ROBERT KNAST

A semigroup characterization of dot-depth one languages


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A SEMIGROUP CHARACTERIZATION OF DOT-DEPTH ONE LANGUAGES (*)

by Robert KNAST (1)

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Abstract. — It is shown that one can decide whether a language has dot-depth one in the dot-depth hierarchy introduced by Brzozowski. The decision procedure is based on an algebraic characterization of the syntactic semigroup of a language of dot-depth 0 or 1.

Résumé. — On démontre que l'on peut décider si un langage est de hauteur 1 dans la hiérarchie de concaténation introduite par Brzozowski. L'algorithme de décision est basé sur une condition algébrique qui caractérise les semigroupes syntactiques des langages de hauteur inférieure ou égale à 1.

1. INTRODUCTION

Let \( A \) be a non-empty finite set, called alphabet. \( A^+ \) (respectively \( A^* \)) is the free semigroup (respectively free monoid) generated by \( A \). Elements of \( A^* \) are called words. The empty word in \( A^* \) is denoted by \( \lambda \) (the identity of \( A^* \)). The concatenation of two words \( x, y \) is denoted by \( xy \). The length of a word \( x \) is denoted by \( |x| \).

Any subset of \( A^* \) is called a language. If \( L_1 \) and \( L_2 \) are languages, then \( L_1 \cup L_2 \) is their union, \( L_1 \cap L_2 \) is their intersection, and \( \overline{L}_1 = A^* - L_1 \) is the complement of \( L_1 \) with respect to \( A^* \). Also \( L_1 L_2 = \{ w \in A^* \mid w = xy, \ x \in L_1, \ y \in L_2 \} \) is the concatenation of \( L_1 \) and \( L_2 \).

Let \( \sim \) be an equivalence relation on \( A^* \). For \( x \in A^* \) we denote by \( [x]_\sim \) the equivalence class of \( \sim \) containing \( x \). An equivalence relation \( \sim \) on \( A^* \) is a congruence iff for all \( x, y \in A^* \), \( x \sim y \) implies \( uxv \sim uyv \) for any \( u, v \in A^* \).

The syntactic congruence of a language \( L \) is defined as follows: for \( x, y \in A^* \), \( x \equiv_L y \) iff for all \( u, v \in A^* \) \( (uxv \in L \iff uyv \in L) \). The syntactic semigroup of \( L \) is the quotient semigroup \( A^+/\equiv_L \).

Let \( \eta \) be any family of languages. Then \( \eta M (\eta B) \) will denote the smallest family of languages containing \( \eta \) and closed under concatenation (finite union and complementation respectively).

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(1) Institute of Mathematics, Polish Academy of Sciences, 61-725 Poznan, Poland.

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Let ε = \{ \{ λ \}, \{ a \}; a ∈ A \} be the family of elementary languages. Then define:

\[ \mathcal{B}_0 = \varepsilon B, \]
\[ \mathcal{B}_k = \mathcal{B}_{k-1} MB \quad \text{for} \quad k \geq 1. \]

This sequence (\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_k, \ldots) is called the dot-depth hierarchy. A language \( L \) is of dot-depth at most \( k \) if \( L \in \mathcal{B}_k \).

The dot-depth hierarchy was introduced in [3]. It was proved in [2] that it is infinite if the alphabet has two or more letters. In [4] it was shown that (\#0, \#, \ldots) forms a hierarchy of \(+−\) varieties of languages. Therefore, in the rest of the paper we consider languages as subsets of \( A^+ \). For an excellent and general presentation of problems related to this paper the reader is referred to Brzozowski’s survey paper [1] or the above mentioned monograph of Eilenberg [4].

