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TOPOLOGIES ON FREE MONOIDS INDUCED BY FAMILIES OF LANGUAGES (*)

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Abstract. — For $\mathcal{L} \subseteq \mathcal{P}(\Sigma^*)$ the language operator $\text{Anf}_{\mathcal{L}}(A)$ is defined by $\{z \mid z \setminus A \in \mathcal{L}\}$. It was characterized what families \mathcal{L} correspond to closure operators. In this paper the families \mathcal{L} are found out corresponding to interior operators: they are filters with a special property. For the case of principal filters $\mathcal{L} = \{A \mid A \supseteq L\}$ such a family is obtained iff L is a monoid. Thus from every monoid a topology can be constructed. Further results are given.

Résumé. — Étant donné une classe de langages \mathcal{L} , on définit un opérateur sur les langages $\text{Anf}_{\mathcal{L}}(A) = \{z \mid z \setminus A \in \mathcal{L}\}$. On connaissait déjà les familles \mathcal{L} correspondant à des opérateurs de fermeture. Dans cet article on décrit les familles \mathcal{L} correspondant à des opérateurs d'ouverture : ce sont des filtres avec une propriété caractéristique. Pour le cas de filtres principaux $\mathcal{L} = \{A \mid A \supseteq L\}$ cette propriété caractéristique est que L soit un monoïde. Par conséquent on peut construire une topologie pour chaque monoïde L . D'autres résultats sont formulés dans l'article.

1. INTRODUCTION

In [2] there are considered some special topologies on the free monoid Σ^* . For the sake of brevity, the reader is assumed to have a certain knowledge of this paper. If \mathcal{L} is a family of languages, let $\text{Anf}_{\mathcal{L}}(A) = \{z \mid z \setminus A \in \mathcal{L}\}$. It has been characterized in terms of 4 axioms what families \mathcal{L} produce closure operators $\text{Anf}_{\mathcal{L}}$. (So we know what families induce a topology on Σ^* ; from now on we call them \mathcal{L} -topologies.) Furthermore it was possible to know from the family of open sets whether or not the topology on Σ^* was an \mathcal{L} -topology.

In Section 2 we make some further remarks on our former paper.

It is well known that a topology can be described in some ways: closure operator, family of open sets, interior operator, neighbourhood system, etc. (We refer for topological conceptions to [1].) The first two ways with respect to \mathcal{L} -topologies are already considered in [2]; in Sections 3 and 4 the third and fourth possibility of generating an \mathcal{L} -topology are discussed.

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2. ADDITIONAL REMARKS ON OUR FIRST STUDY OF \mathcal{L} -TOPOLOGIES

We present a further example of an \mathcal{L} -topology: Let $A/w = \{z \mid zw \in A\}$ and assume $z \in \Sigma^*$ to be fixed. Let $\varphi_z(A) := \bigcup_{n \geq 0} A/z^n$. It is easy to see that φ_z fulfills the axioms (A1)-(A4) and is therefore a closure operator. Now, since $(x \setminus A)/y = x \setminus (A/y)$, it follows that:

$$\varphi_z(w \setminus A) = w \setminus \varphi_z(A) \quad \text{for all } w \in \Sigma^*.$$

So φ_z is *leftquotient-permutable* and thus by Lemma 2.7 of [2] $\varphi_z = \text{Anf}_{\mathcal{L}_z}$, where $\mathcal{L}_z = \{A \mid \varepsilon \in \varphi_z(A)\} = \{A \mid \text{there exists an } n \in N_0 \text{ such that } z^n \in A\}$. For $z = \varepsilon$ we obtain the *discrete topology*.

It is clear how this situation can be generalized. Let $M \subseteq \Sigma^*$ be a *submonoid* and $\varphi_M(A) := \bigcup_{m \in M} A/m$, then φ_M is the closure operator of an \mathcal{L} -topology with $\mathcal{L}_M = \{A \mid M \cap A \neq \emptyset\}$.

