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## A SIMPLE PROOF OF VALIANT'S LEMMA (\*)

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*Abstract. – Valiant's algorithm for the recognition problem of contextfree languages uses the computation of matrix closures. The matrices in consideration are strictly upper triangular. The crucial point is that multiplication is nonassociative.*

*The main point is to prove a lemma concerning the computation of the transitive closure by dividing matrices into submatrices. We give a very simple proof of this lemma.*

*Résumé. – L'algorithme Valiant pour l'analyse de langages algébriques utilise le calcul de fermetures de matrices. Les matrices considérées sont nilpotentes. Le fait difficile est que la multiplication n'est pas associative.*

*Le point le plus important est la preuve d'un lemme concernant le calcul de la fermeture transitive en partitionnant les matrices en sous-matrices. Nous donnons une preuve très simple de ce lemme.*

### 1. INTRODUCTION

Valiant's algorithm [2], to solve the wordproblem for contextfree languages uses a procedure to determine the transitive closure of a strictly upper triangular matrix. The crucial point of his approach is to design this procedure even in the case, where the product operation is non-associative. His algorithm uses several propositions on dividing a matrix into certain submatrices to obtain the transitive closure by recursiveness. One of these propositions, which is in fact the essential part of the correctness-proof, seems to be very hard to prove.

The reader may consult Harrison [1], where an elaborated version is given. We shall show that a real simple-minded proof can be given.

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## 2. PRELIMINARIES

We consider an algebraic structure with two operations  $+$  and  $*$ . With respect to the addition  $(R, +)$  is a semilattice, this means, we assume the following axiomes:

(A1) (Associativity):

$$(x + y) + z = x + (y + z).$$

(A2) (Commutativity):

$$x + y = y + x.$$

(A3) (Idempotence):

$$x + x = x.$$

(A4) (Neutral element). There exists  $0 \in M$  with:

$$x + 0 = x.$$

As usual we introduce a partial ordering by:

(A5) (Absorption):

$$x \leq y \Leftrightarrow x + y = y.$$

With respect to the multiplication we assume:

(A6) (Distributivity):

$$x * (y + z) = x * y + x * z,$$

$$(x + y) * z = x * z + y * z.$$

and:

(A7) (Zero-element):

$$0 * x = x * 0 = 0.$$

By our axioms, multiplication and addition are monotonous operations:

(A8):

$$x \leq y \ \& \ u \leq v \Rightarrow x + u \leq y + v.$$

(A9):

$$x \leq y \ \& \ u \leq v \Rightarrow x * u \leq y * v.$$

By  $M_{n,n}(R)$  we denote the set of  $(n, n)$ -matrices  $A$  over  $R$ .

By transferring the operations  $+$ ,  $*$  in the usual way to matrices,  $M_{n,n}(R)$  again fulfills all our axioms. Especially, matrix product is defined by:

$$(A * B)[i, j] = \sum_{k=1}^n A[i, k] * B[k, j] \quad (1 \leq i, j \leq n).$$

A matrix  $A \in M_{n,n}(R)$  is strictly upper triangular ( $A \in M_{n,n}^{<}(R)$ ) if and only if:

$$A[i, j] = 0 \quad \text{if } j \leq i.$$

Especially, the null-matrix  $0$  containing only  $0$ -entries, is a strictly upper triangular matrix; hence  $M_{n,n}(R)$  again fulfills our axioms.

Since associativity is not valid in general, the definition of exponentiation has to be altered. We define inductively for  $A \in M_{n,n}(R)$ :

$$A^1 = A,$$

$$A^{i+1} = \sum_{k=1}^i A^k * A^{i-k+1} \quad (i \geq 2).$$

The transitive closure of  $A$  is then defined by:

$$A^* = \sum_{k=1}^{\infty} A^k.$$

To assert existence, we assume the necessary completeness axiom for  $R$ . Since it is not necessary for  $M_{n,n}^{<}(R)$  we omit the details. We summarize some facts on exponentiation.

PROPOSITION 1:

- (i)  $A \leq B \Rightarrow A^i \leq B^i \quad (i=1, 2, \dots)$ ;
- (ii)  $A \leq B \Rightarrow A^* \leq B^*$ ;
- (iii)  $(A^*)^* = A^*$ ;
- (iv)  $A \leq A^*$ ;
- (v)  $A \in M_{n,n}^{<}(R) \Rightarrow A^i \in M_{n,n}^{<}(R) \quad (i=1, 2, \dots)$ ;
- (vi)  $A \in M_{n,n}^{<}(R) \Rightarrow A^* \in M_{n,n}^{<}(R) \quad (i=1, 2, \dots)$ ;
- (vii)  $A \in M_{n,n}^{<}(R) \Rightarrow A^{n+i} = 0 \quad (i=1, 2, \dots)$ .

Given a matrix  $A$  we are interested in dividing  $A$  into submatrices. The most interesting division is into nine submatrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

where:

$$1, 5, 9 \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

are squarematrices. Special cases are:

- 5 is not present (which implies 2, 4, 6 are not present);
- 9 (or 1) is not present [which implies 3, 6, 7, 8 (or 2, 3, 4, 7)] are not present.

It is easy to check the following proposition.