In [6] Simon conjectured that a language \( L \) is in \( \mathcal{B}_1 \) iff its syntactic semigroup \( S_L \) is finite and there exists an integer \( n > 0 \) such that for each idempotent \( e \) in \( S_L \), and any elements \( a, b \in S_L \):

\[ (eaeb)^n eae = (eaeb)^n e = ebe(ebe)^n. \]

Simon also proved that \( L \in \mathcal{B}_1 \) implies this condition. By an example we show that this conjecture fails. We present a necessary and sufficient condition for a syntactic semigroup to be the syntactic semigroup of a language of dot-depth at most one. The main result is as follows: Let \( L \) be a language and let \( S_L \) be its syntactic semigroup. Then \( L \in \mathcal{B}_1 \) iff \( S_L \) is finite and there exists an integer \( n > 0 \) such that for all idempotents \( e_1, e_2 \) in \( S_L \) and any elements \( a, b, c, d \in S_L \):

\[ (e_1 ae_2 b)^n e_1 ae_2 de_1 (ce_2 de_1)^n = (e_1 ae_2 b)^n e_1 (ce_2 de_1)^n. \]

We will refer to this as the “dot-depth one” condition. This semigroup characterization gives a decision procedure for testing whether or not a regular language is in \( \mathcal{B}_1 \).

In the proof of this characterization we use a theorem on graphs from [5].

We will say that a language \( L \subseteq A^+ \) is a \( \sim \) language, if \( L \) is a union of congruence classes of \( \sim \). Let \( L \) be a language and let \( S_L \) be its syntactic semigroup. The class \( [x] \equiv_L \), as an element of \( S_L \), will be also denoted by \( x \), where \( x \in A^+ \). Then \( x \equiv_L y \) iff \( x = y \) in \( S_L \).

2. BASIC CONGRUENCE_\_m \~_k \_6

Let \( k, m \) be integers, \( k \geq 1, m \geq 0 \). Let \( v = (w_1, w_2, \ldots, w_m) \) be an \( m \)-tuple of words \( w_i \) of length \( k \), i.e. \( |w_i| = k, w_i \in A^* i = 1, 2, \ldots, m \). We say that \( v \) occurs in
Let us set:

$$\tau_{m,k}(x) = \{ y \mid y \in (A^*)^m \text{ and } y \subseteq x \}.$$ 

By convention $\tau_{0,k}x = \emptyset$.

For $x \in A^*$ and $n \geq 0$ define $f_n(x)$ as follows: if $|x| \leq n$, then $f_n(x) = x$; otherwise $f_n(x)$ is the prefix of $x$ of length $n$. Similarly, $t_n(x) = x$ if $|x| \leq n$, and $t_n(x)$ is the suffix of length $n$ of $x$ otherwise.

Now, for $x, y \in A^*$ and $k \geq 0$, $m \geq 0$ we define:

$$x \sim_k y \iff x = y \text{ if } |x| \leq m + k - 1$$

or $f_k(x) = f_k(y)$, $t_k(x) = t_k(y)$

and $\tau_{m,k+1}(x) = \tau_{m,k+1}(y)$ otherwise.

In the case $k = 0$ we write $\tau_m$ instead of $\tau_{m,0}$ and $\sim_m$ instead of $\sim_{m,0}$. If $m = 1$, we also write $\tau$ instead of $\tau_1$.

**Proposition 1:** (a) $x \sim_k y$ is a congruence of finite index on $A^*$; (b) $x \sim_k y$ implies $x \sim_{m+1} y$, for $m \geq 1$ and all $x, y \in A^*$; (c) $w(xw)^m \sim_k w(xw)^{m+1}$, for $w, x \in A^*$ and $|w| = k$; (d) $(w_1 x w_2 y)^m \sim_k (w_1 x w_2 y)^m$, for $w_1, w_2, x, y, u, v \in A^*$ and $|w_1| = |w_2| = k$.

**Proof:** The verification of (a), (b) and (c) is straightforward.

(0) By (b):

$$\tau_{m,k+1}(x) = \tau_{m,k+1}(y)$$

implies:

$$\tau_{j,k+1}(x) = \tau_{j,k+1}(y),$$

for all $x, y \in A^*$ and $j \in \{0, 1, \ldots, m\}$. If

$$v_1 = (w_1, \ldots, w_i) \in (A^{k+1})^i$$

and

$$v_2 = (v_1, \ldots, v_j) \in (A^{k+1})^j,$$

we denote by $(v_1, v_2)$ the $i+j$-tuple $(w_1, \ldots, w_i, v_1, \ldots, v_j) \in (A^{k+1})^{i+j}$.