We present in short some examples of topologies which are *not* \mathcal{L} -topologies:

The closure operator $L \mapsto L\Sigma^*$; the closure operator $L \mapsto \Sigma^*L$; the (so called) *left topology*; let us recall that the *right topology* is an \mathcal{L} -topology (with closure operator *Init*).

THEOREM 2.1: *The following 3 statements are equivalent:*

- (i) $X_{\mathcal{L}}$ is a T_1 -space (i. e. each set $\{x\}$ is closed);
- (ii) $\partial(\mathcal{L})$ contains no set of cardinality 1;
- (iii) $\partial(\mathcal{L})$ contains no finite set.

Proof: The equivalence of (i) and (ii) has been already proved in [2]. Trivially, (iii) implies (ii). Now assume that (i) holds and $L \in \partial(\mathcal{L})$ be a finite set. Then, by (i), L is closed. But a set L in $\partial(\mathcal{L})$ can never be closed, because $\varepsilon \setminus L = L \in \partial(\mathcal{L}) \subseteq \mathcal{L}$ and $\varepsilon \notin L$.

3. INTERIOR OPERATORS AND \mathcal{L} -TOPOLOGIES

For a given topology, let I be the *interior operator*, defined by $I(A) = (\overline{A^c})^c$ (sometimes written as A^0).

THEOREM 3.1: *The interior operator of an \mathcal{L} -topology is leftquotient-permutable; the corresponding family \mathcal{L}_I is given by:*

$$\mathcal{L}_I = \{A \mid A^c \notin \mathcal{L}\}.$$

Proof: Since $(x \setminus B)^c = x \setminus B^c$ and $\text{Anf}_{\mathcal{L}}(x \setminus B) = x \setminus \text{Anf}_{\mathcal{L}}(B)$, we have:

$$I(x \setminus A) = [\text{Anf}_{\mathcal{L}}((x \setminus A)^c)]^c = [\text{Anf}_{\mathcal{L}}(x \setminus A^c)]^c = [x \setminus \text{Anf}_{\mathcal{L}}(A^c)]^c = x \setminus [\text{Anf}_{\mathcal{L}}(A^c)]^c = x \setminus I(A).$$

By [2]; Lemma 2.7, $\mathcal{L}_I = \{A \mid \varepsilon \in I(A)\}$. Now we have:

$$\begin{aligned} \varepsilon \in I(A) &\Leftrightarrow \varepsilon \in [\text{Anf}_{\mathcal{L}}(A^c)]^c \\ &\Leftrightarrow \varepsilon \notin \text{Anf}_{\mathcal{L}}(A^c) \Leftrightarrow \varepsilon \setminus A^c \notin \mathcal{L} \Leftrightarrow A^c \notin \mathcal{L}, \end{aligned}$$

thus $A \in \mathcal{L}_I \Leftrightarrow A^c \notin \mathcal{L}$.

Example: For $\mathcal{L} = \mathcal{P}_0(\Sigma^*)$, we have $\mathcal{L}_I = \{\Sigma^*\}$; $z \in I(A) \Leftrightarrow$ for all x holds $zx \in A$.

For $\mathcal{L} = \mathcal{U} \cup \{A \mid \varepsilon \in A\}$, we have $\mathcal{L}_I = \{A \mid A^c \text{ finite and } \varepsilon \in A\}$; $z \in I(A) \Leftrightarrow z \in A$ and for almost all x holds $zx \in A$.

In [2] there are given 4 axioms (T1)-(T4) which characterize the \mathcal{L} 's leading to closure operators [$\alpha(\mathcal{L}) = \mathcal{L}$ is assumed to hold].

A straightforward reformulation of this axioms in terms of \mathcal{L}_I yields:

THEOREM 3.2: *Let $\mathcal{L}_I \subseteq \{A \mid \varepsilon \in A\}$. Then \mathcal{L}_I leads to an interior operator iff (I1)-(I4) hold:*

$$\Sigma^* \in \mathcal{L}_I, \tag{I1}$$

$$A \in \mathcal{L}_I, A \subseteq B \Rightarrow B \in \mathcal{L}_I, \tag{I2}$$

$$A \in \mathcal{L}_I, B \in \mathcal{L}_I \Rightarrow A \cap B \in \mathcal{L}_I, \tag{I3}$$

$$A \in \mathcal{L}_I \Leftrightarrow \text{Anf}_{\mathcal{L}_I}(A) \in \mathcal{L}_I, \tag{I4}$$

REMARK: Similar as for \mathcal{L} in [2], it is possible to drop the condition $\mathcal{L}_I \subseteq \{A \mid \varepsilon \in A\}$ and to formulate other axioms. But this is not too meaningful and therefore omitted.

