smaller than a_n. This implies that $z^*(I) \geqq 4\, a_n$. Hence if the theorem is true for $n \leqq 6$ then it is true for all n.

It is trivial to see that algorithm A finds an optimal solution if there are four or less items in I. Hence algorithm A' finds an optimal solution as well. The following facts complete the proof of the theorem.

Fact 1: Given any instance $I=\{a_1, a_2, \ldots, a_5\}$ of the Partition problem with five items then $z^{A'}(I)/z^*(I) \leqq 9/8$.

Proof: Given $I=\{a_1, a_2, \ldots, a_5\}$ we distinguish two cases:

Case 1: there is an optimal solution such that a_1 and a_2 are in the same subset.

Since algorithm A finds an optimal solution when there are four items, then in this case algorithm A' finds an optimal solution as well.

Case 2: there is not an optimal solution such that a_1 and a_2 are in the same subset.

We consider two subcases:

Subcase a: $a_1 \geqq a_2 + a_3$.

In this subcase it is easy to see that algorithm A' finds an optimal solution.

Subcase b: $a_1 < a_2 + a_3$.

By lemma 1 and by the optimality in the case with four or less items we can limit our attention to the case when $a_5 > 1/8\ W(I)$ (where $W(I) = \sum_{i \leqq n} a_i$).

This implies that at least one of the differences between adjacent elements must be small. In fact either $a_1 - a_2 \leqq W(I)/16$, or $a_2 - a_3 \leqq W(I)/16$, or $a_3 - a_4 \leqq W(I)/16$, otherwise we obtain the following contradiction

$$W(I) = \sum_{i \leqq n} a_i > (5/8)\, W(I) + [(1+2+3)/16]\, W(I) = W(I).$$

Now we observe that the two phases of algorithm A' place the first four items in two different ways:

$$(A) \left\{ \begin{array}{l} S=\{a_1, a_4\} \\ T=\{a_2, a_3\} \end{array} \right\} \text{(first phase)}$$

$$(B) \left\{ \begin{array}{l} S=\{a_1, a_2\} \\ T=\{a_3, a_4\} \end{array} \right\} \text{(second phase)}$$

The last item will be added to the smallest set. Now observe that if algorithm A' does not find an optimal solution then items $a_1 \ldots a_4$ are placed in the optimal solution as follows:

$$(C) \left\{ \begin{array}{l} S=\{a_1, a_3\} \\ T=\{a_2, a_4\}. \end{array} \right.$$

If $a_1 - a_2 \leqq W(I)/16$ or $a_3 - a_4 \leqq W(I)/16$ then by comparing (A) with (C) we have that the approximate solution obtained in the first phase satisfies the condition:

$$z^{A'}(I) \leqq z^*(I) + W(I)/16 \text{ hence } z^{A'}(I)/z^*(I) \leqq 9/8.$$

Analogously, if $a_2 - a_3 \leqq W(I)/16$ then by comparing (B) with (C) we have that the approximate solution obtained in the second phase satisfies the condition: $z^{A'}(I) \leqq z^*(I) + W(I)/16$, hence $z^{A'}(I)/z^*(I) \leqq 9/8$.

Fact 2: Given any instance $I = \{a_1, a_2, \ldots, a_6\}$ of the Partition problem with six items then $z^{A'}(I)/z^*(I) \leqq 9/8$.

Proof: If $a_6 \leqq W(I)/8$ then by lemma 1 and fact 1 the lemma is proved.

If $a_6 > W(I)/8$ then it is easy to see that

$$\max[(a_3 - a_4), (a_4 - a_5), (a_5 - a_6)] < W(I)/8$$

(otherwise $W(I) = \sum_{i \leqq n} a_i \geqq 3(a_4 + W(I)/8) + a_4 + a_5 + a_6 > W(I)$).

Now we distinguish two cases:

Case 1: $a_1 \leqq a_2 + W(I)/8$.