Evidently:

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Using (c), we have:
\[ \tau_{m, k+1}((w_1 \times w_2 y)^m w_1 x w_2) \subseteq \tau_{m, k+1}((w_1 \times w_2 y)^{m+1} w_1). \]

Similarly:
\[ \tau_{m, k+1}(w_2 v w_1 (u w_2 v w_1)^m) = \tau_{m, k+1}(w_1 (u w_2 v w_1)^m). \]

Since \( |w_1| = |w_2| = k \), by the above conclusions from (b) and (c):
\[
\tau_{m, k+1}((w_1 \times w_2 y)^m w_1 x w_2 v w_1 (u w_2 v w_1)^m) = \bigcup_{i+j=m \geq 1, j \geq 0} \{ (v_1, v_2) \mid v_1 \in \tau_{i, k+1}((w_1 \times w_2 y)^m w_1), v_2 \in \tau_{j, k+1}(w_1 (u w_2 v w_1)^m) \}
\]
\[
= \bigcup_{i+j=m \geq 1, j \geq 0} \{ (v_1, v_2) \mid v_1 \in \tau_{i, k+1}((w_1 \times w_2 y)^m w_1), v_2 \in \tau_{j, k+1}(w_1 (u w_2 v w_1)^m) \}
\]
\[
= \tau_{m, k+1}((w_1 \times w_2 y)^m w_1 (u w_2 v w_1)^m). \]

**Theorem 2** (Simon [6]): A language \( L \) is of dot-depth at most one, \( L \in \mathcal{B}_1 \), iff \( L \) is a \( m \sim_k \) language for some \( m, k \geq 0 \).

### 3. Graphs and the Induced Syntactic Graph Congruence

First we briefly recall Eilenberg’s terminology for graphs [4]. A directed graph \( G \) consists of two sets, an alphabet \( A \) and a set of vertices \( V \), along with two functions: \( \alpha, \omega : A \to V \). Elements of \( A \) are also called edges in this case.

Two letters (or edges) \( a, b \in A \) are called consecutive if \( a \alpha = b \alpha \). Let \( D \subseteq A^2 \) be the set of all words \( ab \) such that \( a \) and \( b \) are non-consecutive. Then the set of all paths of \( G \) is:
\[
P = A^+ - A^* DA^*.
\]

Functions \( \alpha, \omega \) can be extended to \( \alpha, \omega : P \to V \) in the following way: if \( p = a_1 a_2 \ldots a_n \in P, a_1, a_2, \ldots, a_n \in A \), then \( p \alpha = a_1 \alpha, p \omega = a_n \omega \). For each vertex \( v \) we adjoin to \( P \) a trivial path \( 1_v \) where \( 1_v \alpha = 1_v \omega = v \). If \( p = a_1 a_2 \ldots a_n \in P \), then the length of \( p, |p| = n \).

A path \( p \) is called a loop if \( p \alpha = p \omega \). We say that two paths \( p_1 \) and \( p_2 \) are consecutive if \( p_1 \omega = p_2 \alpha \). In this case the concatenation \( p_1 p_2 \) is again a path. Two paths \( p_1 \) and \( p_2 \) are coterminal if \( p_1 \alpha = p_2 \alpha \) and \( p_1 \omega = p_2 \omega \).
An equivalence relation ~ on $P$ is called a graph congruence if it satisfies the following conditions:

(i) if $p_1 \sim p_2$, then $p_1$ and $p_2$ are coterminals;

(ii) if $p_1 \sim p_2$ and $p_3 \sim p_4$ and $p_1, p_3$ are consecutive, then $p_1 p_3 \sim p_2 p_4$.