REMARK: Since $\Sigma^* \in \mathcal{L}$, it follows $\emptyset \notin \mathcal{L}_I$. This together with (I1)-(I3) leads to the surprising fact that:

\mathcal{L}_I is a (proper) filter.

So the question arise what filters fulfill the axiom (I4). For the special case of a *principal filter* $\mathcal{L}(L) := \{A \mid A \supseteq L\}$ this can be answered:

THEOREM 3.3: *$\mathcal{L}(L)$ fulfills axiom (I4) iff \mathcal{L} is a monoid.*

Proof: Let us reformulate axiom (I4) for this special situation: $A \in \mathcal{L}(L) \Leftrightarrow \text{Anf}_{\mathcal{L}(L)}(A) \in \mathcal{L}(L)$ means:

$$L \subseteq A \Leftrightarrow L \subseteq \text{Anf}_{\mathcal{L}(L)}(A) \Leftrightarrow L \subseteq \{z \mid L \subseteq z \setminus A\}.$$

Thus axiom (I4) is equivalent to:

$$L \subseteq A \Leftrightarrow [z \in L \Rightarrow L \subseteq z \setminus A]. \quad (*)$$

Setting $A = L$, $(*)$ implies:

$$z \in L \Rightarrow L \subseteq z \setminus L. \quad (**)$$

But a short reflection shows that $(**)$ is also equivalent to $(*)$ [and to (I4)!] Furthermore this means:

$$z \in L \Rightarrow [w \in L \Rightarrow w \in z \setminus L],$$

or:

$$z \in L, w \in L \Rightarrow zw \in L.$$

Since $\mathcal{L}(L) \subseteq \{A \mid \varepsilon \in A\}$ we have $\varepsilon \in L$, and the proof is finished.

REMARK: Each submonoid $M \subseteq \Sigma^*$ leads us to an \mathcal{L} -topology!

Let us recall the following fact from [2]: Let $X = (\Sigma^*, \mathfrak{D})$ be an \mathcal{L} -topology. Then:

$$\mathcal{L} = \mathcal{P}(\Sigma^*) - \{A \mid \text{there is an } 0 \in \mathfrak{D} \text{ such that } \varepsilon \in 0 \text{ and } A \subseteq 0^c\};$$

this family \mathcal{L} is unique subject to the condition $\mathcal{L} = \alpha(\mathcal{L})$. Now let us compute \mathcal{L}_I :

$$\begin{aligned} A \in \mathcal{L}_I &\Leftrightarrow A^c \notin \mathcal{L} \Leftrightarrow A^c \in \{B \mid \exists 0 \in \mathfrak{D} : \varepsilon \in 0 \text{ and } B \subseteq 0^c\} \\ &\Leftrightarrow \exists 0 \in \mathfrak{D} : \varepsilon \in 0 \text{ and } A^c \subseteq 0^c \Leftrightarrow \exists 0 \in \mathfrak{D} : \varepsilon \in 0 \subseteq A; \\ \mathcal{L}_I &= \{A \mid \exists 0 \in \mathfrak{D} : \varepsilon \in 0 \subseteq A\} \end{aligned}$$

and we find:

\mathcal{L}_I is the filter of neighbourhoods of ε !

By [2]; Lemma 2.13, we know A open \Leftrightarrow for all $x \in A$ holds $(x \setminus A)^c \notin \mathcal{L}$, which now simply means:

for all $x \in A$ holds $x \setminus A \in \mathcal{L}_I$!

Altogether it seems that it is easier to work with \mathcal{L}_I instead of \mathcal{L} !