Since $a_1 < a_2 + a_3$ then algorithm A puts a_1 and a_4 (a_3) in the same subset. Note that $|(a_1 + a_4) - (a_2 + a_3)| \leqq W(I)/8$. Hence algorithm A will proceed putting a_5 and a_6 in different sets. As $a_5 - a_6 < W(I)/8$ then the solution found satisfies

$$z^A(I) \leqq 9/16\, W(I) \quad \text{and hence } z^A(I)/z^*(I) \leqq 9/8.$$

Case 2: $a_1 > a_2 + W(I)/8$.

In this case the optimal solution is either

$$S^* = (a_1, a_2), \qquad T^* = (a_3, a_4, a_5, a_6),$$

$$z^*(I) = (a_3 + a_4 + a_5 + a_6), \qquad (\text{since } a_6 > W(I)/8)$$

or

$$S^* = (a_1, a_5, a_6), \qquad T^* = (a_2, a_3, a_4), \qquad z^*(I) = (a_1 + a_5 + a_6).$$

In fact if there is only one element with a_1 then it must be a_2. In the other case since $a_1 > a_2 + 1/8\, W(I)$, with a_1 there are the two smallest remaining items.

In the first case the second phase of algorithm A' finds the optimal solution. In the second case algorithm A finds a solution no worse than $a_1 + a_4 + a_6$.

Hence

$$\frac{z^{A'}(I)}{z^*(I)} \leqq \frac{a_1+a_4+a_6}{a_1+a_5+a_6} = 1 + \frac{a_4-a_5}{a_1+a_5+a_6} \leqq \frac{9}{8}.$$

Q.E.D.

Note that lemma 1 implies that the same bound can be achieved even if only the six largest items of I are ordered.

Finally we observe that the performance of a modified LPT rule can be improved to 10/9 if we consider all the possible partitions of the three largest items. The proof of this theorem, analogous to the proof of theorem 2, is omitted.

3. ANALYSIS OF THE MODIFIED DIFFERENCING ALGORITHM

First of all we give the differencing algorithm proposed by Karp and Karmakar [KK].

Algorithm *B*

```
Input: a set of numbers I={a_1, a_2, ..., a_n} (ordered in non increasing order);
begin
  H: =I; k: =n;
  while |H| > 1
  do begin
    k: =k+1;
    pick the two largest items a_i, a_j in H;
    define pseudo-item a_k: =|a_i - a_j|;
    H: =H-{a_i, a_j} ∪ {a_k}; P(k): = (a_i, a_j)
    end
  ∂: =a_k; S: ={a_k}; T: =∅;
  while (S ∪ T contains pseudo-items)
    do begin
      let be any pseudo-item in S or T and suppose P(t)={a_i, a_j}, a_i ≧ a_j;
      if t ∈ S
        then begin
          S: =S-{a_t} ∪ {a_i};
          T: =T ∪ {a_j}
          end
        else begin
          S :=S ∪ {a_j};
          T: =T-{a_t} ∪ {a_i}
          end
        end
end
Output: z^B(I)=max( Σ_{a_j ∈ S} a_j, Σ_{a_j ∈ T} a_j).
```

THEOREM 3 [FM]: *Given any instance $I=\{a_1, a_2, \ldots, a_n\}$ of the Partition problem then $z^B(I)/z^*(I) \leqq 7/6$. Furthermore, the bound is tight.*

Now we consider the following modification of algorithm B.

Algorithm B'

Input: a set of numbers $I=\{a_1, a_2, \ldots, a_n\}$ (ordered in non increasing order);
begin
/first phase/
  apply algorithm B to I obtaining $z_1(I)=z^B(I)$;
/second phase/
  $a_1: =a_1+a_2$;
  $I': =\{a_1\}$;
  for $i: =2$ **to** $n-1$
  do begin
    $a_i: =a_{i+1}$;
    $I'=I \cup \{a_i\}$
    end
  apply algorithm B to I' obtaining $z_2(I)=z^B(I')$;
end
Output: $z^B(I)=\min(z_1(I), z_2(I))$

THEOREM 4: *Given any instance $I=\{a_1, a_2, \ldots, a_n\}$ of the Partition problem, then $z^B(I)/z^*(I) \leqq 10/9$. Furthermore, the bound is tight.*

Proof: In order to prove that algorithm B' cannot be better than 10/9 it is sufficent to consider the following instance: $I=\{5, 3, 3, 3, 2, 2,\}$.