For trivial paths, by convention we set $\tau_m(1_v) = \emptyset$. Thus the relation $m \sim (m \sim 1)$ is also defined on $P$. In [5] the following theorem is proved:

**Theorem 3:** Let $\sim$ be a graph congruence of finite index on $P$ satisfying the condition:

(A) $$(p_1 p_2)^n (p_3 p_4)^n \sim (p_1 p_2)^n (p_3 p_4)^n,$$

for some $n \geq 1$ and $p_1, p_2, p_3, p_4 \in P$. (Note that $p_1, p_2$ and $p_3 p_4$ must be loops about the same vertex).

Then there exists an integer $m \geq 1$ such that for any two coterminal paths $x$ and $y$, $x m \sim y$ implies $x \sim y$.

We will use this theorem in proving the semigroup characterization of languages of dot-depth at most one ($\mathcal{B}_1$).

Let $A$ be a finite alphabet. Define a graph $G_k = (V, E, \alpha, \omega)$ for $k \geq 0$ as follows:

$V = \{ w \mid w \in A^* \text{ and } |w| = k \}$ is the set of vertices,

$E = \{ (w_1, \sigma, w_2) \mid \sigma \in A, w_1, w_2 \in V \text{ and } t_k(w_1 \sigma) = w_2 \}$,

is the set of edges (letters)

$\alpha, \omega : E \to V, (w_1, \sigma, w_2) \alpha = w_1, (w_1, \sigma, w_2) \omega = w_2$.

Let $P$ be the set of all paths in $G_k$, including the empty path over each vertex from $V$. Now, let us define the mapping:

$$A^k A^* \to P,$$

recursively as follows:

$$x = 1_x \text{ if } x \in A^k,$$

$$x \sigma = \overline{x}(t_k(x), \sigma, t_k(x \sigma)).$$

For $k = 0$, by convention $A^0 = \{ \lambda \}$. One can verify that the mapping $\overline{\cdot}$ is bijective. It follows from the definition that $|x| = k + h, h \geq 0$ iff $|\overline{x}| = h$.

If $\rho$ is a congruence relation on $A^*$, then by $\overline{\rho}$ we will denote the induced congruence on $P$ defined in the following way: for $x, y \in P, x, y \in A^k A^*$, $x \rho y$ if $x, y$ are coterminal paths and $x \rho y$. One can verify that $\overline{\rho}$ is a graph congruence on $P$. vol. 17, n° 4, 1983
PROPOSITION 4: Let \( G_k \) be a graph for \( k \geq 1 \) and \( P \) be the set of all paths of \( G_k \). Let \( x \in A^k A^* \). If \( x = x_1 x_2 \), then \( \bar{x} = \bar{x}_1 t_k(x_1) x_2 \), for \( |x_1| \geq k \).

Proof: If \( |x| = k \), then the only decomposition possible is \( x = x \lambda \). But \( \bar{x} = 1_x = 1_x 1_x = \bar{x} x \lambda = \bar{x} t_k(x) \lambda \). Induction assumption: the proposition is true for \( x \) such that \( |x| = k + h \), \( h \geq 0 \). Suppose \( x = x_1 x_2 \sigma \), where \( |x_1 x_2| = k + h \) and \( |x_1| \geq k \). By definition:

\[
\bar{x} = x_1 x_2 (t_k(x_1 x_2), \sigma, t_k(x_1 x_2 \sigma)).
\]

By the induction assumption:

\[
x_1 x_2 = x_1 t_k(x_1) x_2.
\]

Hence:

\[
\bar{x} = x_1 t_k(x_1) x_2 (t_k(x_1 x_2), \sigma, t_k(x_1 x_2 \sigma)).
\]

Again by definition:

\[
t_k(x_1) x_2 \sigma = t_k(x_1) x_2 (t_k(x_1 x_2), \sigma, t_k(x_1 x_2 \sigma)).
\]

Thus \( \bar{x} = x_1 t_k(x_1) x_2 \sigma \) because \( t_k(x_1 x_2) = t_k(t_k(x_1) x_2) \). Thus the induction step holds. \( \square \)

LEMMA 5: Let \( x \in A^k A^* \) and \( \bar{x} = a_1 a_2 \ldots a_n \), \( a_j \in E \), \( j = 1, 2, \ldots, n \). Then for \( i \in \{ 1, 2, \ldots, n \} \) \( a_i = (w, \sigma, t_k(w \sigma)) \) iff \( x = x_1 w \sigma x_2 \) for some \( x_1, x_2 \in A^* \) and \( |x_1 w \sigma| = k + i \).