PROPOSITION2 (Central submatrix lemma): If  $A \in M_{n,n}^<(R)$  and  $5^* = 5$  then:

$$A^* = \begin{bmatrix} 1' & 2' & 3' \\ 4 & 5 & 6' \\ 7 & 8 & 9' \end{bmatrix}$$

COROLLARY 1: If:

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \in M_{n,n}^<(R), \quad 1^* = 1, \quad 9^* = 9,$$

then:

$$A^* = \begin{bmatrix} 1 & 3' \\ 7 & 9 \end{bmatrix}.$$

COROLLARY 2: If:

$$A = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} \in M_{n,n}^<(R) \quad \text{and} \quad A' = \begin{bmatrix} 1^* & 3' \\ 7 & 9^* \end{bmatrix},$$

then:

$$A^* = (A')^*.$$

3. VALIANT'S LEMMA

As indicated in the introduction Valiant's Lemma is the crucial point of the algorithm.

*Valiant's Lemma*

If:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \in M_{n,n}^<(R),$$

with:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}^* = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}^* = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix},$$

then:

$$A^* = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

where:

$$\begin{bmatrix} 1 & 3+2*6 \\ 7 & 9 \end{bmatrix}^* = \begin{bmatrix} 1 & 3' \\ 7 & 9 \end{bmatrix}.$$

*Remark:* Note, that by the central submatrix lemma the assumption of Valiant's lemma immediately yields:

$$1^* = 1, \quad 5^* = 5, \quad 9^* = 9.$$

The key to prove Valiant's lemma is to deal with matrix equations. Consider first  $A^*$ . We calculate:

$$\begin{aligned} A^* * A^* + A &= \left( \sum_{k=1}^{\infty} A^k \right) * \left( \sum_{l=1}^{\infty} A^l \right) + A \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} A^k * A^l + A \quad (\text{Distributivity}) \end{aligned}$$

$$\sum_{r=2}^{\infty} \sum_{l+k=r} A^k * A^l + A = \sum_{r=2}^{\infty} \sum_{k=1}^{r-1} A^k * A^{r-k} + A = \sum_{r=2}^{\infty} A^r + A = A^*,$$

hence  $A^*$  is a solution of the matrix equation:

$$X = X * X + A.$$

We claim that  $A^*$  is the unique minimal solution of this equations. To show this, we prove that, if  $X$  is a solution then

$$A^r \leq X \quad \text{for all } r=1, 2, \dots$$

and therefore:

$$A^* \leq X \quad (\text{Monotonicity of } +).$$

We proceed by induction. If  $r=1$  we get:

$$A + X = A + A + X * X = A + X * X = X,$$

hence  $A \leq X$ .

Let  $r > 1$ . We assume  $A^i \leq X$  for all  $1 \leq i < r$ . By the monotonicity of  $*$ , we get:

$$A^i * A^{r-i} \leq X * X.$$

Therefore:

$$A^r = \sum_{i=1}^{r-1} A^i * A^{r-i} \leq \sum_{i=1}^{r-1} X * X = X * X \leq X * X + A = X.$$

By this we have proven:

**PROPOSITION3:**  $A^*$  is the unique minimal solution of the equation:

$$X = X * X + A.$$

Now, we deal with the following situation. Consider a linear matrix equation of the form:

$$X = A_1 * X + X * A_2 + A_3,$$

where:

$$A_1^* = A_1, A_2^* = A_2, A_1, A_2$$

are strictly upper triangular.

To solve this equation we consider the transitive closure of:

$$\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}^* = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}$$

(by the central submatrix-lemma). Applying proposition 3 we get:

$$\begin{aligned} \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} &= \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix}^* \times \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 * A_1 + A_1 & A_1 * B + B * A_2 + A_3 \\ 0 & A_2 * A_2 + A_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 & A_1 * B + B * A_2 + A_3 \\ 0 & A_2 \end{bmatrix} \text{ again by Proposition 3.} \end{aligned}$$

Hence  $B$  is a solution of the linear matrix equation. Let  $X$  be an arbitrary solution, then we can build:

$$\begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix},$$

and show by the same calculation:

$$\begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix}^* \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}.$$

Thus:

$$\begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}^* \leq \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix}.$$

This yields  $B \leq X$ .

**PROPOSITION 4:** *If  $B$  is determined by:*

$$\begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}^* = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$

*then  $B$  is the unique minimal solution of:*

$$X = A_1 * X + X * A_2 + A_3,$$

*provided  $A_1, A_2$  are strictly upper triangular and  $A_1^* = A_1$  and  $A_2^* = A_2$ .*

Now, consider Valiant's lemma. Let:

$$A = \begin{bmatrix} \bar{1} & 2 & \bar{3} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} \bar{1} & 2 & \bar{X} \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$



Applying Proposition 3 we calculate:

$$A^* = \begin{bmatrix} 1 & 2 & 2 * 6 + 3 + 1 * X + X * 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Again we can use Proposition 3 to show that  $X$  is the unique minimal solution of:

$$1 * X + X * 9 + 2 * 6 + 3 = X.$$

Since 1, 9 are strictly upper triangular and  $1^* = 1$ ,  $9^* = 9$  we can apply Proposition 4. By this we get immediately Valiant's lemma.

#### REFERENCES

1. M. A. HARRISON, *Introduction to Formal Languages Theory*, Addison-Wesley Pub. Co., Reading, Mass. 1978.
2. L. VALIANT, *General Context-free Recognition In Less Than Cubic Time*, J. Comp. Syst. Sc., Vol. 10, 1975, pp. 308-315.