It is easy to see that $z^*(I)=9$ and that $z^{B'}(I)=10$. Hence $z^{B'}(I)/z^*(I)$ cannot be better than 10/9.

The following lemma generalizes a lemma given by Fischetti and Martello.

LEMMA 2 [FM]: *If at some iteration i during the execution of algorithm B,*

$$a_j \leqq W(I)/9 \quad \textit{for all } j \in H, \quad \textit{then } z^{B'}(I)/z^*(I) \leqq 10/9.$$

LEMMA 3: *Let $I=\{a_1, a_2, \ldots, a_n\}$ be an instance of minimum size such that*

$$z^{B'}(I)/z^*(I) > 10/9, \quad \textit{then } a_n > W(I)/9.$$

Proof of lemma 3: If the lemma does not hold, let $I=\{a_1, a_2, \ldots, a_n\}$ be an instance such that $z^{B'}(I)/z^*(I) > 10/9$ with minimum number of items and $a_n < W(I)/9$.

We define a new istance $L=\{a_1, a_2, \ldots, a_{n-1}\}$ by eliminating the smallest item.

CLAIM: $z^{B'}(I)=z^{B'}(L)$.

Proof of the claim: Suppose that $z^{B'}(I)$ is obtained during the first phase of algorithm B'. Let us analyse the behaviour of the first phase over I and let i be the iteration at which only one item (or pseudo-item) a_j greater than $W(I)/9$ is remaining; let S_j be the set of items and pseudo-items less or equal than $W(I)/9$. Note that $a_n \in S_j$, and

$$(1) \qquad a_j > 1/9 + \sum_{a_i \in S_j} a_i;$$

otherwise by Lemma 2 we will find a solution $z^{B'}(I)/z^*(I) \leqq 10/9$.

Note that the first phase of algorithm B' behaves in the same way for the first i iterations over instances I and L. Hence, at iteration i there exists only one item (or pseudo-item) a_j greater than $W(I)/9$; let T_j be the set (possibly empty) of items and pseudo-items less than $W(I)/9$. By (1) we have:

$$(2) \qquad a_j > 1/9 + \sum_{a_i \in T_j} a_i;$$

this implies that the first phase of the algorithm gives a solution with the same value. In a similar way it is possible to prove that also the solution obtained during the second phase has the same value. Hence $z^{B'}(L)$ is equal to $z^{B'}(I)$.

Since $z^*(L) \leqq z^*(I)$, the claim implies

$$\frac{z^{B'}(L)}{z^*(L)} \geqq \frac{z^{B'}(I)}{z^*(I)} \geqq \frac{10}{9}.$$

This contradicts the hypothesis that I is a minimum size instance for which $z^{B'}(I)/z^*(I) > 10/9$ and completes the proof of the lemma.

Lemma 3 implies that it is sufficient to prove the theorem for instances of the problem with at most eight items. In fact, as soon as there are more than eight items in

$$I = \{a_1, a_2, \ldots, a_n\}, \quad \text{then } a_n \leqq 1/9\ W(I).$$

It is easy to see that if there are five or less items in I, then algorithm B' finds an optimal solution. The following facts consider the remaining cases.

Fact 3: Given any instance $I = \{a_1, a_2, \ldots, a_n\}$ of the partition problem with six items, then $z^{B'}(I)/z^*(I) \leqq 10/9$.

Proof: By lemma 3 and by the optimality for the case with five or less items, we can limit our attention to the case when $a_6 > W(I)/9$.

We distinguish two cases:

Case 1: $a_1 - a_2 \leqq W(I)/9$.

In this case we have that

$$a_3 - a_4 \leqq W(I)/9 \qquad \text{and} \qquad a_5 - a_6 \leqq W(I)/9$$

[otherwise $\sum_{i \leqq n} a_i > (3 \cdot 2/9 + 3 \cdot 1/9) W(I) = W(I)$].

Hence after the first three iterations of algorithm B we obtain only pseudo-items with size less or equal 1/9 $W(I)$. Applying lemma 2 we obtain the thesis.