Proof: Suppose \( f_{k+1}(x) = x_1 w \sigma \). By Proposition 3 \( \bar{x} = x_1 w \sigma x_2 \). By the definition of it follows from Proposition 3 that \( \bar{w} \sigma x_2 = (w, \sigma, t_k(w \sigma)) t_k(w \sigma) x_2 \). Also by the definition of \( |x_1 w \sigma| = i - 1 \), because \( |x_1 w| = k + i - 1 \). Hence \( a_i = (w, \sigma, t_k(w \sigma)) \).

The converse follows in the similar way. \( \square \)

PROPOSITION 6: For any \( x, y \in A^k A^* \):

\[
x_m \sim_k y \implies \bar{x}_m \sim \bar{y},
\]

where \( \bar{x}, \bar{y} \in P \) of \( G_k \).

Proof: If \( |x| \leq m + k \), then \( x = y \) and consequently, \( \bar{x}_m \sim \bar{y} \). Otherwise, let \( \tau_{m, k+1}(x) = \tau_{m, k+1}(y) \neq \emptyset \). It follows from Lemma 5 that \( (\bar{w}_1, \bar{v}_1), \ldots, (\bar{w}_m, \bar{v}_m) \in \tau_m(\bar{x}) \) implies \( (w_1, \sigma_1, \ldots, w_m, \sigma_m) \in \tau_{m, k+1}(x) = \tau_{m, k+1}(y) \). Hen-
ce, again by Lemma 4 \(((w, \sigma, v), \ldots, (w_m, \sigma_m, v_m)) \in \tau_m(y)\). Thus, 
\(\tau_m(x) \subseteq \tau_m(y)\). By symmetry, \(\tau_m(y) \subseteq \tau_m(x)\).

Since \(f_k(x) = f_k(y)\) and \(t_k(x) = t_k(y)\), then \(x\) and \(y\) are coterminial.

Consequently, \(x \sim y\). \(\Box\)

**Proposition 7:** Let \(L \subseteq A^+\) and let \(S_L\) be the finite syntactic semigroup of \(L\), satisfying the condition: there exists \(m, m > 0\), such that for all idempotents \(e_1, e_2\) in \(S_L\) and any elements \(a, b, c, d \in S_L\):

\[(e_1 ae_2 b)^m e_1 ae_2 de_1 (ce_2 de_1)^m = (e_1 ae_2 b)^m e_1 (ce_2 de_1)^m.\]

Then the congruence \(\equiv_L\) on \(P\) of \(G_K\) for \(k = \text{card } S_L + 1\), induced by the syntactic congruence \(\equiv_L\) satisfies condition (A) of Theorem 2 and is of finite index on \(P\).

**Proof:** Since \(G_k\) is finite and \(\equiv_L\) is of finite index on \(A^+\), then \(\equiv_L\) is of finite index on \(P\).

We have to show that there is an integer \(n, n > 0\) such that:

(A) \[(p_1 p_2)^n p_1 p_4 (p_3 p_4)^n \equiv_L (p_1 p_2)^n (p_3 p_4)^n,\]

for \(p_1, p_2, p_3, p_4 \in P\).