Now we are ready to formulate a *general base representation theorem* (generalizing [2]; Theorems 3.3 and 3.4):

THEOREM 3.4: Let $X = (\Sigma^*, \mathfrak{D})$ be an \mathcal{L} -topology. Then:

$$\mathfrak{B} = \{x A \mid x \in \Sigma^*, A \in \mathcal{L}_I\} \text{ is a base for } \mathfrak{D}.$$

Proof: If 0 is open, then for all $x \in 0$ holds $x \setminus 0 \in \mathcal{L}_1$. Thus $x(x \setminus 0) \in \mathfrak{B}$ and $0 = \bigcup_{x \in 0} x(x \setminus 0)$.

4. SYSTEMS OF NEIGHBOURHOODS AND \mathcal{L} -TOPOLOGIES

A further method to generate a topology is to construct a system of neighbourhoods.

THEOREM 4.1: *Let $X = (\Sigma^*, \mathfrak{D})$ be an \mathcal{L} -topology and let $\mathfrak{B}(x)$ be the family of neighbourhoods of x . Then:*

$$\mathfrak{B}(x) = y \setminus \mathfrak{B}(yx).$$

Proof:

$$\begin{aligned} \mathfrak{B}(x) &= \{ A \mid \exists 0 \in \mathfrak{D} : x \in 0 \subseteq A \} = \{ A \mid \exists 0 \in \mathfrak{D} : \varepsilon \in x \setminus 0 \subseteq x \setminus A \}; \\ y \setminus \mathfrak{B}(yx) &= y \setminus \{ A \mid \exists 0 \in \mathfrak{D} : \varepsilon \in yx \setminus 0 \subseteq yx \setminus A \} \\ &= y \setminus \{ A \mid \exists 0 \in \mathfrak{D} : \varepsilon \in x \setminus (y \setminus 0) \subseteq x \setminus (y \setminus A) \} \\ &= \{ y \setminus A \mid \exists 0 \in \mathfrak{D} : \varepsilon \in x \setminus (y \setminus 0) \subseteq x \setminus (y \setminus A) \} \\ &= \{ A \mid \exists 0 \in \mathfrak{D} : \varepsilon \in x \setminus 0 \subseteq x \setminus A \}. \end{aligned}$$

REMARK: The property $\mathfrak{B}(x) = y \setminus \mathfrak{B}(yx)$ implies $y \mathfrak{B}(x) \subseteq \mathfrak{B}(yx)$.

We can prove also a converse of Theorem 4.1.

THEOREM 4.2: *Assume that there is a system of neighbourhoods $\{ \mathfrak{B}(x) \}$ satisfying:*

$$\mathfrak{B}(x) = y \setminus \mathfrak{B}(yx).$$

Then the topology is even an \mathcal{L} -topology.

Proof: By [2]; Theorem 2.16 it is sufficient to show that the system of open sets \mathfrak{D} is left stable.

Let 0 be open, i.e. 0 is neighbourhood of all its points, i.e.:

$$x \in 0 \Rightarrow 0 \in \mathfrak{B}(x).$$

To show: $z \setminus 0$ is open. Let $x \in z \setminus 0$, i.e. $zx \in 0$, i.e. $0 \in \mathfrak{B}(zx)$. By the condition: $z \setminus 0 \in z \setminus \mathfrak{B}(zx) = \mathfrak{B}(x)$.

Furthermore we have to show: $z0$ is open. Let $x \in z0$, i.e. $x = zy$ and $y \in 0$, i.e. $0 \in \mathfrak{B}(y)$. From the last remark: $z0 \in z \mathfrak{B}(y) \subseteq \mathfrak{B}(zy) = \mathfrak{B}(x)$.

REMARK: We know already that \mathcal{L}_I is simply $\mathfrak{B}(\varepsilon)$. So we have for all systems of neighbourhoods by means of the remark after Theorem 4.1:

$$\mathfrak{B}(x) \supseteq x \mathfrak{B}(\varepsilon) = x \mathcal{L}_I.$$

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