Case 2: $a_1 - a_2 > W(I)/9$.

If there is only one item with a_1 in an optimal solution, then it must be a_2 and algorithm B' finds an optimal solution.

If $a_1 + a_2 \geqq 4/9\, W(I)$ then the second phase of algorithm B' finds a solution $z^{B'}(I) \leqq 5/9\ W(I)$. This implies $z^{B'}(I)/z^*(I) \leqq 10/9$.

Hence we are left with the case when, in the optimal partition, there are two other elements in the same set with a_1 and $a_1 + a_2 < 4/9\, W(I)$. In this case the optimal solution is $z^*(I) = a_1 + a_5 + a_6$. To prove the above claim it is sufficient to show that

$$a_1 + a_5 + a_6 > a_2 + a_3 + a_4.$$

In fact the condition $a_1 + a_2 < 4/9\, W(I)$ implies $a_3 < 3/18\, W(I)$; hence:

$$a_1 + a_5 + a_6 > a_2 + 3/9 \geqq a_2 + a_3 + a_4.$$

Since there are two elements with a_1, then this solution must be optimal. On the other side the algorithm finds a solution no worse than $z^{B'}(I) = a_1 + a_3 + a_5$. Hence

$$\frac{z^{B'}(I)}{z^*(I)} \leqq \frac{a_1 + a_3 + a_5}{a_1 + a_5 + a_6} \leqq 1 + \frac{a_3 - a_6}{a_1 + a_5 + a_6} \leqq 1 + \frac{1/18\, W(I)}{1/2\, W(I)} = \frac{10}{9}.$$

Fact 4: Given any instance of the partition problem $I = \{a_1, a_2, \ldots, a_7\}$ with seven items, then $z^{B'}(I)/z^*(I) \leqq 10/9$.

Proof: By lemma 3 and fact 3, if $a_7 \leqq W(I)/9$ then $z^{B'}(I)/z^*(I) \leqq 10/9$.

Hence we will limit our attention to the case $a_7 > W(I)/9$.

If $(a_1 - a_2) > W(I)/9$ it is easy to check that after having generated the first pseudo-item $a_8 = a_1 - a_2 > W(I)/9$ all other pseudo-items generated in the following iterations of algorithm B are less than $W(I)/9$. Applying lemma 3 we obtain the thesis.

Furthermore, if $(a_1-a_2)\leqq W(I)/9$ and $(a_1+a_2-a_3-a_4)>W(I)/9$, during the first two iterations of the second phase, algorithm B' gives pseudo-items $a_8=(a_1+a_2)-a_3$ and $a_9=(a_1+a_2)-a_3-a_4>W(I)/9$. It is easy to check that all subsequent pseudo-items are less than $W(I)/9$. Applying lemma 2 we obtain the thesis.

If $(a_1-a_2)\leqq W(I)/9$ and $(a_1+a_2-a_3-a_4)\leqq W(I)/9$ then the second phase of algorithm B' generates pseudo-items $a_8=(a_1+a_2)-a_3$, $a_9=a_8-a_4\leqq W(I)/9$, $a_{10}=a_5-a_6$ [note that $a_{10}<W(I)/9$]. Hence the solution found during the second phase of algorithm B' is equal to $a_3+a_4+a_6+a_7$.

If $a_7\leqq(a_1+a_2)-a_3-a_4+a_5-a_6+W(I)/9$ then this solution satisfies the thesis. Otherwise we distinguish two possibilities:

Case 1: $a_3-a_5\leqq 1/18\ W(I)$.

In this case $z(I)=a_4+a_5+a_6+a_7$.

To prove this claim it is sufficient to observe that $a_4+a_5+a_6+a_7\geqq W(I)/2$. In fact we have

$$a_4+a_5+a_6+a_7>a_4+a_5+a_6+(a_1+a_2)-a_3-a_4+a_5-a_6+W(I)/9$$
$$=a_1+a_2+2a_5-a_3+W(I)/9\geqq a_1+a_2+a_3.$$

This implies:

$$\frac{z^{B'}(I)}{z^*(I)}\leqq\frac{a_3+a_4+a_6+a_7}{a_4+a_5+a_6+a_7}\leqq 1+\frac{a_3-a_5}{1/2\,W(I)}\leqq\frac{10}{9}.$$

Case 2: $a_3-a_5>W(I)/18$.