Since \(p_1 p_2\) and \(p_3 p_4\) are loops about the same vertex and since paths \(p_1\) and \(p_4\) are consecutive by (A), then \(p_1 \alpha = p_2 \omega = p_3 \alpha = p_4 \omega = w\), and \(p_1 \omega = p_2 \alpha = p_3 \omega = p_4 \alpha = v\) for some \(w, v \in A^K\). Therefore we may assume that \(p_1 = wu_1, p_2 = vu_2, p_3 = wu_3, p_4 = vu_4\) for some \(u_1, u_2, u_3, u_4 \in A^*\) such that \(t_k(wu_1) = t_k(wu_3) = v, t_k(vu_2) = t_k(vu_4) = w\). Consequently:

\[(p_1 p_2)^n p_1 p_4 (p_3 p_4)^n = w(u_1 u_2)^n u_1 u_4 (u_3 u_4)^n.\]

Similarly:

\[(p_1 p_2)^n (p_3 p_4)^n = w(u_1 u_2)^n (u_3 u_4)^n.\]

By the definition of \(\equiv_L\) it is sufficient to show that there exists \(n, n > 0\), such that:

\[w(u_1 u_2)^n u_1 u_4 (u_3 u_4)^n \equiv_L w(u_1 u_2)^n (u_3 u_4)^n,\]

i.e.:

(1) \[w(u_1 u_2)^n u_1 u_4 (u_3 u_4)^n = w(u_1 u_2)^n (u_3 u_4)^n.\]

Let \(s \in S_L\). Since \(S_L\) is finite, then \(s^r\) is an idempotent for some \(r \geq 1\) ([4], vol. 17, n° 4, 1983)
Proposition 4.2, p. 68). Now, since $S_L$ satisfies the dot-depth one condition, there is $m \geq 1$ such that:

$$s^*(ss^*)^m = s^*(ss^*)^{m+1}$$

i.e. $s^* s^m = s^* s^m s$. It follows that there exists an integer $q$ such that for any $s \in S_L$

$$s^q = s^{q+1}$$

i.e. $S_L$ is aperiodic.

We claim that (1) holds for $n > m, q$. First we will show that if $|u_1 u_2| > 0 (|u_3 u_4| > 0)$ then we may consider $u_1, u_2 (u_3, u_4$ respectively) such that $|u_1|, |u_2| \geq k (|u_3|, |u_4| > k$ respectively). Since $n > q$, then by the aperiodicity of $S_L$:

$$w(u_1 u_2)^n = w(u_1 u_2)^n(2k+1).$$

Let us define:

$$\tilde{u}_1 = (u_1 u_2)^k u_1, \tilde{u}_2 = (u_1 u_2)^k.$$ 

Evidently:

$$|\tilde{u}_1|, |\tilde{u}_2| \geq k, \quad t_k(w \tilde{u}_1) = v, \quad t_k(v \tilde{u}_2) = w$$

and:

$$w(u_1 u_2) = w(\tilde{u}_1 \tilde{u}_2)^n.$$

Similarly, we may proceed for $u_3$ and $u_4$.

Now, we consider the full case if $|u_1 u_2|, |u_3 u_4| > 0$. The other cases if $|u_1 u_2| = 0$ or $|u_3 u_4| = 0$ follow in the same way. By the above, instead of proving (1) it is sufficient to show that:

(2) $$w(u_1 v u_2 w)^n u_1 v u_4 w (u_3 v u_4 w)^n = w(u_1 v u_2 w)^n(u_3 v u_4 w)^n,$$

holds.

Now, since $|w| = |v| = k > \text{card } S_L + 1$, then $w = w_1 w_2 w_3$ and $v = v_1 v_2 v_3$ for $w_1, w_3, v_1, v_3 \in A, w_2, v_2 \in A^+$ such that $w_1 = w_1 w_2^i, v_1 = v_1 v_2^i$ for any $i \geq 0$. So as before, we can choose $i$ such that $w_2^i$ and $v_2^i$ are idempotents in $S_L$. Thus (2) can be rewritten in a form:

$$w_1 e_1 (ae_1 be_1)^n ae_2 de_1 (ce_2 de_1)^n w_3 = w_1 e_1 (ae_2 be_1)^n(ca_2 de_1)^n w_3,$$

where:

$$e_1 = w_2^i, \quad e_2 = v_2^i, \quad a = w_3, u_3, v_3,$$

$$b = u_3 u_2, \quad c = u_3 u_3,$$

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and \( d = v_3 u_4 w_1 \). Thus by the dot-depth one condition, (2) holds. \( \square \)

4. SEMIGROUP CHARACTERIZATION OF \( \mathcal{B}_1 \)

Now we are in a position to prove our main result.