In this case a_1, a_2 and a_3 are greater than $3/18\ W(I)$. Hence

$$a_1+a_2+a_3>W(I)/2,$$
$$a_4+a_5+a_6+a_7<W(I)/2,\qquad a_3+a_5+a_6+a_7>W(I)/2.$$

This implies that the optimal solution is

$$\min(a_1+a_2+a_3,\ a_3+a_5+a_6+a_7).$$

If

$$z^*(I)=a_1+a_2+a_3\quad\text{then}\quad a_4+a_6+a_7\leqq 7/18\ W(I)$$

[otherwise

$$\sum_{i \leqq n} a_i = (a_1 + a_2 + a_3) + a_5 + (a_4 + a_7 + a_8)$$

$$> 9/18\, W(I) + W(I)/9 + 7/18\, W(I) = W(I)].$$

This implies:

$$\frac{z^{B'}(I)}{z^*(I)} \leqq \frac{a_3 + a_4 + a_6 + a_7}{a_1 + a_2 + a_3} = 1 + \frac{a_4 + a_6 + a_7 - a_1 - a_2}{a_1 + a_2 + a_3}$$

$$\leqq 1 + \frac{7/18\, W(I) - 6/18\, W(I)}{1/2\, W(I)} \leqq \frac{10}{9}$$

If $z^*(I) = a_3 + a_5 + a_6 + a_7$ then $a_4 - a_5 \leqq W(I)/18$
[otherwise

$$\sum_{i \leqq n} a_i = (a_1 + a_2 + a_3) + a_4 + (a_5 + a_6 + a_7)$$

$$> 9/18\, W(I) + 3/18\, W(I) + 6/18\, W(I) = W(I)].$$

This implies:

$$\frac{z^{B'}(I)}{z^*(I)} \leqq \frac{a_3 + a_4 + a_6 + a_7}{a_3 + a_5 + a_6 + a_7} = 1 + \frac{a_4 - a_5}{a_3 + a_5 + a_6 + a_7} \leqq 1 + \frac{1/18\, W(I)}{1/2\, W(I)} \leqq \frac{10}{9}.$$

Fact 5: Given any instance $I = \{a_1, a_2, \ldots, a_8\}$ of the partition problem with eight items, then $z^{B'}(I)/z^*(I) \leqq 10/9$.

Proof: By lemma 3, and facts 5 and 6 if $a_8 \leqq W(I)/9$ then $z^{B'}(I)/z^*(I) \leqq 10/9$. If $a_8 > W(I)/9$ all the following inequalities hold:

$$(a_1 - a_2) \leqq W(I)/9; \qquad (a_3 - a_4) \leqq W(I)/9;$$
$$(a_5 - a_6) \leqq W(I)/9; \qquad (a_7 - a_8) \leqq W(I)/9;$$

otherwise $W(I) = \sum_{i \leqq n} a_i \geqq 1/9 + 8\, a_8 > W(I)$.

Hence all pseudo-items generated during the execution of algorithm B are less or equal to $W(I)/9$. Hence applying lemma 2 we obtain $z^B(I)/z^*(I) \leqq 10/9$. This trivially implies that $z^{B'}(I)/z^*(I) \leqq 10/9$.

This completes the proof of theorem 4.

Q.E.D.

Fischetti and Martello observed that the differencing method achieves the

7/6 bound even if only the six largest items are ordered. Analogously algorithm B' achieves the 10/9 bound even if only the eight largest items are ordered. The proof of this fact is a simple extension of the proof of theorem 4 and it is omitted.

4. CONCLUSION

In this paper we have shown how a limited amount of enumeration allows to improve considerably the worst case performance of approximate algorithms for the Partition problem. It would be interesting to answer the following questions:

(i) what is the worst case ratio of algorithms that use a large amount of enumeration?

(ii) is there any trade-off between enumeration used and approximation obtained better than the one provided by Johnson's algorithm [J]?

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