**Theorem 8:** Let \( L \) be a language, \( L \subseteq A^+ \) and let \( S_L \) be its syntactic semigroup. Then the following are equivalent:

(i) \( L \in \mathcal{B}_1 \);

(ii) \( L \) is a \( m \sim_k \) language for some \( m, k \geq 1 \);

(iii) \( S_L \) is finite and there is an integer \( n > 0 \) such that for all idempotents \( e_1, e_2 \) in \( S_L \) and any elements \( a, b, c, d \) in \( S_L \):

\[
(e_1 ae_2 b)^n e_1 ae_2 de_1 (ce_2 de_1)^n = (e_1 ae_2 b)^n e_1 (ce_2 de_1)^n.
\]

**Proof:**

(i) \( \Leftrightarrow \) (ii) by Theorem 2;

(ii) \( \Rightarrow \) (iii) : by (a) of Proposition 1 \( S_L \) is finite.

Now, let \( e_1 = z_1, e_2 = z_2, a = x, b = y, c = u, d = v \) for some \( z_1, z_2, x, y, u, v \in A^+ \).

Define \( w_1 = z_1^h, w_2 = z_2^h \) for \( h \) such that \( |w_1|, |w_2| \geq k \). Consequently, \( e_1 = w_1, e_2 = w_2 \). By (d) of Proposition 1 for \( m \sim_k \):

\[
(\overline{w_1 xw_2 y})^m \overline{w_1 xw_2 y} = (\overline{w_1 xw_2 y})^m \overline{w_1 (uw_2 v w_1)}^m.
\]

Thus \( S_L \) satisfies the dot-depth one condition with \( n = m \).

(iii) \( \Rightarrow \) (ii): suppose \( S_L \) satisfies the dot-depth one condition with \( n \). Let \( k = \text{card} S + 1 \). By Proposition 7 the induced syntactic congruence \( \equiv_L \) on \( P \) of \( G_k \), satisfies the condition (A) of the theorem on graphs with some \( n_1 > n, q, \) and is of finite index on \( P \). Hence by Theorem 3 there exists \( m \) such that for any two coterminal paths \( x, y \):

\[
\overline{x}_m \sim_k \overline{y} \text{ implies } \overline{x} \equiv_L \overline{y}.
\]

Now, consider \( x, y \in A^k A^* \), and the congruence \( m \sim_k \). We have that \( x_m \sim_k y \) implies \( \overline{x} \sim_k \overline{y} \) and that \( \overline{x}, \overline{y} \) are coterminal. Hence, \( x_m \sim_k y \) implies \( \overline{x} \equiv_L \overline{y} \) and consequently, \( x \equiv_L y \). If \( |x| \leq k \), then \( x_m \sim_k y \) implies \( x = y \) and consequently, \( x \equiv_L y \). Thus \( L \) is a \( m \sim_k \) language. \( \square \)

It is easy to see that if a syntactic semigroup satisfies the dot-depth one condition, then it also satisfies the condition: there exists an integer \( n > 0 \) such that for any idempotent \( e \) in \( S_L \) and any elements \( a, b \) in \( S_L \):

\[
(eaeb)^n eae = (eaeb)^n e = ebe(ebe)^n.
\]
The following example shows that the converse is not true.

Let \( A = \{0, 1, 2, 3\} \) and let \( L = (01^+ \cup 02^+)^*01^+3(2^+ 3 \cup 1^+ 3)^* \). The syntactic semigroups \( S_L \) of \( L \) satisfies the above condition, but it fails the dot-depth one condition. By Theorem 8 \( L \notin \mathcal{B}_1 \). On the other hand one can verify that \( L \notin \mathcal{B}_1 \), apart from Theorem 8, using \((d)\) of Proposition 1 and proving that for any \( m, k \), \( L \) cannot be a \( m \sim_k \) language.

REFERENCES